# Tableau Proofs. Preliminaries

A *partial order* ... a set $S$ with a binary relation (”less than”, written $<$, on $S$ which is *transitive* and *irreflexive*.

A *tree*

**König’s lemma**: *If a finitely branching tree $T$ is infinite, it has an infinite path.*

**Proof:** Logic for Applications, p. 9

# Tableau Proofs

- start with a signed formula, $F\alpha$, as the root of a tree
- analyze it into its components to see that any analysis leads to a contradiction

# Tableau Proofs in Propositional Calculus I

- signed formula $F\alpha$, $T\alpha$
- atomic tableau, $\alpha$-rules, $\beta$-rules

# Tableaux I

A *finite tableau* is a binary tree, labeled with signed formulas called entries, that satisfies the following inductive definition:

1. All atomic tableaux are finit tableaux.
2. If $\tau$ is a finite tableau, $P$ a path on $\tau$, $E$ an entry of $\tau$ occurring on $P$, and $\tau^{'}$ is obtained from $\tau$ by adjoining the unique atomic tableau with root entry $E$ to $\tau$ at the end of the path $P$, then $\tau^{'}$ is also a finite tableau.

# Tableaux II

Let $\tau$ be a tableau, $P$ a path on $\tau$ and $E$ an entry occuring on $P$.

1. $E$ has been *reduced* on $P$ if all the entries on one path through the atomic tableau with root $E$ occur on $P$.
2. $P$ is *contradictory* if, for some proposition $\alpha$, $T\alpha$ and $F\alpha$ are both entries on $P$. $P$ is *finished* if it is contradictory or every entry on $P$ is reduced on $P$.
3. $\tau$ is finished if every path through $\tau$ is finished.
4. $\tau$ is contradictory if every path through $\tau$ is contradictory.

# Tableau proof

A *tableau proof* of a proposition $\alpha$ is a contradictory tableau with root entry $F\alpha$.

A *tableau refutation* for a proposition $\alpha$ is a contradictory tableau with root entry $T\alpha$.

tableau provable/refutable proposition

# Complete systematic tableaux

Let $R$ be a signed propositin. We define the *complete systematic tableau* (CST) with root entry $R$ by induction.

1. Let $\tau_0$ be the unique tableau with $R$ at its root.
2. Assume that $\tau_m$ has been defined.
3. Let $n$ be the smallest level of $\tau_m$ containing an entry that is unreduced on some noncontradictory path in $\tau_m$ and let $E$ be the leftmost such entry of level $n$.
4. Let $\tau_{m+1}$ be gotten by adjoining the unique atomic tableau with root $E$ to the end of every noncontradictory path of $\tau_m$ on which $E$ is unreduced.

5. The union of the sequence $\tau_m$ is the desired systematic tableau.

# Complete systematic tableaux II

1. Every CST is finished.
2. Every CST is finited.
3. Soundness: $\vdash \Rightarrow \models$

   If $\alpha$ is tableau provable, it is valid.
4. Completness: $\models \Rightarrow \vdash$

   If $\alpha$ is valid, then $\alpha$ is tableau provable.

$\models$ validity $\vdash$ provability

# Tableaux from premises

$\Sigma$ - a possibly infinite set of propositions

1. Every atomic tableau is a finite tableau from $\Sigma$ .
2. If $\tau$ is a finite tableau from $\Sigma$ and $\alpha \in \Sigma$, then the tableau formed by putting $T\alpha$ at the end of every noncontradictory path not containing it is also a finite tableau from $\Sigma$ .
3. If $\tau$ is a finite tableau from $\Sigma$, $P$ a path on $\tau$, $E$ an entry of $\tau$ occurring on $P$, and $\tau^{'}$ is obtained from $\tau$ by adjoining the unique atomic tableau with root entry $E$ to $\tau$ at the end of the path $P$, then $\tau^{'}$ is also a finite tableau from $\Sigma$.

# Tableaux from premises II

Every CST is finished.

Both soundness and completness of deduction from premises hold.

Every CST is finite ... ?

If a CST from $\Sigma$ is a proof, it is finite

Compactness: $\alpha$ is a conequence of $\Sigma$ iff $\alpha$ is a consequence of some finite subset of $\Sigma$.

# Tableaux in predicate calculus

$T(\exists x)\phi(x)$

$\mathcal{L}_\mathcal{C}$ - adding on a set of constants $c_0, c_1, \ldots$

$\gamma$ -rules, $\delta$ -rules

Tableaux in predicate calculus

1. All atomic tableaux are tableaux. The requirement that $c$ be new in (7b) and (8a) means that $c$ is one of the constants $c_i$ added on to $\mathcal{L}$ to get $\mathcal{L}_\mathcal{C}$ (which therefore does not appear in $\phi$).

2. ... adjoining an atomic tableau with root entry $E$ to $\tau$ at the end of the path $P$: $c$ did not appear in any entries on $P$.

# Tableau is finished:

$T(\exists x)\phi(x)$, $F(\forall x)\phi(x) \rightarrow$ a witness $c$

$T(\forall x)\phi(x)$, $F(\exists x)\phi(x)$

$\rightarrow$

add $T\phi(t)$ ($F\phi(t)$) for **any ground term** $t$ ... ?

# Tableau is finished II

$P$ a path in $\tau$, $E$ an entry on $P$ and $W$ the i$^{th}$ occurence of $E$ on $P$.

$w$ is reduced

1. ...

2. $E$ is of the form $T(\forall x)\phi(x)$ or $F(\exists x)\phi(x)$, $T\phi(t_i)$ or $F\phi(t_i)$, respectively, is an entry on $P$ and there is an (i+1)$^{st}$ occurence of $E$ on $P$.

   **Note:** *Signed sentences like $T(\forall x)\phi(x)$ must be instantiated for each term $t_i$ in our language before we can say that we have finished with them.*

3. finished ...