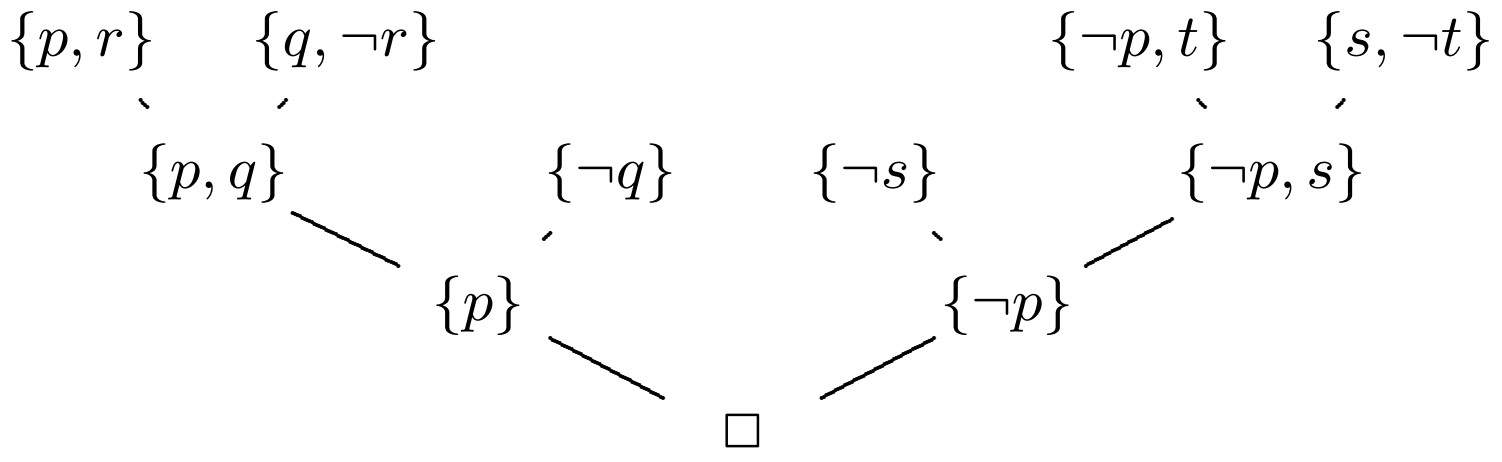
Resolution in propositional logic – example

- $S \vdash_R \square$?

  $S = (p \vee r) \wedge (q \Leftarrow r) \wedge \neg q \wedge (t \Leftarrow p) \wedge \neg s \wedge (s \Leftarrow t)$

  $S = (p \vee r) \wedge (q \vee \neg r) \wedge \neg q \wedge (\neg p \vee t) \wedge \neg s \wedge (s \vee \neg t)$

  $S = \{\{p, r\}, \{q, \neg r\}, \{\neg q\}, \{\neg p, t\}, \{\neg s\}, \{s, \neg t\}\}$



Refining resolution I

- to narrow search space - SAT=$\{S \mid S$ is satisfiable $\}$ is NP-complete
  - to terminate the search along paths that are unpromising
  - to specify the oder in which to go down alternative paths

Refining resolution II

- if there is a literal that is only positive(negative), remove all clauses that contain such a literal
- *T-resolution* : no parent clause is a tautology
- *Semantic resolution*. Let $\mathcal{I}$ be an interpretation. Semantic resolution with respect to $\mathcal{I}$ permits applications of the resolution rule only when at least one of their premises has a ground instance which is not satisfied by $\mathcal{I}$
- *Ordered resolution.* The propositional letters are indexed resolve on the literal with the higher indexthan any othe in the parent clauses

- *Lock resolution.* Each occurence of a literal has a distinct index the literal resolved on has in each parent the lowest index

Resolution in predicate logic – introduction

- based on *refutation*
- suitable for automated theorem proving
- formulas in Skolem normal form
  - clause = disjunction of literals (atoms or negation of atoms), represented as a set
  - formula = conjunction of clauses, represented as a set
- Example:
  $\forall x \forall y((P(x, f(x)) \vee \neg Q(y)) \wedge (\neg R(f(x)) \vee \neg Q(y)))$
  $\rightarrow$
  $\{\{P(x, f(x)), \neg Q(y)\}, \{\neg R(f(x)), \neg Q(y)\}\}$

Unification

- a substitution ϕ is a *unifier* for $S = \{E_1, \ldots, E_2\}$ if $E_1\phi = E_2\phi = \ldots = E_n\phi$, i.e., $S\phi$ is singleton.

  S is said to be *unifiable* if it has a unifier.

- a unifier ϕ for S is a *most general unifier (mgu) for* S if, for every unifier ψ for S, there is a substitution λ such that $\phi\lambda = \psi$

  up to renaming variables there is only one result applying an mgu

Unification – Examples

1. a unifier for $\{P(x, c), P(b, c)\}$ is
   $\phi = \{x/b\}$; is there any other?

2. a unifier for $\{P(f(x), y), P(f(a), w)\}$ is
   $\phi = \{x/a, y/w\}$
   but also $\psi = \{x/a, y/a, w/a\}$,
   $\sigma = \{x/a, y/b, w/b\}$ etc.

3. $\{P(x, a), P(b, c)\}, \{P(f(x), z), P(a, w)\}$,
   $\{P(x, w), \neg P(a, w)\}$,
   $\{P(x, y, z), P(a, b)\}, \{R(x), P(x)\}$
   are not unifiable

mgu?

in (2.) ϕ is the mgu: $\psi = \phi\{w/a\}, \sigma = \phi\{w/b\}$

Resolution in predicate logic – preliminaries

- variables are local for a clause (pozn.: $\forall x(A(x) \wedge B(x)) \Leftrightarrow (\forall x A(x) \wedge \forall x B(x)) \Leftrightarrow (\forall x A(x) \wedge \forall y B(y))$)
  i.e. there is no relation between variables equally named

- standardization of vars = renaming, necessary
  $\{\{P(x)\}, \{\neg P(f(x))\}\}$ is unsatisfiable. Without renaming a variable no unification can be performed

Resolvent – Examples

Example 1: $\{P(x,a)\}, \{\neg P(x,x)\}$

- rename vars: $\{P(x_1,a)\}$
- $mgu(\{P(x_1,a), P(x,x)\}) = \{x_1/a, x/a\}$
- resolvent $\square$

Example 2: $\{P(x,y), \neg R(x)\}, \{\neg P(a,b)\}$

- $mgu(\{P(x,y), P(a,b)\}) = \{x/a, y/b\}$
- apply mgu to $\{\neg R(x)\}$
- resolvent $\{\neg R(a)\}$

Resolution rule in predicate logic

C_1, C_2 clauses that have no variables in common in the form

$$C_1 = C_1' \sqcup \{P(\vec{x}_1), \ldots, P(\vec{x}_n)\},$$
$$C_2 = C_2' \sqcup \{\neg P(\vec{y}_1), \ldots, \neg P(\vec{y}_m)\}$$

respectively. If ϕ is an mgu for

$\{P(\vec{x}_1), \ldots, P(\vec{x}_n), P(\vec{y}_1), \ldots, P(\vec{y}_m)\}$,

then $C_1'\phi \cup C_2'\phi$ is a *resolvent* of C_1 and C_2

(also called the *child* of *parents* C_1 and C_2).

Resolution rule in predicate logic II

- *Resolution proofs of C from S* is a finite sequence $C_1, C_2, ..., C_N = C$ of clauses such that each C_i is either a member of S or a resolvent of clauses C_j, C_k for $j, k < i$
- *resolution tree proof* C from S is a labeled binary tree
  the root is labeled C
  the leaves are labeled with elements of S and
  if any nonleaf node is labeled with C_2 and its immediate successors are labeled with C_0, C_1 then C_2 is a resolvent C_0 and C_1
- *(resolution) refutation* of S is a deduction of $\square$ from S

Resolution – Examples II

Ex. 3: $C_1 = \{Q(x), \neg R(y), P(x, y), P(f(z), f(z))\}$ a
$C_2 = \{\neg N(u), \neg R(w), \neg P(f(a), f(a)), \neg P(f(w), f(w))\}$

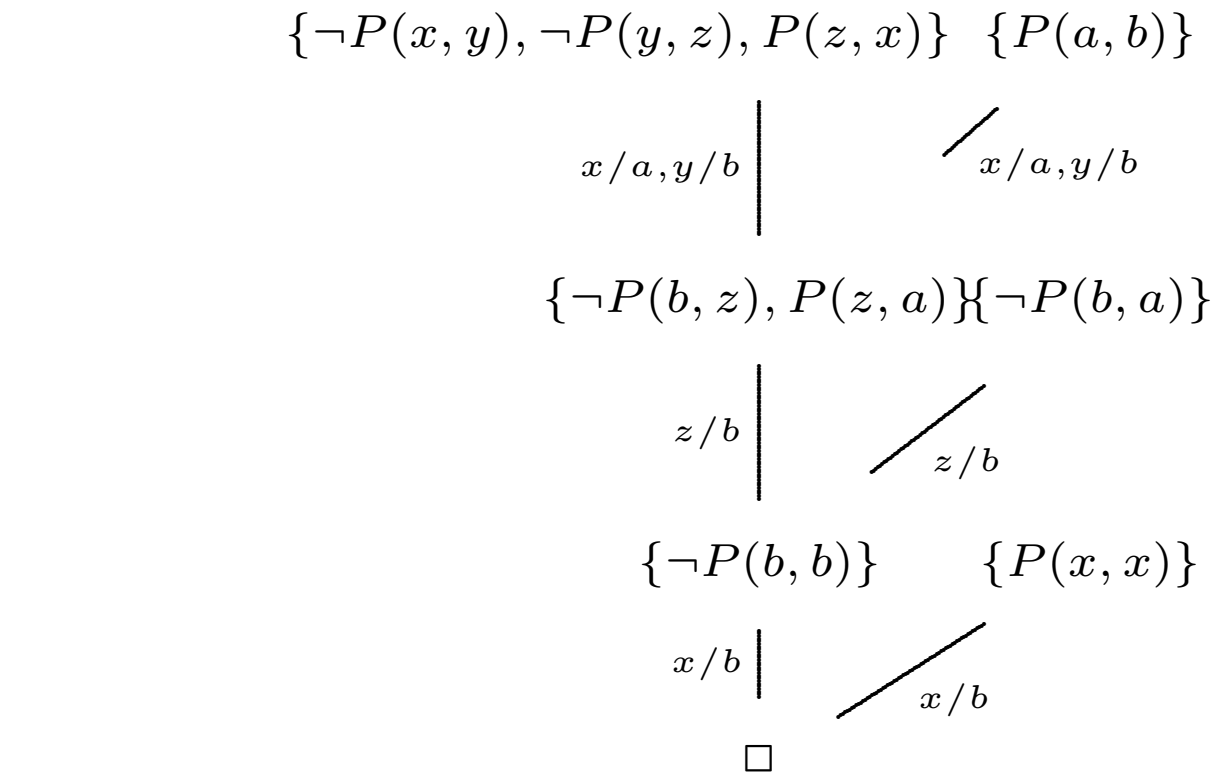
- choose the set of literal
  $\{P(x, y), P(f(z), f(z)), P(f(a), f(a)), P(f(w), f(w))\}$
- mgu $\phi = \{x/f(a), y/f(a), z/a, w/a\}$
- $C_1' = \{Q(x), \neg R(y)\}, C_1'\phi = \{Q(f(a)), \neg R(f(a))\}$
- $C_2' = \{\neg N(u), \neg R(w)\}, C_2'\phi = \{\neg N(u), \neg R(a)\}$
- the resolvent
  $C_1'\phi \cup C_2'\phi = \{Q(f(a)), \neg R(f(a)), \neg N(u), \neg R(a)\}$

Resolution in the predicate logic

- is sound (soundness) and complete
- systematic attempts at generating resolution proofs possible but redundant and inefficient: the search space is too huge
- what strategy of generating resolvents to choose?

Linear resolution

$$\{\{P(x,x)\},\{\neg P(x,y),\neg P(y,z),P(z,x)\},\{P(a,b)\},\{\neg P(b,a)\}\}$$

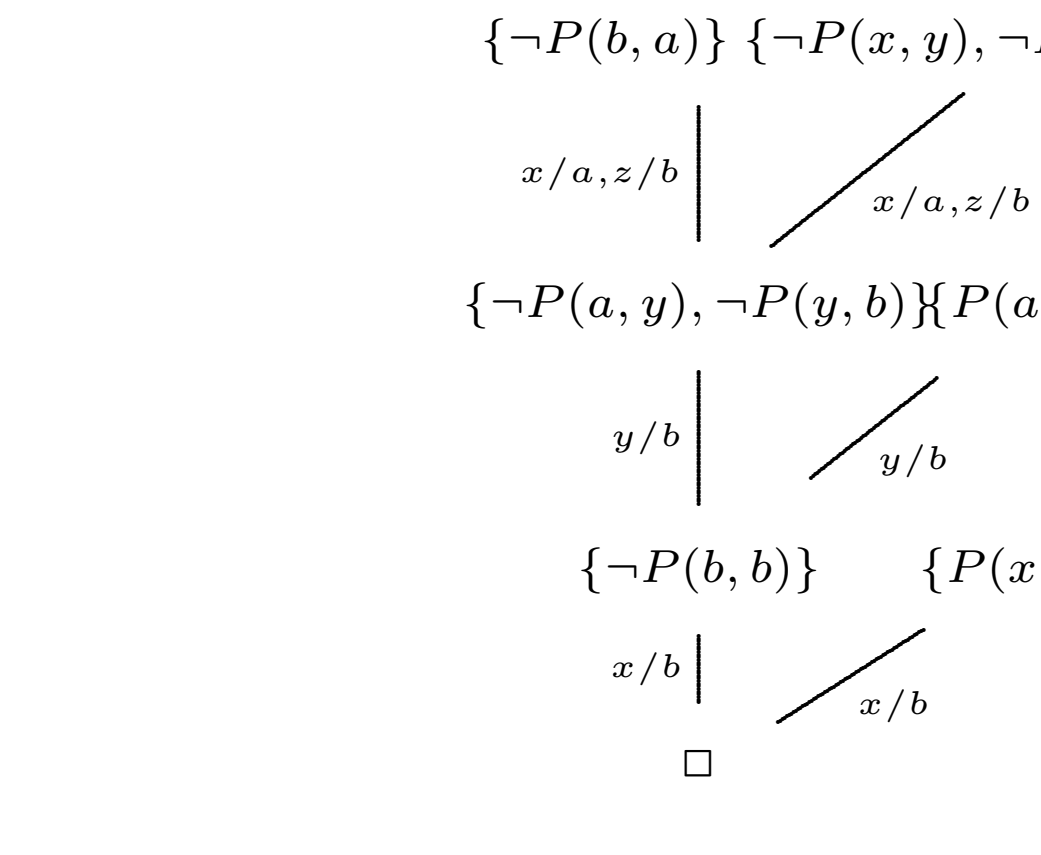


sound and complete

LI-resolution

linear input resolution

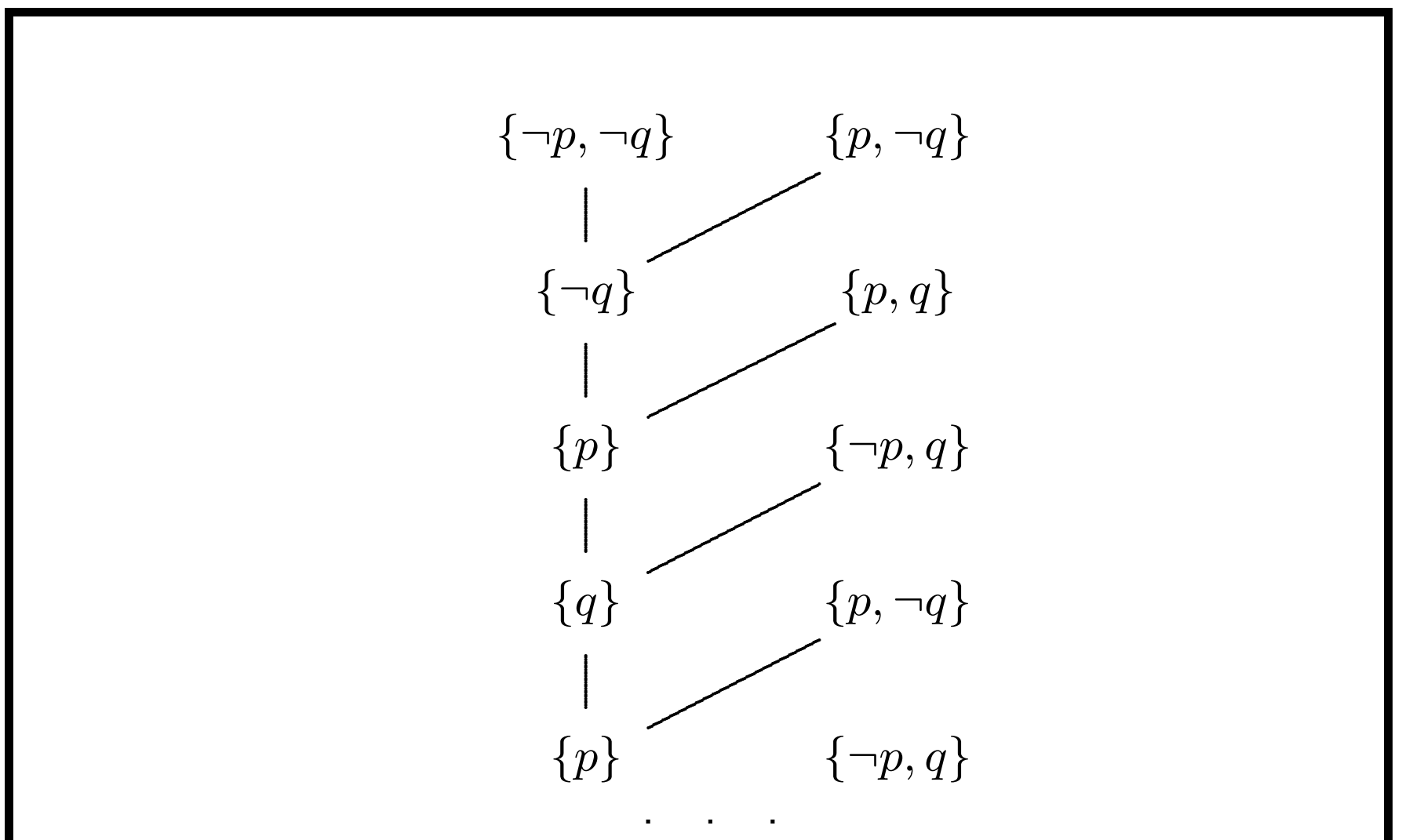
$$\{\{P(x,x)\},\{\neg P(x,y),\neg P(y,z),P(z,x)\},\{P(a,b)\},\{\neg P(b,a)\}\}$$



LI-resolution II

sound but not complete in general

Ex.: : $S = \{\{p, q\}, \{p, \neg q\}, \{\neg p, q\}, \{\neg p, \neg q\}\}$



LI-resolution is complete for Horn clauses

Horn clause

- max. one positive literal
  which of $\{\{p, q\}, \{p, \neg q\}, \{\neg p, q\}, \{\neg p, \neg q\}, \{p\}\}$ are Horn clauses?
- an alternative notation
  $\{p \leftarrow q\}, \{p \rightarrow q\}, \{true \rightarrow p\}$
- the Prolog notation
- rule `p :- q.`
  fact `p.`
  goal `?- p,q.`

LD-resolution

- from LI-resolution to an ordered resolution
- works with an *ordered clauses*; $[P(x), \neg R(x, f(y)), \neg Q(a)]$

If $G = [\neg A_1, \neg A_2, \ldots, \neg A_n]$ and
$H = [B_0, \neg B_1, \neg B_2, \ldots, \neg B_m]$ are ordered clauses and ϕ an mgu for B_0 and A_i),
then the *(ordered) resolvent* of G a H is the ordered clause
$[\neg A_1\phi, \neg A_2\phi, \ldots, \neg A_{i-1}\phi, \neg B_1\phi, \neg B_2\phi, \ldots, \neg B_m\phi, \neg A_{i+1}\phi, \ldots, \neg A_n\phi]$

LD – Linear Definite

LD-resolution

$$\{[P(x,x)], [P(z,x), \neg P(x,y), \neg P(y,z)], [P(a,b)], [\neg P(b,a)]\}$$

$[\neg P(b,a)]$ $\qquad$ $[P(z,x), \neg P(x,y), \neg P(y,z)]$

$x/a, z/b$ $\qquad$ $x/a, z/b$

$[\neg P(a,y), \neg P(y,b)]$ $\qquad$ $[P(a,b)]$

y/a $\qquad$ y/a

$[\neg P(a,a)]$ $\qquad$ $[P(x,x)]$

x/a $\qquad$ x/a

$\square$

LD-resolution is sound and complete for Horn clauses.

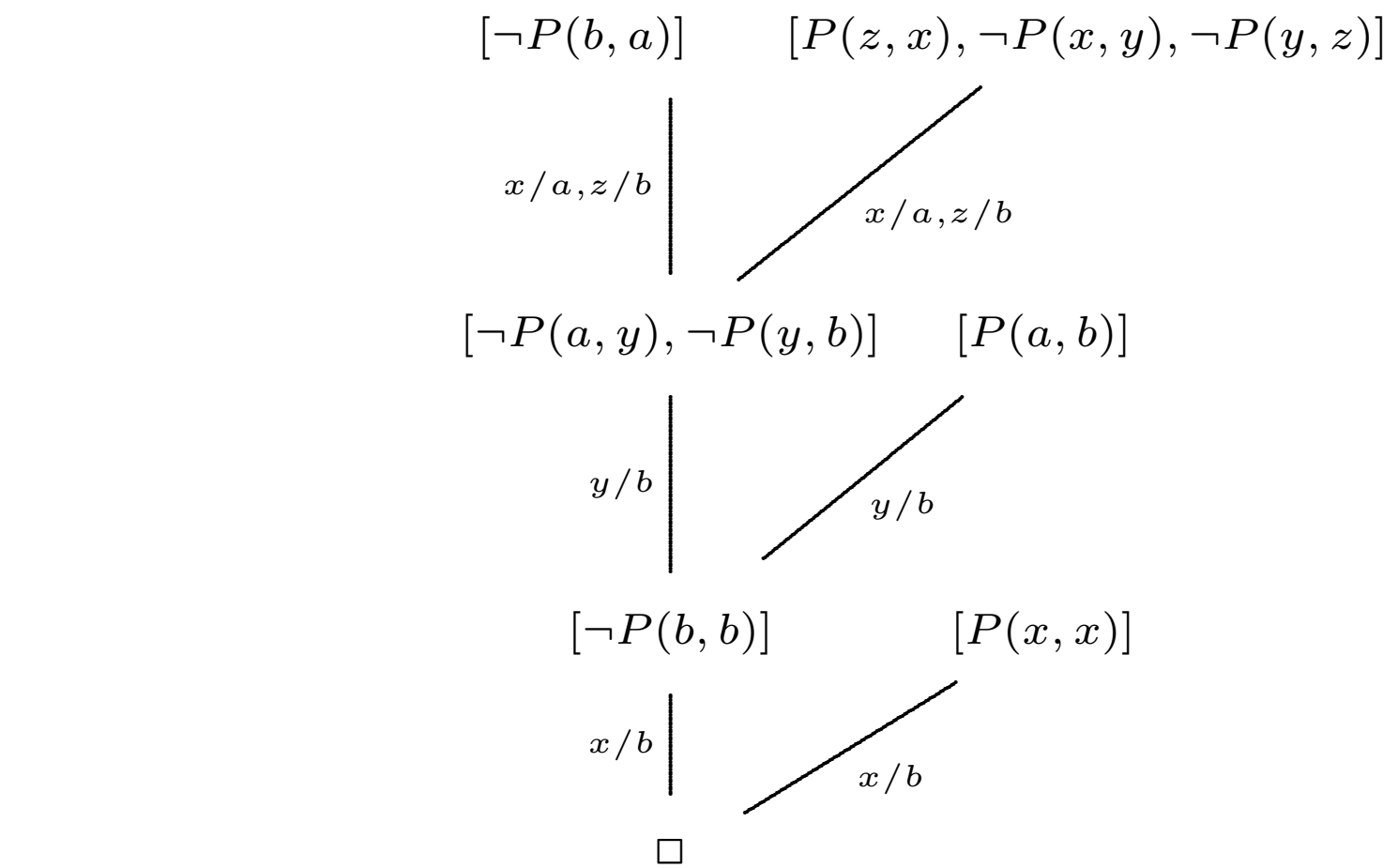
SLD-resolution

- LD–resolution with a selection rule
- A *selection rule* R s a function that chooses a literal from every nonempty ordered clause C.
- If no R is mentioned we assume that the standard one of choosing the leftmost literal is intended.
- Example: $G = [\neg A_1, \neg A_2, \ldots, \neg A_n]$,
  $H = [B_0, \neg B_1, \neg B_2, \ldots, \neg B_m]$,
  The resolvent of G and H for $\phi = mgu(B_0, A_1)$ is
  $[\neg B_1\phi, \neg B_2\phi, \ldots, \neg B_m\phi, \neg A_2\phi, \ldots, \neg A_n\phi]$

SLD-resolution is sound and complete for Horn clauses

SLD-resolution

selection rule = the leftmost literal



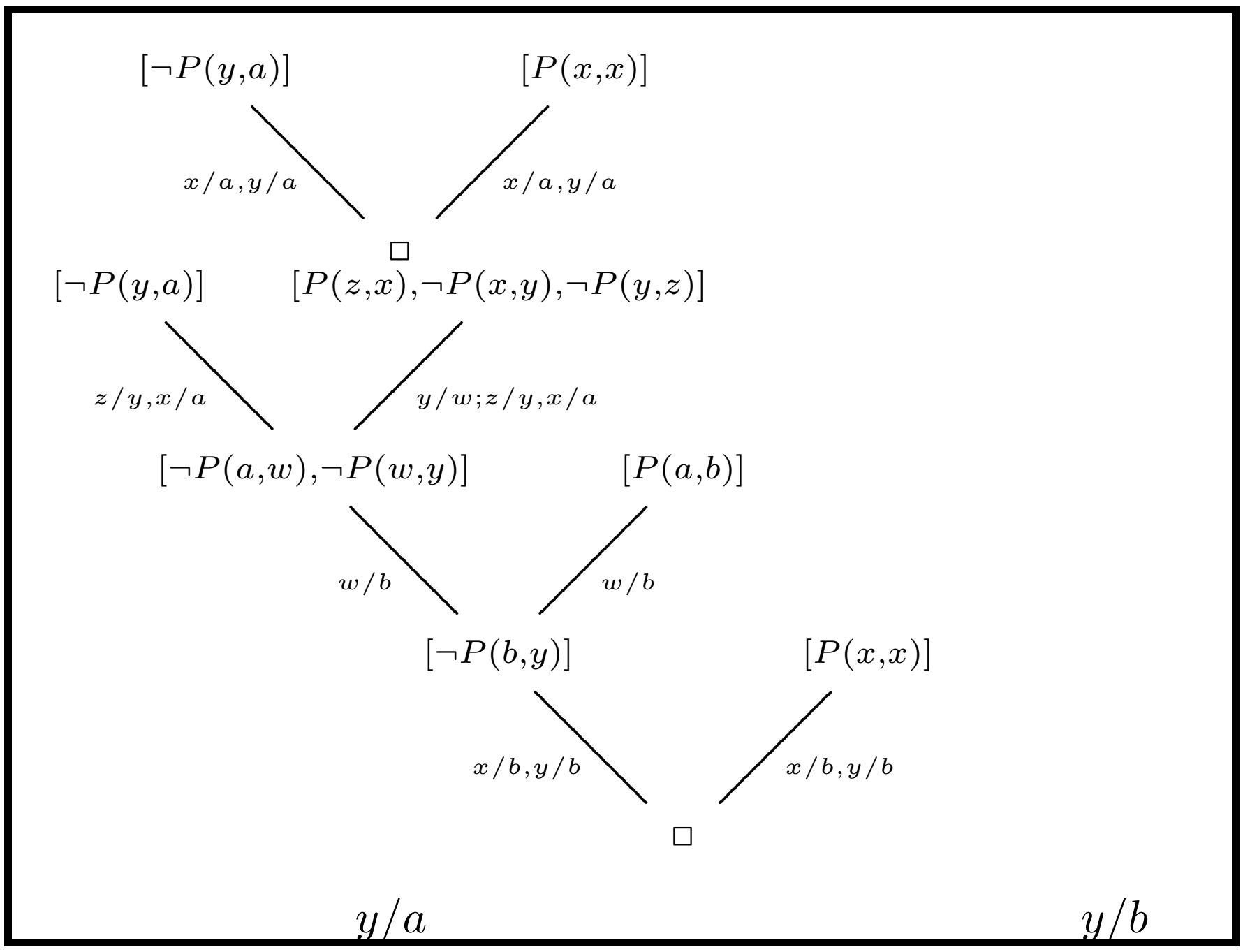
Example

For

$P = \{[P(a, b)], [P(x, x)], [P(z, x), \neg P(x, y), \neg P(y, z)]\},$

find all solutions (i.e. substitutions of variables) of the goal

$[\neg P(y, a)]$

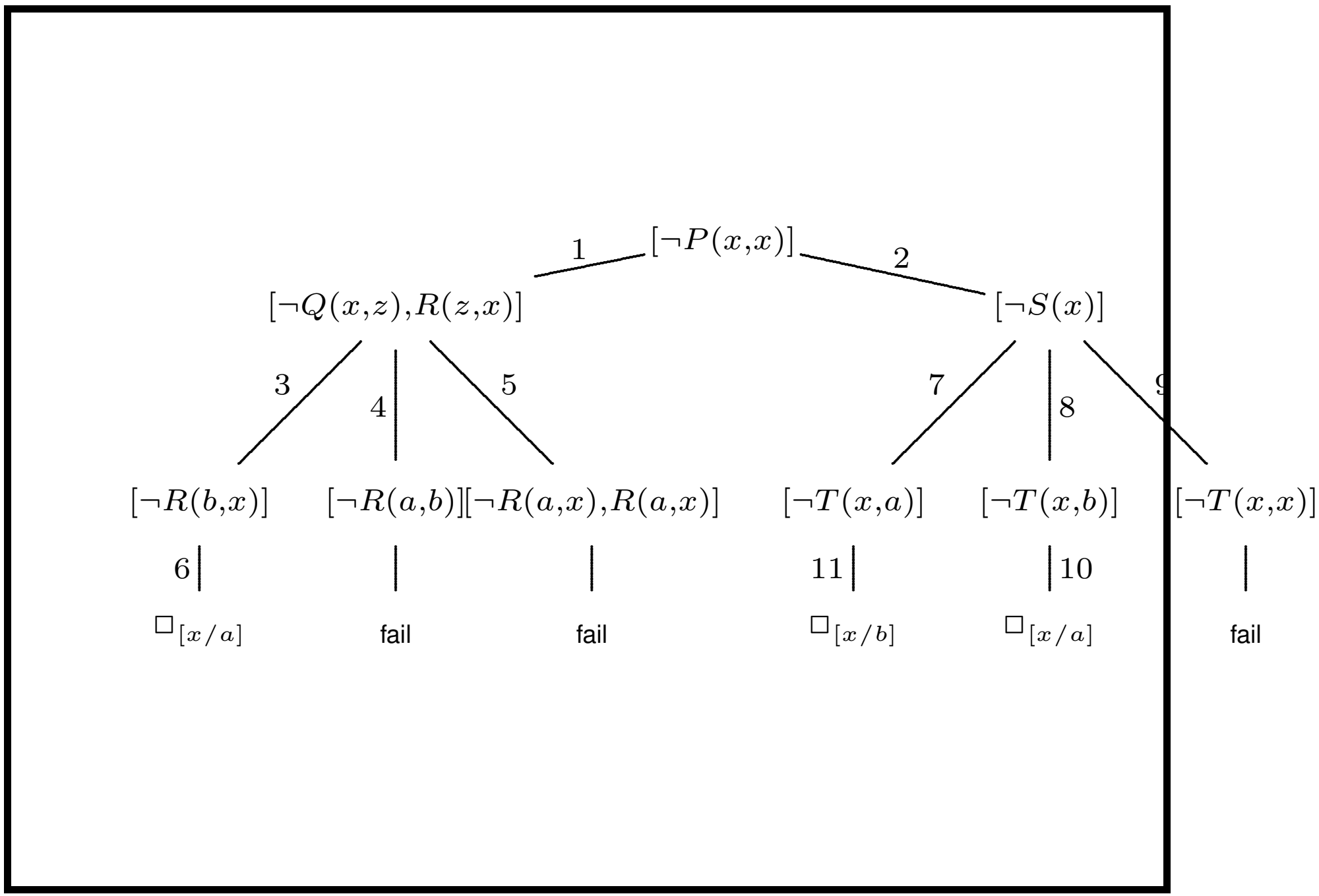

$[\neg P(y,a)]$
$[P(x,x)]$
$x/a,y/a$
$x/a,y/a$
$\square$
$[\neg P(y,a)]$
$[P(z,x),\neg P(x,y),\neg P(y,z)]$
$z/y,x/a$
$y/w;z/y,x/a$
$[\neg P(a,w),\neg P(w,y)]$
$[P(a,b)]$
w/b
w/b
$[\neg P(b,y)]$
$[P(x,x)]$
$x/b,y/b$
$x/b,y/b$
$\square$
y/a
y/b

SLD-trees

all SLD-derivations for a given goal G and the program P

1. $[P(x,y),\neg Q(x,z),\neg R(z,y)]$
2. $[P(x,x),\neg S(x)]$
3. $[Q(x,b)]$
4. $[Q(b,a)]$
5. $[Q(x,a),\neg R(a,x)]$
6. $[R(b,a)]$
7. $[S(x),\neg T(x,a)]$
8. $[S(x),\neg T(x,b)]$
9. $[S(x),\neg T(x,x)]$
10. $[T(a,b)]$
11. $[T(b,a)]$

cíl: $[\neg P(x,x)]$


$[\neg P(x,x)]$
1
2
$[\neg Q(x,z), R(z,x)]$
$[\neg S(x)]$
3
4
5
7
8
9
$[\neg R(b,x)]$
$[\neg R(a,b)]$
$[\neg R(a,x), R(a,x)]$
$[\neg T(x,a)]$
$[\neg T(x,b)]$
$[\neg T(x,x)]$
6
11
10
$\square_{[x/a]}$
fail
fail
$\square_{[x/b]}$
$\square_{[x/a]}$
fail