Chapter 1

Introduction and basic definitions

Exercise 1.1

Prove that $\mathcal{P}(\emptyset) = 0$.

Exercise 1.2

Prove that $\mathcal{P}(\overline{A}) = 1 - \mathcal{P}(A)$ for every event A.

Exercise 1.3

Prove that $\mathcal{P}(A \cup B) = \mathcal{P}(A) + \mathcal{P}(B) - \mathcal{P}(A \cap B)$.

Exercise 1.4

Consider a random experiment "tossing a fair coin twice". Define a sample space, an algebra of events (σ-field) and a probability for this random experiment. Give an example of two independent events, an example of two events that are not independent, an example of two mutually exclusive events and an example of two events that are not mutually exclusive. What is the probability that the head occurred in the second toss, given that it occurred in at least one toss.

Exercise 1.5

Give an example of two events A and B that are mutually exclusive, but not independent.

Exercise 1.6

Prove the following statement from the lecture: If A and B are mutually exclusive, then $\mathcal{P}(A \cap B) = 0$. If they are also independent, we obtain either $\mathcal{P}(A) = 0$ or $\mathcal{P}(B) = 0$.

Exercise 1.7

Is there a probability space $(\mathbf{S}, \mathcal{F}, \mathcal{P})$ such that

1. $\mathbf{S} = \mathbb{N}$ and $\mathcal{P}(n) > 0$ for all $n \in \mathbb{N}$?
2. $\mathbf{S} = \mathbb{N}$ and $\mathcal{P}(n) = 0$ for all $n \in \mathbb{N}$?
3. $\mathbf{S} = \mathbb{R}$ and $\mathcal{P}(r) > 0$ for all $r \in \mathbb{R}$?

We use the convention that $0 \notin \mathbb{N}$.

Exercise 1.8

Give an example of two independent events A and B, such that the conditional probability $\mathcal{P}(A|B)$ is not defined.

Exercise 1.9

Prove that if $\mathcal{P}(A \cap B) = \mathcal{P}(A)\mathcal{P}(B)$, then $\mathcal{P}(\overline{A} \cap B) = \mathcal{P}(\overline{A})\mathcal{P}(B)$.

Exercise 1.10

Let us consider a binary relation $\sim$ defined on algebra of events by $A \sim B$ iff A is independent of B. Is $\sim$ reflexive, symmetric or transitive?

Exercise 1.11

Throwing a (six-sided) die until the ace is reached, what is the probability that it will be reached after at least 4 throws, on the condition that it will not be reached at the first throw?

Exercise 1.12

Assume we are throwing a die as in Exercise 1.11. On the condition that the ace is reached in an even number of throws, what is the probability of reaching it in the second throw?

Exercise 1.13

Machines A, B, C produce 25%, 35% and 40% of a given commodity. Among the products of machine A, 5% are defective, for B there are 4% and 2% for C. On the condition of taking a defective product, what is the probability that it was produced by machine A, B, C?

Exercise 1.14

Let $\{B_1, B_2, \ldots, B_n\}$ be a partition of S such that $\mathcal{P}(B_i) > 0$ for all $1 \leq i \leq n$. Prove that $\mathcal{P}(A) = \sum_{i=1}^{n} \mathcal{P}(A|B_i)\mathcal{P}(B_i)$.

Exercise 1.15

Let A, B be two independent events with probabilities $\mathcal{P}(A), \mathcal{P}(B)$. Compute the probability of $A \cup B$.

Exercise 1.16

Three players a,b,c play rock-paper-scissors in a following way: First, a plays with b. Then the winner plays with c. In third round winner of the second round plays with the one who did not play in second round. Generally, in $(i+1)$-th round the winner of i-th round plays with the one who did not play in i-th round. The game proceeds until one of the players wins two consecutive rounds. If we write the play as a sequence of names of players who won respective rounds, atomic events are words of a language $(acb)^*(aa+acc+acbb)+(bca)^*(bb+bcc+bcaa)$ together with two independent words $(acb)^\omega$, $(bca)^\omega$. This defines the sample space S. Now consider a σ-algebra 2^S. Let us have a function that assigns 2^{-k} to each word of length k and 0 to an infinite word. Does this function induce a probability measure on this space? What is the probability that the player a (b,c) wins a play?

Exercise 1.17

Let us consider the problem to decide whether the two polynomials $P(x)$ and $Q(x)$ are equal. Polynomials can be specified in a number of ways, namely e.g. in the canonical form as

$$P(x) = p_d x^d + p_{d-1} x^{d-1} + \cdots + p_1 x + p_0, p_i \in \mathbb{R}.$$

In general any expression can be considered to be a polynomial if it is built up from a variable and constants using only addition, subtraction, multiplication, and raising expressions to constant positive integer, e.g. $(x+2)^3 - 11$. An obvious algorithm to decide equality of two polynomials is to transform both input polynomials to the canonical form and compare respective coefficients. This algorithm takes $\mathcal{O}(d^2)$ multiplications, with d being the degree of the polynomial.

Consider the following randomized algorithm to test the polynomial equality:

Computing the values of $P(r)$ and $Q(r)$ is in $\mathcal{O}(d)$. If uniform and random integer in line 1 is generated in $\mathcal{O}(d)$, than the algorithm is in $\mathcal{O}(d)$.

Algorithm 1 Randomized Polynomial Equality Testing

Require: Two polynomials $P(x)$ and $Q(x)$ of degree at most d.

Select uniformly and randomly an integer $r \in_R \{1, \ldots, 100d\}$
if $P(r) = Q(r)$ **then**
    return 'YES'
else
    return 'NO'
end if

What is the probability that Algorithm 1 gives the wrong answer?