Propositional logic

- propositional letters, propositions
- connectives $\wedge, \vee, \rightarrow, \leftrightarrow, \neg$
- truth tables
- adequacy, adequate set of truth connectives
- truth assignement

  assigns to each propositional letter a unique truth value

  truth valuation

  assigns to each proposition a unique truth value
- valid proposition, tautology
- logically equivalent propositions

Propositional logic: axiomatic approach

- axioms (A, B, C = formulas):

  $\mathbf{A_1}$ $A \Rightarrow (B \Rightarrow A)$

  $\mathbf{A_2}$ $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$

  $\mathbf{A_3}$ $(\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$

- inference rule *modus ponens (MP)*

$$\frac{A \qquad A \Rightarrow B}{B}$$

Predicate logic

Language

- constants
- variables
- connectives
- quantifiers – universal $\forall$, existential $\exists$
- predicate symbols – predicate = n-ary relation
- function symbols
- punctuation

Formulas

- terms = constants, variables, $f(t_1, ..., t_n)$

  ground terms = variable–free terms

- atomic formula $R(t_1, ..., t_n)$, arity, arguments

- formulas

  - atomic formulas
  - $\neg F$, $F\ OP\ G$ (OP = $\wedge, \vee, \rightarrow$.)
  - $\exists F, \forall F$

- sentence = no free occurence of any variable (all variables are bound)

- open formula = without quantifiers

Substitution

- only free variables
- If the term t contains an occurence of some variable x (which is necessarily free in t) we say that t is *substitutable* for the free variable v in the formula $A(v)$ if all occurencies of x in t remains free in $A(v/t)$
- Example:$A = \exists x P(x, y)$
  $A(y/z) = \exists x P(x, z)$
  $A(y/2) = \exists x P(x, 2)$
  $A(y/f(z, z)) = \exists x P(x, f(z, z))$.
  but not
  $A(y/f(x, x)) = \exists x P(x, f(x, x))$

Axiomatic system for predicate calculus

- axioms (A, B, C = formulas):

  $\mathbf{A_1}$ $A \Rightarrow (B \Rightarrow A)$

  $\mathbf{A_2}$ $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$

  $\mathbf{A_3}$ $(\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$

  $\mathbf{A_4}$ $(\forall x)\alpha(x) \rightarrow \alpha(t)$ for any term that is substitutable for x in α

  $\mathbf{A_5}$ $(\forall x)(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow (\forall x)\beta)$ if α contains no occurence of x

- two inference rules

  Modus Ponens

$$\frac{A \qquad A \Rightarrow B}{B}$$

Generalization From $\forall x \alpha$ infer α

- proof = a finite sequence of formulas

Prenex normal forms

- DNF, CNF

  $$Qx_1 \ldots Qx_n((A_{1_1} \vee \ldots \vee A_{1_{l_1}}) \wedge (A_{2_1} \vee \ldots \vee A_{2_{l_2}}) \wedge \ldots \wedge (A_{m_1} \vee \ldots \vee A_{m_{l_m}}))$$

- Example:

  $\forall x \forall y \exists z \forall w((P(x,y) \vee \neg Q(z)) \wedge (R(x,w) \vee R(y,w)))$

- Every formula ϕ has a prenex equivalent.

Algorithm

1. Remove the quantifiers that are not used
2. Rename variables so that each quantifier has a unique variable
3. Eliminate all connectives but $\neg$, $\wedge$ a $\vee$
4. Move negation to the right
   $\neg\forall x A$ --> $\exists x \neg A$
   $\neg(A \wedge B)$ --> $\neg A \vee \neg B$ apod.
5. Move quantifiers to the left ($\mathtt{op} \in \{\wedge, \vee\}, Q \in \{\forall, \exists\}$):
   $A\ \mathtt{op}\ QxB$ --> $Qx(A\ \mathtt{op}\ B)$
   $QxA\ \mathtt{op}\ B$ --> $Qx(A\ \mathtt{op}\ B)$

6. Use distributive laws

$A \vee (B \wedge C)$ --> $(A \vee B) \wedge (A \vee C)$

$(A \wedge B) \vee C$ --> $(A \vee C) \wedge (B \vee C)$

Skolemization

- Skolem Normal Form - NF with universal quantifiers
- $\forall x_1 \ldots \forall x_n \exists y P(x_1, \ldots, x_n, y)$
  $-->$
  $\forall x_1 \ldots \forall x_n P(x_1, \ldots, x_n, f(x_1, \ldots, x_n))$
- Example:

  $\forall x \exists y (x + y = 0) --> \forall x (x + f(x) = 0)$

  For the domain of integers with the operation $+$: f = inverse number,
- not equivalent but equisatisfiable

Skolemization: Algorithm

1. transform the formula into NF
2. replace all existentially quantified variables with Skolem functions.
   Arguments of SF = all universally quantified vars that have appeared before the variable.

- Example 2: $\forall x \exists y \neg (P(x,y) \Rightarrow \forall z R(y)) \vee \neg \exists x Q(x)$

  1. $\forall x_1 \exists y \forall x_2 ((P(x_1, y) \vee \neg Q(x_2)) \wedge (\neg R(y) \vee \neg Q(x_2)))$
  2. $\forall x_1 \forall x_2 ((P(x_1, f(x_1)) \vee \neg Q(x_2)) \wedge (\neg R(f(x_1)) \vee \neg Q(x_2)))$

- Example 3: $\forall x \exists y \forall z \exists w (P(x,y) \vee \neg Q(z,w))$

$$\forall x \forall z (P(x, f_1(x)) \vee \neg Q(z, f_2(x, z)))$$

Herbrand's Theorem I

- looking for the simplest interpretation; Skolem normal form, all the constants (maybe +1), functions and predicate symbols
- *Herbrand universe* $U(S)$ = all such terms

  Example:

  For $S = \{P(f(0))\}$,

  $U(S) = \{0, f(0), f(f(0)), f(f(f(0))), \ldots\}$
- *Herbrand base* $B(S)$ = all atomic formulas build upon $U(S)$;

  $B(S) = \{P(t_1, \ldots, t_n) | t_i \in$
  $U(S), P$ ... a predicate symbol from$S\}$

  Example:

  For $S = \{P(f(0))\}$,

  $B(S) = \{P(0), P(f(0)), P(f(f(0))), \ldots\}$

Herbrand's Theorem II

- *Herbrand structure* (in Czech interpretace) is a subset of $B(S)$.
- *Herbrand model* $M(S)$ of S is an Herbrand structure which is model of S, i.e. every sentence f S s true in $M(S)$.
- **Herbrand's Theorem:** Let S be a set of open formulas of a language L. Either
  1. S has an Herbrand model or
  2. S is unsatifiable and, in particular, there are finitely many ground instances of elements of S whose conjunction is unsatisfiable.

Consequence: we do not need to explore any other structures but Herbrand