Notes on satisfiability

Skolemization

Resolution

Skolemization. An example

Example 1: Prove that $\forall x \phi(x, f(x)) \Rightarrow \forall x \exists y \phi(x, y)$ holds..

Skolemization. An example

Example 1: Prove that $\forall x \phi(x, f(x)) \Rightarrow \forall x \exists y \phi(x, y)$ holds..

Example 2: Prove that $\forall x \exists y \phi(x, y) \Rightarrow \forall x \phi(x, f(x))$ not.

Skolemization I

(Nerode, Shore, Logic for Applications)

Theorem 9.4 *For every sentence ϕ in a given language $\mathcal{L}$ there is a universal formula ϕ' in an expanded language $\mathcal{L}'$ gotten by the addition of new function symbols such that ϕ and ϕ' are equisatisfiable.*

Lemma 9.5 *For any sentence $\phi = \forall x_1 ... \forall x_n \exists y \psi$ of a language $\mathcal{L}$ ϕ and $\phi' = \forall x_1 ... \forall x_n \psi(y/f(x_1, ..., x_n))$ are equisatisfiable when f is a function symbol not in $\mathcal{L}$.*

Proof:$\mathcal{L}'$ ... $\mathcal{L}$ extended with the function symbol f
If $\mathcal{A}'$ is a structure for $\mathcal{L}'$ and $\mathcal{A}$ is a structure obtained from $\mathcal{A}'$ by omitting the function interpreting f, and $\mathcal{A}' \models \phi'$ then $\mathcal{A} \models \phi$.

On the other hand, if $\mathcal{A}$ is a structure for $\mathcal{L}$ and $\mathcal{A} \models \phi$, we can extend $\mathcal{A}$ to a structure $\mathcal{A}'$ by defining $f^{\mathcal{A}'}$ so that for every $a_1, ..., a_n \in A = A', \mathcal{A} \models \psi(y/f(a_1, ..., a_n))$. Then $\mathcal{A}' \models \phi'$. ($n$ may be 0, f be a constant).

Skolemization II

Theorem 9.4 *For every sentence ϕ in a given language $\mathcal{L}$ there is a universal formula ϕ' in an expanded language $\mathcal{L}'$ gotten by the addition of new function symbols such that ϕ and ϕ' are equisatisfiable.*

Lemma 9.5 *For any sentence $\phi = \forall x_1 ... \forall x_n \exists y \psi$ of a language $\mathcal{L}$ ϕ and $\phi' = \forall x_1 ... \forall x_n \psi(y/f(x_1, ..., x_n))$ are equisatisfiable when f is a function symbol not in $\mathcal{L}$.*

Resolution. An example

$\{\{p, q\}, \{r, \neg q\}\}$

Resolution. An example

$\{\{p, q\}, \{r, \neg q\}\}$ satisfiable $\Rightarrow$ the resolvent $\{p, r\}$ satisfiable

$\{p, r\}$ unsatisfiable $\Rightarrow \{\{p, q\}, \{r, \neg q\}\}$ unsatisfiable

Resolution

Lemma 8.12 *If the formula (i.e., set of clauses)* $S = \{C_1, C_2\}$ *is satisfiable and* C *is a resolvent of* C_1 *and* C_2*, then* C *is satisfiable. Any assignement* $\mathcal{A}$ *satisfying* S *satisfies* C*.*

Proof: $C_1 = \{l\} \sqcup C_1'$, $C_2 = \{\neg l\} \sqcup C_2'$, the resolvent is $C = C_1' \cup C_2'$. As $\mathcal{A}$ is an assignement that satisfies $S = \{C_1, C_2\}$ it cannot be that both $l \in \mathcal{A}$ and $\neg l \in \mathcal{A}$. Say $\neg l \notin \mathcal{A}$. As $\mathcal{A} \models C_2$ and $\neg l \notin \mathcal{A}$, $\mathcal{A} \models C_2^{'}$ and so $\mathcal{A} \models C$. Similarily for $l \notin \mathcal{A}$.