Notes on satisfiability

Skolemization

Resolution

Skolemization. An example

Example 1: Prove that $\forall x \phi(x, f(x)) \Rightarrow \forall x \exists y \phi(x, y)$ holds..

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Example 2: Prove that $\forall x \exists y \phi(x, y) \Rightarrow \forall x \phi(x, f(x))$ not.

Skolemization I

(Nerode, Shore, Logic for Applications)

Theorem 9.4 *For every sentence* ϕ *in a given language* $\mathcal{L}$ *there is a universal formula* $\phi^{'}$ *in an expanded language* $\mathcal{L}^{'}$ *gotten by the addition of new function symbols such that* ϕ *and* $\phi^{'}$ *are equisatisfiable.*

Skolemization II

Theorem 9.4 *For every sentence ϕ in a given language $\mathcal{L}$ there is a universal formula $\phi^{'}$ in an expanded language $\mathcal{L}^{'}$ gotten by the addition of new function symbols such that ϕ and $\phi^{'}$ are equisatisfiable.*

Lemma 9.5 *For any sentence $\phi = \forall x_1 ... \forall x_n \exists y \psi$ of a language $\mathcal{L}$ ϕ and $\phi^{'} = \forall x_1 ... \forall x_n \psi(y/f(x_1, ..., x_n))$ are equisatisfiable when f is a function symbol not in $\mathcal{L}$.*

Lemma 9.5 *For any sentence* $\phi = \forall x_1 ... \forall x_n \exists y \psi$ *of a language* $\mathcal{L}$ ϕ *and* $\phi' = \forall x_1 ... \forall x_n \psi(y/f(x_1, ..., x_n))$ *are equisatisfiable when* f *is a function symbol not in* $\mathcal{L}$*.*

Proof: $\mathcal{L}'$ ... $\mathcal{L}$ extended with the function symbol f
If $\mathcal{A}'$ is a structure for $\mathcal{L}'$ and $\mathcal{A}$ is a structure obtained from $\mathcal{A}'$ by omitting the function interpreting f, and $\mathcal{A}' \models \phi'$ then $\mathcal{A} \models \phi$.

On the other hand, if $\mathcal{A}$ is a structure for $\mathcal{L}$ and $\mathcal{A} \models \phi$, we can extend $\mathcal{A}$ to a structure $\mathcal{A}'$ by defining $f^{\mathcal{A}'}$ so that for every $a_1, ..., a_n \in A = A', \mathcal{A} \models \psi(y/f(a_1, ..., a_n))$. Then $\mathcal{A}' \models \phi'$. ($n$ may be 0, f be a constant).

Resolution. An example

$\{\{p, q\}, \{r, \neg q\}\}$

Resolution. An example

$\{\{p, q\}, \{r, \neg q\}\}$ satisfiable $\Rightarrow$ the resolvent $\{p, r\}$ satisfiable

$\{p, r\}$ unsatisfiable $\Rightarrow$ $\{\{p, q\}, \{r, \neg q\}\}$ unsatisfiable

Resolution

Lemma 8.12 *If the formula (i.e., set of clauses) $S = \{C_1, C_2\}$ is satisfiable and C is a resolvent of C_1 and C_2, then C is satisfiable. Any assignement $\mathcal{A}$ satisfying S satisfies C.*

Proof: $C_1 = \{l\} \sqcup C_1', C_2 = \{\neg l\} \sqcup C_2'$, the resolvent is $C = C_1' \cup C_2'$. As $\mathcal{A}$ is an assignement that satisfies $S = \{C_1, C_2\}$ it cannot be that both $l \in \mathcal{A}$ and $\neg l \in \mathcal{A}$. Say $\neg l \notin \mathcal{A}$. As $\mathcal{A} \models C_2$ and $\neg l \notin \mathcal{A}$, $\mathcal{A} \models C_2^{'}$ and so $\mathcal{A} \models C$. Similarily for $l \notin \mathcal{A}$.