

BULLETIN

OF THE

AUSTRALIAN MATHEMATICAL SOCIETY

[\(BibTeX\)](#)

[\(XML metadata\)](#)

[\(RSS feed\)](#)

pp. 17–30: Rainer Löwen and Burkard Polster
Linear geometries on the Moebius strip: a theorem of Skornjakov type.

**Volume 72, Number 1
August, 2005**

[\(MathReviews\)](#)

[\(Zentralblatt\)](#)

*The Australian Mathematical Publishing Assoc. Inc.
Australian National University, ACT 0200, Australia*

Printed in Australia by Pirion Printing

Print Post approved - PP229219/00095

ISSN 0004-9727

THE AUSTRALIAN MATHEMATICAL SOCIETY

President: M.G. Cowling
Department of Pure Mathematics,
The University of New South Wales,
Sydney NSW 2052, Australia.

Secretary: E.J. Billington
Department of Mathematics,
The University of Queensland,
Queensland 4072, Australia.

Treasurer: A. Howe
Department of Mathematics,
The Australian National University,
Canberra ACT 0200, Australia

Membership and correspondence.

Applications for membership, notices of change of address or title or position, members' subscriptions and correspondence related to accounts should be sent to the Treasurer. Correspondence about the distribution of the Society's BULLETIN, GAZETTE, and JOURNALS, and orders for back numbers should be sent to the Treasurer. All other correspondence should be sent to the Secretary.

The Bulletin.

The Bulletin of the Australian Mathematical Society began publication in 1969. Normally two volumes of three numbers are published annually. The BULLETIN is published for the Australian Mathematical Society by the Australian Mathematical Publishing Association Inc.

Editor: Alan S. Jones
Deputy Editor: Graeme A. Chandler
Department of Mathematics,
The University of Queensland,
Queensland 4072, Australia.

ASSOCIATE EDITORS

Robert S. Anderssen	B.D. Craven	B.D. Jones	M. Murray
R. Bartnik	Brian A. Davey	Owen D. Jones	J.H. Rubinstein
Elizabeth J. Billington	J.R. Giles	G.I. Lehrer	Jamie Simpson
G. Cairns	J.A. Hempel	K.L. McAvaney	Brailey Sims
J. Clark	B.D. Hughes	A.G.R. McIntosh	Ross Street
G.L. Cohen	G. Ivanov	Terry Mills	R.P. Sullivan
John Cossey	B.R.F. Jefferies	Sidney A. Morris	H.B. Thompson
N.S. Trudinger		A.J. van der Poorten	

©Copyright Statement Where necessary, permission to photocopy for internal or personal use or the internal or personal use of specific clients is granted by the Treasurer, Australian Mathematical Publishing Association, Inc., for libraries and other users registered with the Copyright Clearance Center (CCC), provided that the base fee of \$A2.00 per copy of article is paid directly to CCC, 21 Congress Street, Salem, MA 01970, U.S.A. Special requests should be addressed to the Treasurer, Australian Mathematical Publishing Association, Inc., School of Mathematical Sciences, ANU, Canberra ACT 0200 Australia. Serial-fee code: 0004-9727/05 \$A2.00 + 0.00.

INFORMATION FOR AUTHORS

The Bulletin of the Australian Mathematical Society aims at quick publication of original research in all branches of mathematics. The Editors receive more than three times as much material as can be published in the BULLETIN; many meritorious papers can, therefore, not be accepted. Authors are asked to avoid, as far as possible the use of mathematical symbols in the title. Manuscripts are accepted for review with the understanding that the same work is not concurrently submitted elsewhere.

To ensure speedy publication, editorial decisions on acceptance or otherwise are taken quickly, normally within a month of receipt of the paper. Papers are accepted only after a careful evaluation by the Editor and an Associate Editor or other expert in the field. As even minor revisions are generally not permitted, authors should read carefully all the details listed below. For a paper to be acceptable for publication, not only should it contain new and interesting results but also

- (i) the exposition should be clear and attractive;
- (ii) the manuscript should be in publishable form, without revision.

Authors should submit three clean, high quality copies to

The Editorial Office, Bulletin of the Australian Mathematical Society,
Department of Mathematics, The University of Queensland,
Queensland 4072, Australia.

Unless requested at the time, material submitted to the BULLETIN will usually not be returned.

EDITORIAL POLICY

1. References. Arrange references alphabetically (by surname of the first author) and cite them numerically in the text. Ensure the accuracy of the references: authors' names should appear as in the work quoted. Include in the list of references only those works cited, and avoid citing works which are "in preparation" or "submitted". Where the work cited is not readily accessible (for example, a preprint) a photocopy of the title page and relevant sections of the copy that you have used should be included with your submission.

2. Abstracts.

1. Each paper must include an abstract of not more than 200 words, which should contain a brief but informative summary of the contents of the paper, but no inessential details.
2. The abstract should be self-contained, but may refer to the title.
3. Specific references (by number) to a section, proposition, equation or bibliographical item should be avoided.

3. Subject Classification. Authors should include in their papers one or more classification numbers, following the 2000 Mathematics Subject Classification. Details of this scheme can be found in each Annual Index of Mathematical Reviews or on the web at <http://www.ams.org/msc> .

4. Abstracts of Ph.D. Theses. The Bulletin endeavours to publish abstracts of all accepted Australasian Ph.D. theses in mathematics. One restriction, however, is that the abstract must be received by the Editor within 6 months of the degree being approved.

5. Electronic Manuscripts. The Bulletin is produced using $\mathcal{A}\mathcal{M}\mathcal{S}\text{-T}\mathcal{E}\mathcal{X}$. Authors who are able to do so are invited to prepare their manuscripts using $\text{T}\mathcal{E}\mathcal{X}$. (We accept Plain $\text{T}\mathcal{E}\mathcal{X}$, $\mathcal{A}\mathcal{M}\mathcal{S}\text{-T}\mathcal{E}\mathcal{X}$ or $\text{L}\mathcal{A}\text{T}\mathcal{E}\mathcal{X}$.) Hard copy only should be submitted for assessment, but if the paper is accepted the author will be asked to send the text on an IBM PC compatible diskette or via e-mail to ams@maths.uq.edu.au. [Typed manuscripts are, of course, still acceptable.]

Bulletin of the Australian Mathematical Society

A note on the lattice of density preserving maps	
Sejal Shah and T.K. Das	1
A strong excision theorem for generalised Tate cohomology	
N. Mramor Kosta	7
Linear geometries on the Moebius strip: a theorem of Skornyakov type	
Rainer Löwen and Burkhard Polster	17
Div-curl type theorems on Lipschitz domains	
Zengjian Lou	31
A nonlinear map for midpoint locally uniformly rotund renorming	
S. Lajara and A.J. Pallarés	39
A remarkable continued fraction	
David Angell and Michael D. Hirschhorn	45
A new variational method for the $p(x)$-Laplacian equation	
Marek Galewski	53
Boundary unique continuation theorems under zero Neumann boundary conditions	
Xiangxing Tao and Songyan Zhang	67
On the Ky Fan inequality and related inequalities II	
Edward Neuman and József Sándor	87
Finite presentability of some metabelian Hopf algebras	
Dessislava H. Kochloukova	109
On the monotonicity properties of additive representation functions	
Yong-Gao Chen, András Sárközy, Vera T. Sós and MinTang	129
Generation of diagonal acts of some semigroups of transformations and relations	
Peter Gallagher and Nik Ruškuc	139
Subalgebras of free restricted Lie algebras	
R.M. Bryant, L.G. Kovács and Ralph Stöhr	147
A multiple character sum evaluation	
Dae San Kim	157
Implicit vector equilibrium problems via nonlinear scalarisation	
Jun Li and Nan-Jing Huang	161

ABSTRACTS OF AUSTRALASIAN Ph.D. THESES

Numerical methods for quantitative finance	
Jamie Alcock	173

LINEAR GEOMETRIES ON THE MOEBIUS STRIP: A THEOREM OF SKORNYAKOV TYPE

RAINER LÖWEN AND BURKARD POLSTER

We show that the continuity properties of a stable plane are automatically satisfied if we have a linear space with point set a Moebius strip, provided that the lines are closed subsets homeomorphic to the real line or to the circle. In other words, existence of a unique line joining two distinct points implies continuity of join and intersection. For linear spaces with an open disk as point set, the same result was proved by Skornyakov.

1. INTRODUCTION

A *flat stable plane* (E, \mathcal{L}) consists of a *point space* E , which is a surface (topological 2-manifold), and a system \mathcal{L} of *lines*, which are closed subsets of E , such that any two points are joined by a unique line and that the operations of join and intersection are continuous. Moreover, it is required that intersection is *stable*, that is, the set of pairs of distinct intersecting lines is open. For a comprehensive introduction to general stable planes (not only flat ones), see [6].

The lines of a flat stable plane are 1-manifolds ([10]), and if they are connected, then the point surface E is either a topological open disk D , or the compact nonorientable surface P of genus 1 (the real projective plane), or an open (that is, boundaryless) Moebius strip M , see [8, 11]. The flat stable planes with $E = P$ and lines homeomorphic to the circle are of particular importance and are called *flat projective planes*.

Conversely, if (E, \mathcal{L}) is a linear space (that is, two points are joined by a unique line) and lines are connected 1-manifolds and closed subsets of E , then it has been proved by Skornyakov for $E = D$ and by Salzmann for $E = P$ (see [10, 2.5] and [12, 32.5]), that (E, \mathcal{L}) is a flat stable plane, that is, that join and intersection are automatically continuous. Here we prove a similar result for $E = M$:

THEOREM. *Let M be an open Moebius strip and suppose we are given a system \mathcal{L} of closed subsets $L \subseteq M$, called lines, such that lines are connected 1-manifolds, and such that any two points are joined by a unique line.*

Received 17th January, 2005

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9729/05 \$A2.00+0.00.

Then one may define a topology on \mathcal{L} such that (M, \mathcal{L}) becomes a flat stable plane, that is, the operations of join and intersection of point and line pairs, respectively, are continuous and intersection is stable.

Similar characterisations have been derived for circle planes on surfaces in which any three points in ‘general position’ are connected by a unique circle, see [4, 5, 13, 16].

Flat stable planes with point set M and connected 1-manifolds as lines have been investigated by Betten [2], who completely determined the most homogeneous ones among them. Examples of such flat stable planes arise from flat projective planes ($E = P$) in two ways: either by deleting a single point from P or by deleting a closed disk which is convex in the sense that it intersects every line in an interval. This disk could be a conic together with its interior points in the real projective plane, yielding the most homogeneous example. The two constructions produce planes with slightly different behaviour. In the first case, every point is incident with a unique noncompact line, and the missing point of the original projective plane can be restituted in a unique way. Such planes were called pointwise coaffine in [7]. In the other case, every point is incident with infinitely many compact and noncompact lines, and it is not clear that all examples arise from projective planes by deletion of disks. The problem of embedding such planes into projective ones amounts to constructing an ideal boundary and pasting a compact disk in the sense of Stroppel [15] to this boundary; this has to be done in a way compatible with the line system. We suspect that this is not always possible.

The proof of our theorem uses the theorem of Skornyakov for the disk. Our strategy is to locate enough subplanes with point set homeomorphic to $D \approx \mathbb{R}^2$ and lines homeomorphic to \mathbb{R} (so-called $(\mathbb{R}^2, \mathbb{R})$ -planes) within the given plane, to apply Skornyakov’s result to them and to derive our result.

We would like to prove a Skornyakov type theorem in full generality: The point set would be an arbitrary surface, and the lines would be closed subsets which are 1-manifolds, connected or not. We roughly got as far as showing that every point is on the boundary of some $(\mathbb{R}^2, \mathbb{R})$ -subplane, but we would need that every point actually belongs to such a subplane.

2. PROOF OF THE THEOREM

We think of the point set M as the real projective plane P minus one point ∞ , even if our geometry is not pointwise coaffine. We depict P as a circular disk, whose boundary points are identified in antipodal pairs, that is, $|x| \leq 1$ holds for all points, and $x = -x$ if $|x| = 1$. The point ∞ will always be represented by the pair $\{(0, 1), (0, -1)\}$, as in Figure 1.

Since lines are closed subsets $L \subseteq M$, their closure \bar{L} in the one-point compactification P will always be homeomorphic to a circle. This circle contains the point ∞ if and only if L is not compact.

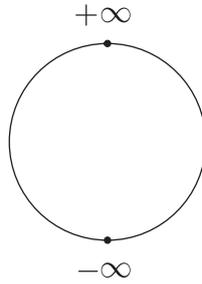


FIGURE 1: The Moebius strip.

Consider the two-sheeted covering $\pi : \mathbb{S}_2 \rightarrow P$ and the inverse image $\pi^{-1}(\bar{L})$ of one of these circles. This set is a covering of \bar{L} , and there are two possibilities: either $\pi^{-1}(\bar{L})$ consists of two disjoint circles $C, -C$ which are interchanged by the antipodal map, or $\pi^{-1}(\bar{L})$ is a single circle $C = -C$, and π restricted to C is the unique two-fold connected covering of \bar{L} .

Consider the case $C = -C$. Applying the theorem of Schoenflies [9] to each of the two closed disks in \mathbb{S}_2 having $C = -C$ as their boundary, we obtain a homeomorphism of each disk onto a disk defined by the equator of the sphere. The two homeomorphisms may be adjusted (for example, using polar coordinates) such that they agree on C and are compatible with the antipodal map, which shows that the pair (P, \bar{L}) is homeomorphic to the pair formed by the real projective plane and one of its lines. In this case, the line L will be called *essential*. The complement $P \setminus \bar{L}$ is an open disk.

A similar procedure (needing extra corrections on the annulus defined by C and $-C$) may be applied in the case of two disjoint circles and results in a proof that (P, \bar{L}) then is homeomorphic to the pair formed by the real projective plane with a conic. In this case, the line L will be called *inessential*. The complement of \bar{L} in P is a disjoint union of an open disk and an open Moebius strip in this case. Our first aim is to show that inessential lines do not occur (Proposition 5).

Each of the two cases splits into subcases depending on the position of ∞ relative to \bar{L} : whether ∞ is in \bar{L} or not in \bar{L} , and in the latter case, which of the complementary components of \bar{L} contains ∞ . The resulting five possibilities are depicted in Figures 2 through 6.

The figures also show the connected components of the complement of a line relative to M . Essential lines do not separate the point set M ; their complement is a punctured disk or an intact disk depending on whether the line is compact or not. An inessential line L separates M into two components. If L is noncompact, then one complementary component is a disk and the other is a Moebius strip. If L is compact, then one of these components is punctured.

PROPOSITION 1. *There are no compact inessential lines.*

PROOF: Such a line L would bound a subset $D \subseteq M$ homeomorphic to a compact disk or to a compact Moebius strip, and the line $u \vee v$ joining an interior point $u \in D$ to a point v outside D would intersect the boundary ∂D twice (remember that the closure of $u \vee v$ in P is a topological circle). This is a contradiction. \square

PROPOSITION 2. *The geometry induced on the open disk bounded by a noncompact inessential line is an $(\mathbb{R}^2, \mathbb{R})$ -plane. According to Skornyakov's Theorem, it is a stable plane.*

Remember that our aim is to exclude this particular type of lines. The proposition is merely a step towards this goal, it cannot contribute to our construction of a topology on \mathcal{L} .

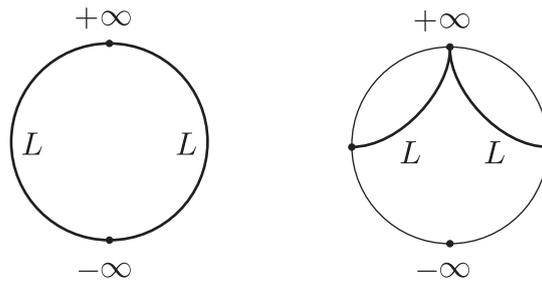


FIGURE 2: Two noncompact essential lines.

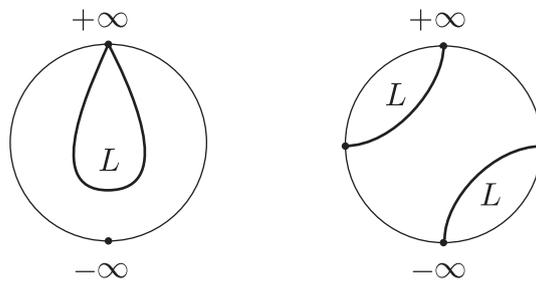


FIGURE 3: Two noncompact inessential lines.

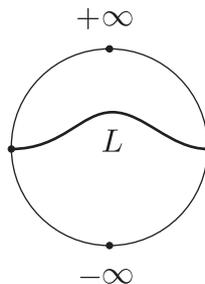


FIGURE 4: A compact essential line.

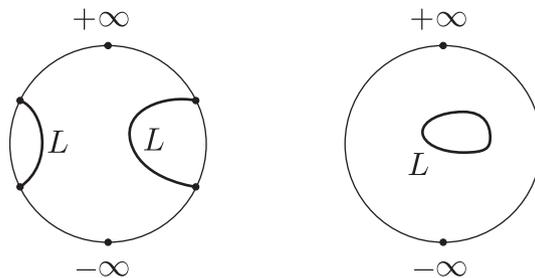


FIGURE 5: Two compact inessential lines of first type.

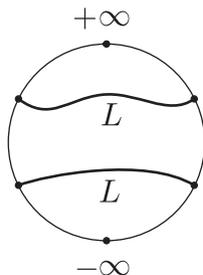


FIGURE 6: A compact inessential line of second type.

PROOF: Let $D \approx \mathbb{R}^2$ be this disk and L the bounding line. We have to show that each line intersects D either in an open interval or in the empty set. A compact line entirely contained in the closure \bar{D} would be inessential, contrary to [Proposition 1](#). A compact line intersecting \bar{D} nontrivially would have to meet the boundary L of D twice, again a contradiction.

The only remaining problem concerns a noncompact line, K , which meets L (in a single point, x) and is entirely contained in the closure $\bar{D} = D \cup L$. Such a line would have a disconnected intersection with D . As indicated in [Figure 7](#), the complement $D \setminus K$ then has two connected components whose boundaries meet both K and L , and whose only common boundary point is x . The line joining two points u, v , one in each of these components, has to pass through x , or else it would meet one of the lines K, L twice. Therefore, the line $u \vee x = u \vee v$ has to contain all points of the component containing v , a contradiction. \square

PROPOSITION 3. *If x is a point in the open disk D bounded by a noncompact inessential line, then there is an essential line passing through x .*

PROOF: In view of [Proposition 1](#), we have to exclude the possibility that all lines passing through x are noncompact inessential. So assume that this is the case. Near x we are in a stable plane (with point set D), and we topologise the pencil \mathcal{L}_x^+ of oriented lines as prescribed by this stable plane. An oriented line L in this pencil separates M into a component L^+ which lies to the right of L and a component L^- . One of L^+, L^- is a disk and the other, a Moebius strip. For L with opposite orientation, the complementary components are switched, hence we have lines where L^+ is a disk and others where L^- is

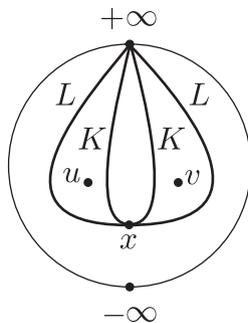


FIGURE 7:

a disk. Possibly after exchanging ‘right’ and ‘left’, we find a convergent sequence $L_n \rightarrow L$ in \mathcal{L}_x^+ such that L^+ and all L_n^- are disks. The intersection $L^+ \cap L_n^-$ is a half plane in both of the stable planes L^+ and L_n^- , and it follows that the union $L^+ \cup L_n^-$ is again a disk. (Observe that lines intersect transversally at x since we are in a stable plane.) We may assume that the sequence L_n is monotone in some order of a small neighbourhood of L in \mathcal{L}_x^+ compatible with the topology.

Figure 8 shows the situation within D : as n increases, the union $L^+ \cup L_n^-$ becomes

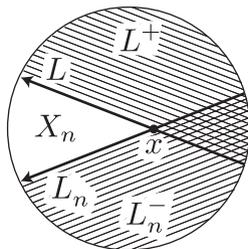


FIGURE 8:

larger, and its complement X_n in M becomes smaller. This remains true even outside the depicted neighbourhood D , because L^+ , for example, is the union of all half lines with end point x that meet $D \cap L^+$. In fact, the intersection of all X_n is the positive half line X of L . The complement $M \setminus X$ thus is the union of an increasing sequence of open disks, hence it is an open disk by the following lemma. On the other hand, a look at Figure 3 shows that the complement in M of a half line of L is still a Moebius strip. This contradiction proves our claim. \square

LEMMA 4. *Let \mathcal{D} be a set of open 2-disks in some surface S and suppose that \mathcal{D} is totally ordered with respect to inclusion. Then the union $D = \cup \mathcal{D}$ is a 2-disk.*

We remark that there is a far more general result due to Brown, see [3].

PROOF: Every loop in D is covered by finitely many $X \in \mathcal{D}$, hence it is contained in the largest one among them and can be contracted therein, hence D is simply connected. The only simply connected surfaces are \mathbb{R}^2 and the sphere \mathbb{S}_2 , see [1]. If D were compact, then the empty set $D \setminus D$ would be the intersection of the nonempty compact complements

$D \setminus X$, $X \in \mathcal{D}$, a contradiction because any finite number of those complements has a nonempty intersection. \square

PROPOSITION 5. *In a geometry on the Moebius strip, as considered here, all lines are essential.*

PROOF: As we have ruled out compact inessential lines ([Proposition 1](#)), the existence of an inessential line provides us with an open disk D on which a stable $(\mathbb{R}^2, \mathbb{R})$ -plane is induced ([Proposition 2](#)). A point $x \in D$ is incident with some essential line L by [Proposition 3](#). All lines containing x are noncompact, because a compact line would have to meet the bounding line of D twice by [Proposition 1](#). In particular, L is noncompact essential and can be represented as the boundary of the disk which yields M upon antipodal identification. We split all lines passing through x up into half lines with end point x . Figure 9 represents these half lines.

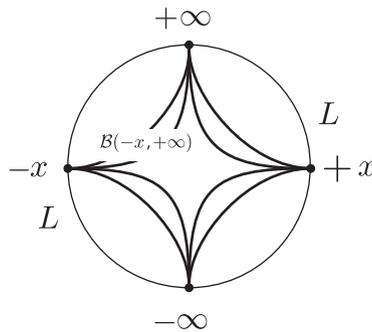


FIGURE 9:

Each of them connects x with ∞ . Observe that it makes no difference to us whether these half lines come from essential or inessential lines (this concerns only the way lines are combined from two halves).

In our disk model \tilde{P} of the projective plane, we have obtained four bundles of half lines; for example, we let $\mathcal{B}(-x, +\infty)$ denote the bundle of half lines joining the left copy of x to the upper copy of ∞ ; some of these bundles may be empty, however. Moreover, we let \mathcal{B}_{\pm} be the set of all half lines with end point $\pm x$. On \mathcal{B}_{\pm} there is a natural order, obtained as follows. Every half line H separates the disk \tilde{P} into an upper disk H^+ and a lower disk H^- , and we say that $H_1 \leq H_2$ if $H_1^- \subseteq H_2^-$.

Since x is a point of a stable plane, there are convex neighbourhoods U of x , see [[12](#), 31.22]. Here, convex means that the intersection of any line with U is connected. This implies that the set of points covered by \mathcal{B}_{\pm} is a neighbourhood of $\pm x$ in the disk \tilde{P} . It further implies that the order on \mathcal{B}_{\pm} has no gaps, that is, between any two half lines with end point $\pm x$ there lies another one having the same end point $\pm x$. Finally, we see that the two bundles \mathcal{B}_{\pm} are both nonempty.

If one of the two bundles $\mathcal{B}(\pm x, -\infty)$ has a largest element H , then we replace the disk \tilde{P} by the upper closed disk H^+ . Similarly, if $\mathcal{B}(\pm x, +\infty)$ has a smallest element H ,

we pass to H^- . Note that at most two times among the four bundles a largest or smallest element can occur, because \mathcal{B}_\pm has no gaps as we have seen.

The set of points in the interior of the remaining disk is completely covered by the remaining arcs. Because the order of \mathcal{B}_\pm has no gaps, this set can be described as the total union of at least two totally ordered families of open disks (the interiors of suitable disks H^\pm). By Lemma 4, the resulting set is a disjoint union of at least two disjoint open disks. On the other hand, it is itself a connected open disk. The contradiction shows that there can be no inessential lines. \square

Our next aim is to determine the possible positions of pairs of lines, up to homeomorphism of M . First consider the case where two lines K, L are given and one of them is compact. As both lines are essential, their closures in P are covered by two circles C_K and C_L in \mathbb{S}_2 which are invariant under the antipodal map. The two circles have at most one antipodal pair of common points $x, -x$. None of the circles can be contained in one of the two complementary disks defined by the other, because the antipodal map interchanges those disks. Therefore, the circles must intersect transversally in a pair of antipodal points, and the configuration in \mathbb{S}_2 is homeomorphic to a pair of great circles. The position of $\pm\infty$ can be on one of the circles or outside. On the Moebius strip, we obtain two possible configurations, represented by Figures 10 and 11.

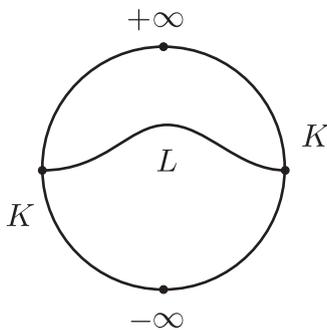


FIGURE 10: Two intersecting lines, one compact and one noncompact.

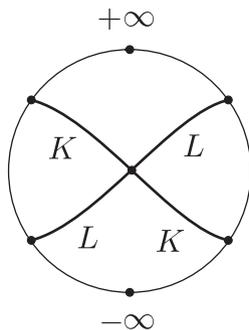


FIGURE 11: Two compact intersecting lines.

What we have proved is the following.

PROPOSITION 6. *Let K, L be two lines and assume that one of them is compact. Then the lines K, L intersect in M transversally, and the configuration is homeomorphic to the one represented in Figure 10 or 11. In particular, two compact lines together bound an open disk in M .*

Note that the closure of the open disk bounded by two compact lines is a closed disk with two boundary points identified.

We turn to pairs of noncompact lines K, L . Here the situation is more complicated because the two circles C_K, C_L have the points $\infty, -\infty$ in common and may have one more antipodal pair of intersection points. We prefer to look at the picture in M , representing K as the boundary circle. The case of disjoint lines is easily dealt with (see Figure 16 below). The case where there is an intersection point $x \in M$ splits into four subcases depending on whether the intersections at x and ∞ are transversal or not. If both intersections are of the same kind, then we get the configurations in Figures 12 and 13, and we see that L is inessential.

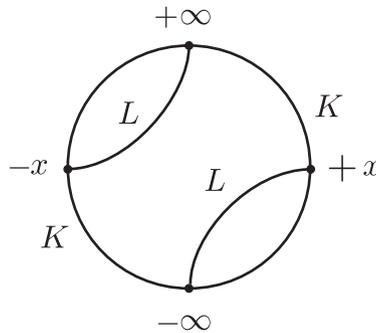


FIGURE 12: Two noncompact intersecting lines. Both intersections are transversal.

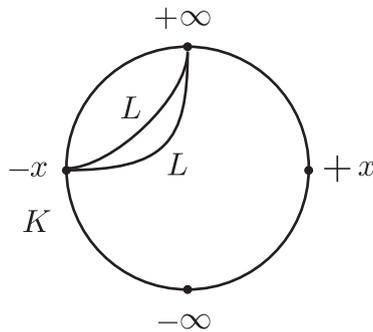


FIGURE 13: Two noncompact intersecting lines. Both intersections are not transversal.

If the intersection at ∞ is the only transversal one, then the configuration is as in Figure 14.

The line joining the points u and v has two ends at $\pm\infty$ and it must enter the two disks bounded by K, L and containing u and v , respectively. It cannot be shaped like

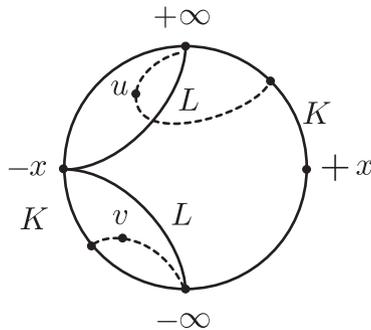


FIGURE 14: Two noncompact intersecting lines. The intersection at ∞ is transversal, the one at x is not.

the dotted curve in Figure 14, because this curve is inessential. Hence all lines joining points in the two disks pass through the point x , and we have a contradiction. The only remaining possibility is that the lines intersect transversally at x and touch at ∞ , as shown in Figure 15.

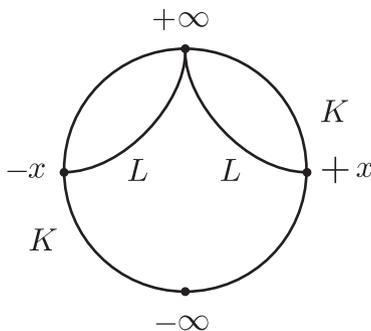


FIGURE 15: Two noncompact intersecting lines. The intersection at x is transversal, the one at ∞ is not.

We summarise:

PROPOSITION 7. *The configuration of two noncompact lines is as shown in Figure 15 or 16. In particular, if they intersect, they do so transversally.*

REMARK 8. *Ordering a line pencil.* Let x be a point and L a noncompact line passing through x . We want to define an order on the set $\mathcal{L}_x \setminus \{L\}$ in the same way as we did in the proof of Proposition 5. In the disk model \tilde{P} with boundary circle L , every line $H \in \mathcal{L}_x \setminus \{L\}$ separates \tilde{P} into an upper part H^+ and a lower part H^- (the parts may be disconnected). Since we know from Propositions 6 and 7 that lines always intersect transversally, it follows that for two such lines, one of the respective lower parts is entirely contained in the other. We say that $H_1 \leq H_2$ if $H_1^- \subseteq H_2^-$, compare Figure 17.

There are no gaps in this order: if $H_1 \leq H_2$, choose any point $y \in H_2^- \setminus H_1^-$ and join it to x in order to obtain a line between H_1 and H_2 . We remark that this fact was obtained in a different way in the proof of Proposition 5. There, we used a stable

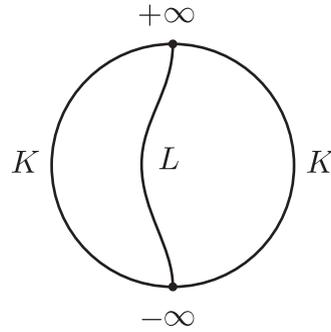


FIGURE 16: Two noncompact lines that do not intersect in M .

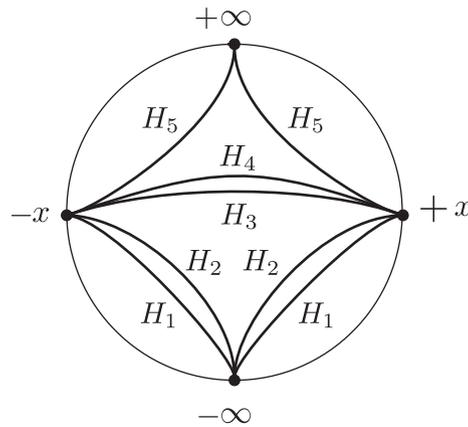


FIGURE 17: $H_1 < H_2 < H_3 < H_4 < H_5$

plane containing x . Here, we used transversality of intersection which we have proved independently.

PROPOSITION 9. *For every point x , the compact lines passing through x form a nonempty proper subset \mathcal{C}_x of the line pencil \mathcal{L}_x .*

PROOF: First we show that there are noncompact lines passing through x . Assume that this is not the case, and consider three distinct lines in \mathcal{C}_x . By Proposition 6, any two of them together bound an open disk in M . Looking at the corresponding disks in the sphere covering P , we see that one of the three disks in M contains the other two. We keep the two lines bounding that disk and add another line not contained in it. This produces a larger open disk.

Repeating the last argument, we obtain that any two of the disks bounded by pairs of lines in \mathcal{C}_x are contained in another such disk. Inductively, we obtain that any finite number of these disks is contained in a single one. Therefore, we can apply the proof of Lemma 4 to the union D of all those disks, and D is a disk. By construction of D , the complement $M \setminus D$ is either empty (impossible, since M is not a disk) or a single compact line. However, the complement of a compact line is an annulus (a punctured disk), and we have again obtained a contradiction.

Now suppose that there are no compact lines passing through x . In order to reach a contradiction, we proceed as in the proof of [Proposition 5](#). Our starting point differs from [Proposition 5](#) in that we know all lines to be essential, and the nonexistence of compact lines through x is now an assumption. Since we do not assume the existence of an inessential line, we do not have a stable plane containing x yielding a convex neighbourhood of x . We used this in [Proposition 5](#) in order to show that the order considered there had no gaps. This is now obtained from [Remark 8](#), and the proof of [Proposition 5](#) goes through unchanged. \square

PROPOSITION 10. *For every point $x \in M$, the compact lines passing through x form a convex subset $\mathcal{C}_x \neq \emptyset$ of the ordered set $\mathcal{L}_x \setminus \{L\}$ introduced in [8](#), and this set has no smallest or largest element. In particular, there are infinitely many compact lines containing x .*

PROOF: Given $C_1, C_2 \in \mathcal{C}_x$, $C_1 < C_2$, we have to show that every line $C \in \mathcal{L}_x$ such that $C_1 < C < C_2$ is compact. Now C is contained in the compact closure of the open disk bounded by C_1 and C_2 , hence it is indeed compact.

If there is a largest compact line $C \in \mathcal{C}_x$, for example, then we may repeat the exhaustion arguments indicated in the proofs of [Propositions 5](#) and [9](#) on the disk C^+ , where we are considering M in the disk model with boundary L . Again, this leads to a contradiction. \square

PROPOSITION 11. *Let $K, L \in \mathcal{L}$ be two compact lines. On the open disk $D = D_{K,L}$ bounded by these lines, (M, \mathcal{L}) induces a (stable) $(\mathbb{R}^2, \mathbb{R})$ -plane.*

PROOF: That two compact lines together bound a disk was shown in [Proposition 6](#). A line G meeting this disk D may be contained in its closure. Then G is compact and intersects K and L transversally. This can happen only at the point $x = K \wedge L$, and then $G \cap D = G \setminus \{x\} \approx \mathbb{R}$. A line meeting D and passing through x is entirely contained in the closure of D , because lines intersect transversally. Therefore, a line H meeting both the disk D and the complement of its closure has to meet the boundary $K \cup L$ at least twice, in points distinct from x . Since there can be at most one point of intersection with each of the lines K, L , we have that $H \cap D \approx \mathbb{R}$. Thus, the induced structure is an $(\mathbb{R}^2, \mathbb{R})$ -plane; it is stable by [[12](#), 31.22]. \square

PROPOSITION 12. *Given a compact proper subset A of a line H , we can always find two compact lines K, L such that A is contained in the open disk $D_{K,L}$ bounded by K and L .*

PROOF: If H is noncompact, we may simply choose two points u, v on H such that A is contained in the open interval $I \subseteq L$ between these points, and take arbitrary compact lines K and L containing u and v , respectively. Then I must be contained in the bounded complementary component of $K \cup L$, which is $D_{K,L}$.

If H is compact, then we choose a point $x \in H \setminus A$ and a noncompact line G passing through x . We order $\mathcal{L}_x \setminus \{G\}$ according to [Remark 8](#) and choose $K, L \in \mathcal{C}_x$ such that H

lies between them in that order; this is possible by [Proposition 10](#). Then $D_{K,L}$ contains $H \setminus \{x\}$. \square

DEFINITION 13. We are now in position to define a topology on the line set \mathcal{L} . Let $L \in \mathcal{L}$ be a line and let H_1, H_2 be lines meeting L in distinct points $x_i = L \wedge H_i$. Let A_i be a neighbourhood of x_i in H_i , $i \in \{1, 2\}$. If $A_1 \cap A_2 = \emptyset$, then the set

$$A_1 \vee A_2 = \{u_1 \vee u_2; u_i \in A_i\} \subseteq \mathcal{L}$$

will be called an (H_1, H_2) -neighbourhood of L .

By [Propositions 11](#) and [12](#), there is an $(\mathbb{R}^2, \mathbb{R})$ -subplane of (M, \mathcal{L}) containing the points x_1, x_2 , and $A_1 \vee A_2$ is an open set in the line space of that subplane, provided that A_1 and A_2 are small enough. It follows that the sets $A_1 \vee A_2$ form the basis of a topology on \mathcal{L} , which we call the *natural topology*.

LEMMA 14. Given two pairs of distinct lines, (H_1, H_2) and (H'_1, H'_2) , meeting the line L , every (H_1, H_2) -neighbourhood of L contains an (H'_1, H'_2) -neighbourhood.

PROOF: Using [Proposition 12](#), we can find an $(\mathbb{R}^2, \mathbb{R})$ -subplane containing the four intersection points of L with the given lines, and we may also assume that the linear neighbourhoods A_i, A'_i are all contained in this subplane. Then the assertion is a direct consequence of the continuity of operations in the subplane. \square

PROPOSITION 15. If (D, \mathcal{D}) is an $(\mathbb{R}^2, \mathbb{R})$ -subplane of (M, \mathcal{L}) , then \mathcal{D} is open with respect to the natural topology on \mathcal{L} , and the topology induced on \mathcal{D} by \mathcal{L} is the topology which makes (D, \mathcal{D}) a stable plane.

PROOF: The topology on \mathcal{D} belonging to the stable plane (D, \mathcal{D}) is generated by all (H_1, H_2) -neighbourhoods of lines $L \in \mathcal{D}$ such that the intersection points with L and the defining sets A_i are contained in D . It follows from [Lemma 14](#). that this agrees with the topology induced by the natural topology on \mathcal{L} . \square

Now it is easy to prove our main result:

THEOREM 16. With respect to the natural topology, (M, \mathcal{L}) is a stable plane.

PROOF: Since any pair of points and any pair of intersecting lines is contained in some $(\mathbb{R}^2, \mathbb{R})$ -subplane, which is a stable plane, our assertion is a direct consequence of [Proposition 15](#). \square

REFERENCES

- \mathcal{M} [1] L.V. Ahlfors and L. Sario, *Riemann surfaces*, Princeton Mathematical Series **26** (Princeton University Press, Princeton, N.J., 1960).
- \mathcal{M} [2] D. Betten, ‘Topologische Geometrien auf dem Möbiusband’, *Math. Z.* **107** (1968), 363–379.
- \mathcal{M} [3] M. Brown, ‘The monotone union of open n -cells is an open n -cell’, *Proc. Amer. Math. Soc.* **12** (1961), 812–814.

- \mathcal{M} [4] H. Groh, ‘Topologische Laguerreebenen I’, *Abh. Math. Sem. Univ. Hamburg* **32** (1968), 216–231.
- \mathcal{M} [5] H. Groh, ‘Topologische Laguerreebenen II’, *Abh. Math. Sem. Univ. Hamburg* **34** (1970), 11–22.
- \mathcal{M} [6] T. Grundhöfer and R. Löwen, ‘Linear topological geometries’, (Chapter 23), in *Handbook of incidence geometry* (North Holland, Amsterdam, 1995), pp. 1255–1324.
- \mathcal{M} [7] R. Löwen, ‘Central collineations and the parallel axiom in stable planes’, *Geom. Dedicata* **10** (1981), 283–315.
- \mathcal{M} [8] R. Löwen, ‘Ends of surface geometries, revisited’, *Geom. Dedicata* **58** (1995), 175–183.
- \mathcal{MM} [9] M.H.A. Newman, *Elements of the topology of plane sets of points*, (reprinted by Dover Publications, New York, 1992) (Cambridge University Press, New York, 1961).
- \mathcal{M} [10] H. Salzmann, ‘Topological planes’, *Adv. Math.* **2** (1967), 1–60.
- \mathcal{M} [11] H. Salzmann, ‘Geometries on surfaces’, *Pacific J. Math.* **29** (1967), 397–402.
- \mathcal{M} [12] H. Salzmann, D. Betten, T. Grundhöfer, H. Hähl, R. Löwen and M. Stroppel, *Compact projective planes* (de Gruyter, Berlin, New York, 1995).
- \sim [13] A. Schenkel, *Topologische Minkowski-Ebenen*, Dissertation (Erlangen–Nürnberg, 1980).
- \mathcal{M} [14] L.A. Skornyakov, ‘Systems of curves on a plane’, (in Russian), *Trudy Moskov. Mat. Obshch.* **6** (1957), 135–164.
- \mathcal{M} [15] M. Stroppel, ‘A note on Hilbert and Beltrami systems’, *Results Math.* **24**, 242–247.
- \mathcal{M} [16] R.D. Wölk, ‘Topologische Möbiusebenen’, *Math. Z.* **93** (1966), 311–333.

Institut für Analysis und Algebra
 Technische Universität
 Pockelsstr. 14
 38106 Braunschweig
 Germany
 e-mail: r.loewen@tu-bs.de

School of Mathematical Sciences
 Monash University
 Victoria 3800
 Australia
 e-mail: Burkard.Polster@sci.monash.edu.au