Lecture 4 - Random walk, ruin problems and random processes

Jan Bouda

FI MU

April 19, 2009

Part I

Random Walk and the Ruin Problem

Classical ruin problem

- Consider a gambler who wins or loses a euro with probability p and $q = 1 - p$, respectively.
- Let his initial capital be z and he plays against an adversary with initial capital $a - z$, i.e. their combined capital is a.
- The game continues until the gambler is ruined (his capital is 0) or he wins, i.e. his capital grows to a. In other words, the game continues until one of the players is ruined.
- We are interested in the probability of gambler's ruin and the probability distribution of the duration of the game.
- This is the classical ruin problem.

Random walk

- Physical applications include e.g. position of a particle on the x-axis. This particle starts from the position z and moves in each step in positive or negative direction.
- The position of the particle after n steps is the gambler's capital after n trials.
- The process terminates when the particle reaches first time position 0 or a.
- We say that the particle performs a **random walk with absorbing barriers at** 0 **and** a.
- This random walk is restricted to positions $1, 2, \ldots a - 1$.
- In the absence of barriers the random walk is called **unrestricted**.

Random walk - generalizations

- In the limiting case $a \to \infty$ we get a random walk on a semi-infinite line $(0, \infty)$. In this case a particle starting at $z > 0$ performs a random walk until the moment it reaches the origin first time. This is called the first-passage time problem.
- A possible generalization is to replace absorbing barrier by either **reflecting** or **elastic barrier**.
- When particle reaches position next to a **reflecting barrier**, say e.g. 1, then it has the probability p to move to 2 and the probability q to stay at 1 (particle tries to go to 0, but is reflected back).
- In gambling terminology this means that player is never ruined. His adversary generously grants him one euro in case the player looses his last one.

Random walk - generalizations

- An **elastic barrier** is a generalization uniting both absorbing and reflecting barrier.
- The elastic barrier at the origin works in the following way. From position 1 the particle
  - moves with probability p to position 2.
  - stays in 1 with probability δq at 1.
  - with probability $(1-\delta)q$ it moves to 0 and is absorbed (process terminates).
- For $\delta = 0$ we get the absorbing barrier.
- For $\delta = 1$ we get the reflecting barrier.

Classical ruin problem

Let us revisit the problem of the classical ruin. Let q_z be the probability of gambler's ultimate ruin and p_z of his winning, given his initial capital is z. In the random walk terminology q_z and p_z are probabilities that particle is absorbed at 0 or a, respectively. We will calculate these probabilities and verify that they sum to 1 to rule out the possibility of an unending walk of nonzero probability.

After the first trial the gambler's capital is either $z+1$ or $z-1$ and therefore

$$q_z = pq_{z+1} + qq_{z-1} \tag{1}$$

provided $1 < z < a-1$. Note that this equation is linear, i.e. for $A, B \in \mathbb{R}$ and solutions q_z and q'_z of Eq. (1) $Aq_z + Bq'_z$ is a solution of Eq. (1) too.

To complete our derivations, we observe that

$$q_0 = 1 \text{ and } q_a = 0. \tag{2}$$

Systems of equations of the form (1) are known as **difference equations** and (2) represents **boundary conditions** on q_z.

Suppose that $p \neq q$. It is easy to verify that the system of equations (1) admits solutions $q_z = 1$ and $q_z = (q/p)^z$ (substitute these values to (1) to verify). From linearity it follows that for any parameters A and B

$$q_z = A + B\left(\frac{q}{p}\right)^z \tag{3}$$

is solution of (1). On the other hand, the previous equation describes all possible solutions of the system of equations (1). To see this, observe that any particular solution is fully defined by two values of q_i, e.g. q_0 and q_1 (when we fix these two values we can calculate all remaining recursively). We substitute (3) into two equations of the form (1) for two chosen values of q_i. This gives exactly one solution of eq. (3). Therefore, (3) specifies all possible solution of the system of equations (1).

Classical ruin problem

Using the boundary conditions (2) we get $A + B = 1$ and $A + B(q/p)^a = 0$. Combining with (3) we get

$$q_z = \frac{(q/p)^a - (q/p)^z}{(q/p)^a - 1} \tag{4}$$

as a solution of (1) satisfying the boundary (2).
The previous derivation fails if $p = q = 1/2$ since probabilities $q_z = 1$ and $q_z = (q/p)^z$ are identical. However, $q_z = z$ is a solution of Eq. (1) in this case, and therefore, using analogous arguments as in the previous case, $q_z = A + Bz$ are all solutions of the system (1). Using the boundary conditions (2) we get

$$q_z = 1 - \frac{z}{a}. \tag{5}$$

The same solution we obtain from (4) by calculating the limit for $p \to 1/2$.

Classical ruin problem

The probability that player wins is equal to the probability that his adversary is ruined and therefore we obtain it from equations (4) and (5) when replacing p, q and z by q, p and $a-z$, respectively. It follows that $p_z + q_z = 1$.

In this game gambler's payoff is a random variable G attaining values $a-z$ and $-z$ with probabilities $1-q_z$ and q_z, respectively. The expected gain is

$$E(G) = a(1-q_z) - z. \tag{6}$$

Clearly, $E(G) = 0$ if and only if $p = q = 1/2$. This means that using such a sequence of games, a fair game remains fair and no unfair game can become a fair one.

Classical ruin problem - remarks

- In case $p = q$ the player having 999 euros has the probability 0.999 to win one euro before loosing all his capital. With probabilities $q = 0.6$ and $p = 0.4$ the game is unfair, but still the probability of winning a euro is around $2/3$.
- In general, a gambler with large capital z has a reasonable chance to win a small amount $a - z$ before being ruined.
- Interpretation!

Example

A certain man used to visit Monte Carlo year after year and was always successful to win enough to cover his vacations. He believed in a magical power of his luck. Assuming that he started with ten time the cost of vacation, he had chance 0.9 in each year to cover his vacations from the wins. Is it really the ultimate way how to pay your vacations?

Classical ruin problem - changing the stakes

Let us suppose that we have fixed capitals, i.e. z and a, and we are changing the stakes, e.g. instead of one euro we bet in each trial 50 cents. We obtain the corresponding probability of ruin q'_z from (4) on replacing z by $2z$ and a by $2a$

$$q'_z = \frac{(q/p)^{2a} - (q/p)^{2z}}{(q/p)^{2a} - 1} = q_z \frac{(q/p)^a + (q/p)^z}{(q/p)^a + 1}. \tag{7}$$

For $q > p$ the last fraction is greater than unity and $q'_z > q_z$. Therefore, is stakes are decreased, the probability of ruin increases for gambler with $p < 1/2$. On contrary, if stakes are doubled, the probability of ruin decreases for player with success probability $p < 1/2$ and increases for the adversary.

Classical ruin problem - changing the stakes

Example

Suppose that Alice owns 90 euros and Bob 10 and let $p = 0.45$ (the game is unfavorable for Alice). In case in each trial the bet is 1 euro, the probability of Alice's ruin is appx. 0.866. If the same game is played with bet 10 euros, then the probability of Alice's ruin drops to 0.210. In general, multiplying the bet by k results in equation where we replace z by z/k and a by a/k.

Classical ruin problem - rich adversary:-)

The case when $a = \infty$ corresponds to the case when the adversary is infinitely rich. Sending $a \to \infty$ in Equations (4) and (5) we get

$$q_z = \begin{cases} 1 & \text{if } p \leq q \\ (q/p)^z & \text{if } p > q. \end{cases} \tag{8}$$

q_z is the probability of ultimate ruin of a gambler with initial capital z playing against an infinitely rich adversary. In random walk terminology it is the probability that the particle ever reaches the origin. Alternatively, it is the probability that a random walk starting at the origin reaches the position $z > 0$ (with no limitation towards $-\infty$). Therefore, this probability equals 1 if $p \geq q$ and $(p/q)^z$ when $p < q$.

Classical ruin problem - expected duration of the game

Let us return to the original example with absorbing barriers at 0 and a. Suppose the duration of the game (number of trials until one of the barriers is reached) has a finite expectation (indeed it has) D_z. We would like to calculate it.

If the first trial is success, the game continues as if it started from the point $z+1$ and analogously if it is a failure. The conditional expectation assuming that the first trial was success is $d_{z+1}+1$. Therefore the expected duration satisfies the difference equation

$$D_z = pD_{z+1} + qD_{z-1} + 1 \tag{9}$$

with the boundary conditions

$$D_0 = 0 \quad D_a = 0. \tag{10}$$

Classical ruin problem - expected duration of the game

The term '+1' makes Equation (9) nonhomogenous and we cannot apply the same solution as in the case of the ruin probability. If $p \neq q$, then $D'_z = z/(q-p)$ is a solution of (9).

Let D_z and D'_z be solutions of Eq. (9). We will examine the dependence of $\Delta_z = D_z - D'_z$ on Δ_{z-1} and Δ_{z+1}. We get

$$\begin{aligned} \Delta_z = D_z - D'_z &= (pD_{z+1} + qD_{z-1} + 1) - (pD'_{z+1} + qD'_{z-1} + 1) = \\ &= p(D_{z+1} - D'_{z+1}) + q(D_{z-1} - D'_{z-1}) = p\Delta_{z+1} + q\Delta_{z-1} \end{aligned} \tag{11}$$

We know that all solutions of this equation are of the form $A + B(q/p)^z$. For $p \neq q$ all solutions of (9) are

$$\frac{z}{q-p} + \Delta_z = D_z = \frac{z}{q-p} + A + B\left(\frac{q}{p}\right)^z. \tag{12}$$

The boundary conditions give that

$$A + B = 0 \quad A + B(q/p)^a = -a/(q-p). \tag{13}$$

Classical ruin problem - expected duration of the game

Solving for A and B we get

$$D_z = \frac{z}{q-p} - \frac{a}{q-p}\frac{1-(q/p)^z}{1-(q/p)^a}. \tag{14}$$

It remains to perform an independent analysis for the case $p = q = 1/2$. In this case we replace $z/(q-p)$ by $-z^2$ (as a solution of (9) for $p = q = 1/2$). It follows that in this case all solutions of (9) are of the form $D_z = -z^2 + A + Bz$. Using the boundary conditions (10) we get the solution (all solutions of Δ_z are $A + Bz$)

$$D_z = z(a-z). \tag{15}$$

Classical ruin problem - expected duration of the game

Example

Duration of the game may be considerably longer than one may expect. Consider two players with 500 euros each. They toss the coin until one of them is ruined. Average duration of the game is 250000 trials.

Example

In the limiting case $a \to \infty$ we get expected duration $z/(q-p)$ for $p < q$, but when $q = p$ the expected duration is infinite. In case $p > q$ the game may go on forever and the average value has no sense.

Part II

Random processes

Random process

Definition (Random process)

A **random process** (or **stochastic process**) is a family of random variables $\{X(t) | t \in \mathbf{T}\}$, defined on a common probability space and indexed by the parameter t, where t varies over an index set $\mathbf{T}$.

The values assumed by the random variable $X(t)$ (i.e. image) are called **states** and the set of all possible values forms the **state space** of the process. The state space is often denoted by $\mathbf{I}$. Interpretation is that some random process is in one of possible states $\mathbf{I}$ and this state is changing in time.

Recalling that a random variable is a function defined on a sample space S of the underlying experiment, the above defined family of random variables is a family of real values $\{X(t,s) | s \in S, t \in \mathbf{T}\}$. For a fixed value $t_1 \in \mathbf{T}$, $X_{t_1} = X(t_1, s)$ is a random variable as s varies over S. For a fixed sample point $s_1 \in S$, the expression $X_{s_1}(t) = X(t, s_1)$ is a function of time t, called a **sample function** of a **realization** of the process.

Classification of random processes

- If the state space of a stochastic process is discrete, then it is called a **discrete-state process** or **chain**. The state space is often assumed to be $\{0, 1, 2 \ldots\}$. If the state space is continuous, we have **continuous-state process**.
- If the index set $\mathbf{T}$ is discrete, we have a **discrete-time (parameter) process**. Otherwise we have a **continuous-time (parameter) process**.
- A discrete time process is also called a **stochastic sequence**.

Classification of random processes

- For a fixed time t_1, the term $X(t_1)$ denotes a random variable that describes the state of the process at time t_1. For a fixed number $x_1 \in \mathbb{R}$ the probability of the event $[X(t_1) \leq x_1]$ gives the distribution function of $X(t_1)$ by

$$F(x_1; t_1) = F_{X(t_1)}(x_1) = P[X(t_1) \leq x_1].$$

- $F(x_1; t_1)$ is known as the *first-order distribution* of the process $\{X(t)|t \in \mathbf{T}\}$.
- Given two fixed time instants $t_1, t_2 \in \mathbf{T}$, $X(t_1)$ and $X(t_2)$ are two random variables on the same probability space. Their joint distribution is known as the *second-order distribution* of the process and is given by

$$F(x_1, x_2; t_1, t_2) = P[X(t_1) \leq x_1, X(t_2) \leq x_2].$$

Classification of random processes

- In general, we define the *n–th order joint distribution* of the random process $\{X(t)\}$ by

$$F(\vec{x};\vec{t}) = P[X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n]$$

  for all $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\vec{t} = (t_1, \ldots, t_n) \in \mathbf{T}^n$ such that $t_1 < t_2 < \cdots < t_n$
- In practice we are interested in processes with more compact description. For instance, the n–th order joint distribution is often invariant under shifts of the time origin.

Classification of random processes

Definition (Strictly stationary process)

A random process $\{X(t)|t \in \mathbf{T}\}$ is said to be stationary in the strict sense if for all $n \geq 1$ its n–th order joint distribution function satisfies

$$F(\vec{x}; \vec{t}) = F(\vec{x}; \vec{t} + \tau)$$

for all vectors $\vec{x} \in \mathbb{R}^n$, all $\vec{t} \in \mathbf{T}^n$, and all scalars τ such that $t_i + \tau \in \mathbf{T}$. The notation $\vec{t} + \tau$ means that τ is added to each component of $\vec{t}$.

- Let $\mu(t) = E[X(t)]$ denotes the time-dependent mean of the stochastic process. It is often called the **ensemble average** of the stochastic process.
- Applying on the first-order distribution function of strictly stationary process, we get $F(x; t) = F(x; t + \tau)$ or, equivalently, $F_{X(t)} = F_{X(t+\tau)}$ for all τ. It follows that strict-sense stationary random process has a time-independent mean, i.e. $\mu(t) = \mu$ for all $t \in \mathbf{T}$.

Classification of random processes

- Another possibility to simplify the joint distribution function is to restrict the nature of dependence between random variables $\{X(t)\}_{t\in\mathbf{T}}$. The simplest is the case of independent random variables.

Definition (Independent process)

A random process $\{X(t)|t\in\mathbf{T}\}$ is said to be an **independent process** provided that its n–th order joint distribution function satisfies the condition

$$F(\vec{x};\vec{t})=\prod_{i=1}^{n}F(x_i;t_i)=\prod_{i=1}^{n}P[X(t_i)\leq x_i]. \tag{16}$$

Classification of random processes

A special case of independent process is

Definition (Renewal process)

A **renewal process** is defined as the discrete-time independent process $\{X_n | n = 1, 2, \dots\}$, where $X_1, X_2, \dots$ are independent, identically distributed, nonnegative random variables.

The assumption that random variables are mutually independent is often too restrictive and more practical is the simplest form of dependence - the first-order, or **Markov dependence**.

Classification of random processes

Definition (Markov process)

A random process $\{X(t)|t \in \mathbf{T}\}$ is called a **Markov process** if for any $t_0 < t_1 < \cdots < t_n < t$ the conditional distribution of $X(t)$ given the values of $X(t_0), \ldots, X(t_n)$ depends only on $X(t_n)$, i.e. for all $x_0, \ldots, x_n, x$

$$P[X(t) \leq x|X(t_n) = x_n, \ldots, X(t_0) = x_0] = P[(X(t) = x|X(t_n) = x_n]. \tag{17}$$

In many interesting problems the conditional probability distribution (17) is invariant with respect to the time origin:

$$P[(X(t) = x|X(t_n) = x_n] = P[(X(t - t_n) = x|X(0) = x_n] \tag{18}$$

In this case the Markov chain is **(time-)homogenous**. Note that the stationarity of the conditional distribution does not imply the stationarity of the joint distribution function. A homogenous Markov process need not be a stationary process.

Part III

Bernoulli and Binomial Process

Bernoulli process

Definition (Bernoulli process)

Consider a sequence of Bernoulli trials and let the random variable Y_i denotes the result of the i–th trial (e.g. $[Y_i = 1]$ denotes success). Further assume that the probability of success on the i–th trial $P(Y_i = 1) = p$ is independent of i. Then $\{Y_i | i = 1, 2, \dots\}$ is a discrete-time, discrete-state random process, which is stationary in the strict sense. It is known as the **Bernoulli process**.

Recall that Y_i is a Bernoulli random variable and therefore

$$\begin{aligned} E(Y_i) &= p \\ E(Y_i^2) &= p \\ Var(Y_i) &= p(1-p). \end{aligned} \tag{19}$$

Binomial process

Based on random variables Y_i we can form another process by considering the sequence of partial sums $\{S_n | n = 1, 2, \dots\}$, where $S_n = Y_1 + Y_2 + \cdots Y_n$. From the property $S_n = S_{n-1} + Y_n$ it is easy to see that $\{S_n\}$ is a discrete-time, discrete-state Markov process, with

$$P(S_n = k | S_{n-1} = k) = P(Y_n = 0) = 1 - p \tag{20}$$

and

$$P(S_n = k | S_{n-1} = k - 1) = P(Y = 1) = p.$$

S_n is a binomial random variable and therefore this random process is called a **binomial process**.