Tree-depth and vertex minors

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and O-Joung Kwon (KAIST Korea)

0 Structural Measures of Graphs

- **Being close to a TREE – "$\star$-width"**

SPARSE

tree-width / branch-width
– showing a structure

DENSE

clique-width / rank-width
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- **Being close to a STAR – "$\star$-depth"**

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tree-depth
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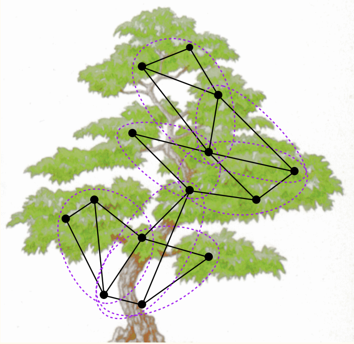
DENSE

???
(will show)

1 Width Measures

Tree-width $tw(G) \leq k$ if whole G can be covered by bags of size $\leq k+1$, arranged in a "tree-like fashion".



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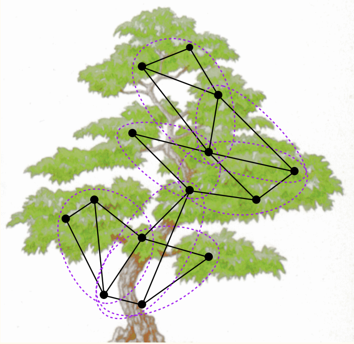

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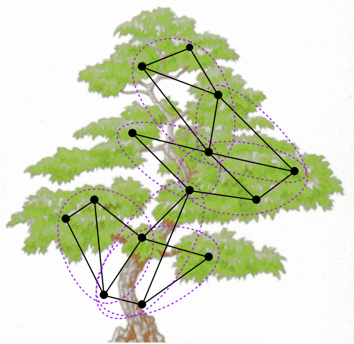

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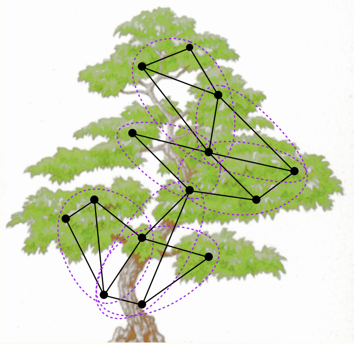

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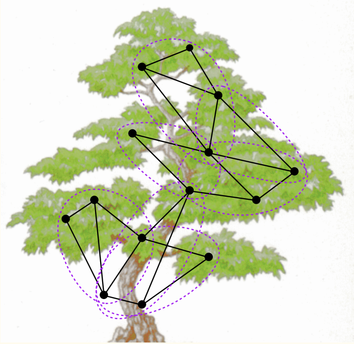

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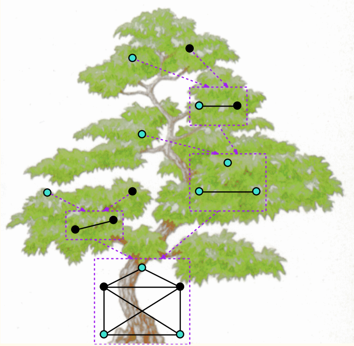

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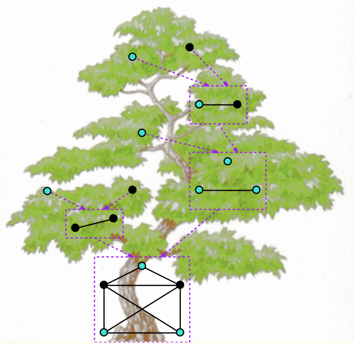

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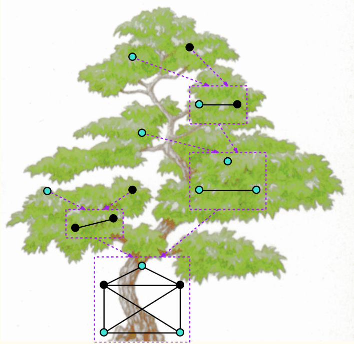

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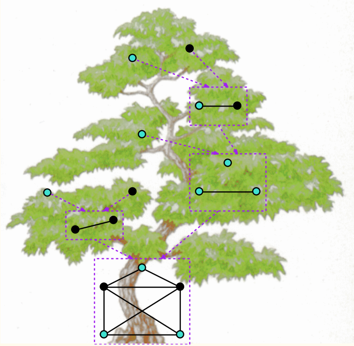

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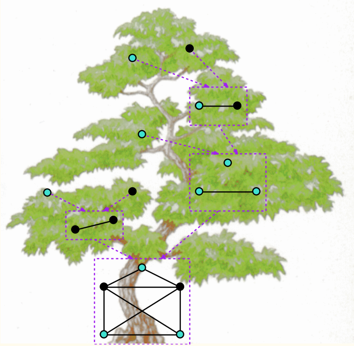

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- no simple "excluded something" characterization known so far.

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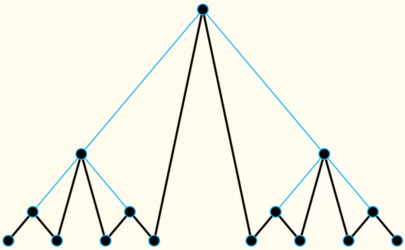

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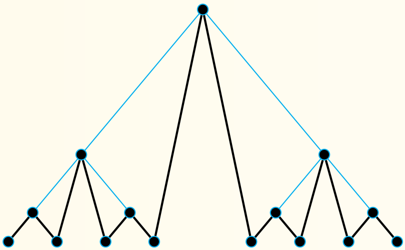

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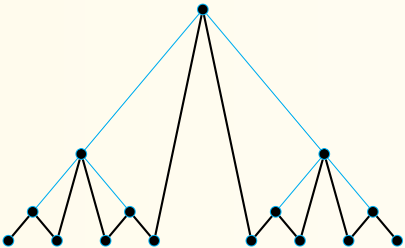

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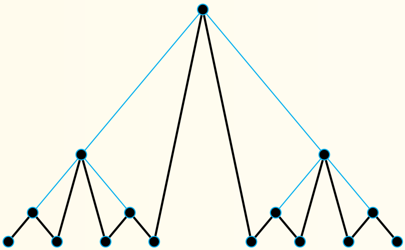

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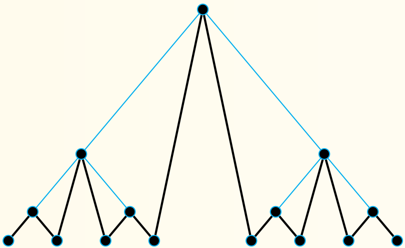

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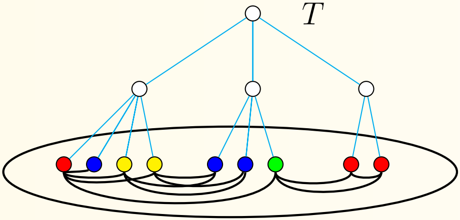

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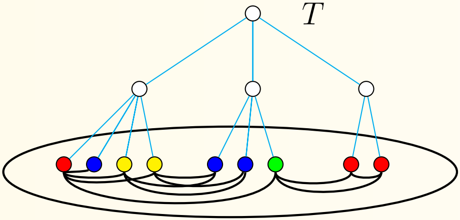

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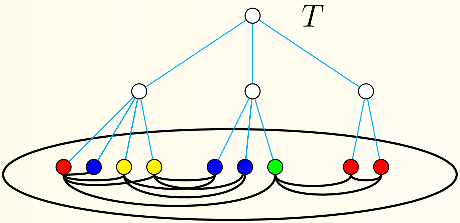

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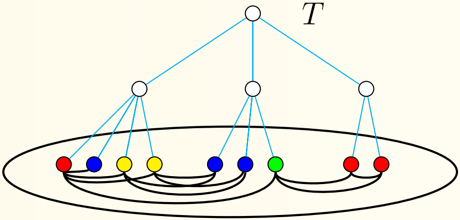

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– whether $\{u, v\} \in E(G)$ depends solely on the labels of u, v and the distance between u, v in T.

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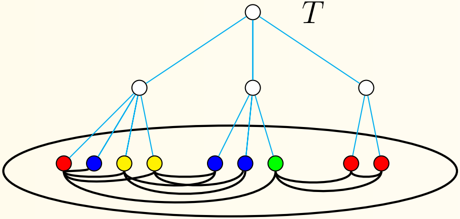

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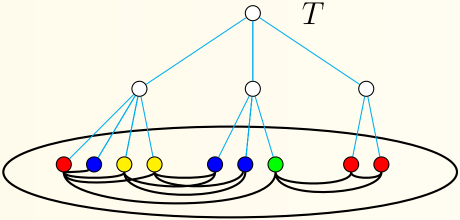

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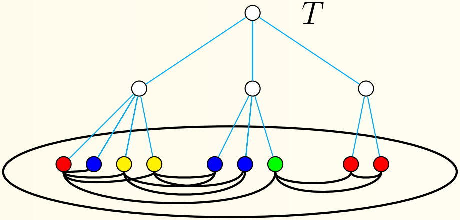

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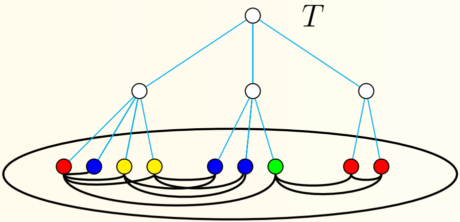

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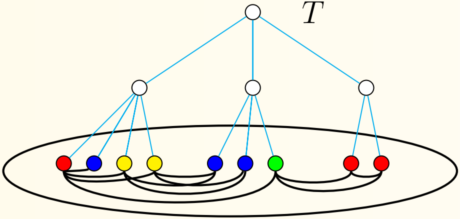

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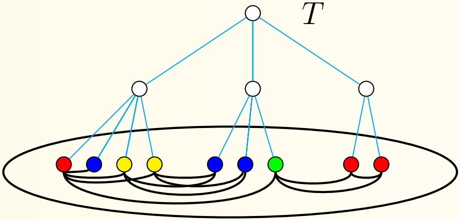

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A graph class $\mathcal{G}$ is of shrub-depth d iff

there exists m such that $\mathcal{G} \subseteq \mathcal{TM}_m(d)$ (same m for all $\mathcal{G}$!).

E.g., the shrub-depth of $\{K_n\}$ is one.

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- So, can we nicely relate tree-depth to shrub-depth ?

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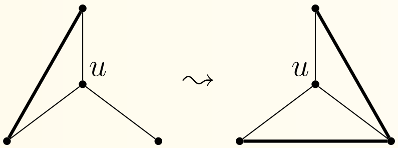

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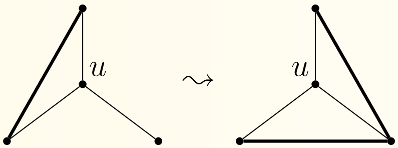

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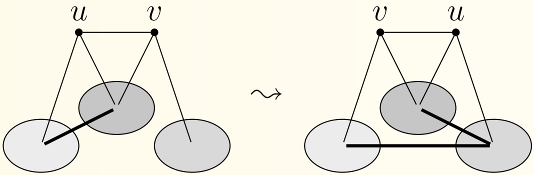

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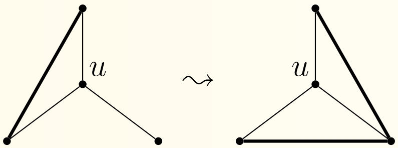

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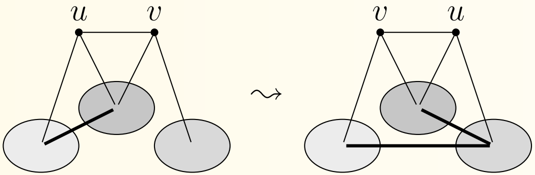

Definition.

Vertex-minor / pivot-minor results as an induced subgraph after a sequence of local complementations / edge pivoting.

(pivot-minor $\subsetneq$ vertex-minor)

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– the effect of vertex-minors can be "paid for" by a limited increase of labels (not the depth); proved by established logical means.

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- "Complement on X" $=$ local complementation of v_X now! $\square$

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Proving II; CMSO interpretation

- This direction follows from some previous results...

Theorem [Ganian et al, 2012 +]
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Local complementations are expressible by C$_2$MSO$_1$ interpretation.

(Note, this holds for arbitr. seq. of local complementations at once.)

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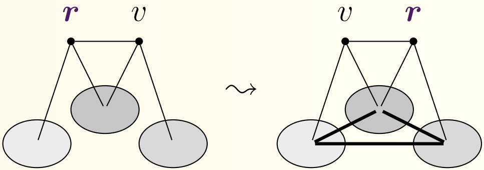

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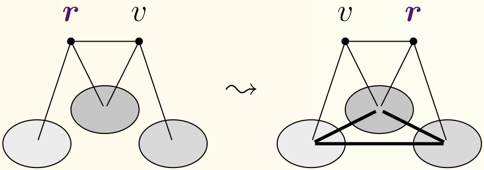

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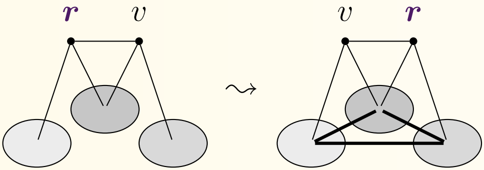

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- and $G \setminus r$ has tree-depth $d-1$ by the definition. $\square$

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THANK YOU FOR ATTENTION.