On a surprising fall of Fellows' Conjecture

Petr Hliněný

Faculty of Informatics
Masaryk University

Botanická 68a, 602 00 Brno, Czech Rep.

e-mail: `hlineny@fi.muni.cz`
`http://www.fi.muni.cz/~hlineny`

*Partly based on joint works with Markus Chimani and Martin Derka.

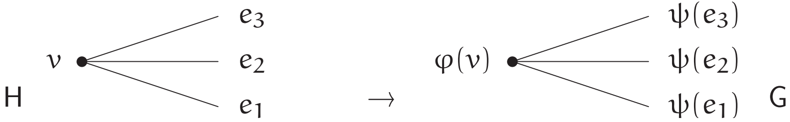
1 Definitions

Motivation: Exploring the two graphs locally, we cannot see any difference...

A graph H is a *cover* of a graph G if there exists a pair of onto mappings

(a *projection*) $\qquad \varphi : V(H) \to V(G), \qquad \psi : E(H) \to E(G)$

such that ψ maps the edges incident with each vertex v in H
bijectively onto the edges incident with $\varphi(v)$ in G.



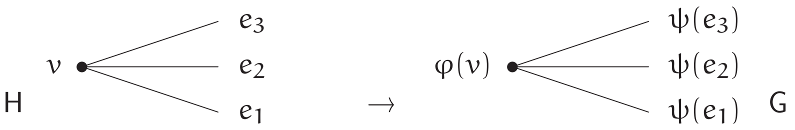
1 Definitions

Motivation: Exploring the two graphs locally, we cannot see any difference...

A graph H is a *cover* of a graph G if there exists a pair of onto mappings

(a *projection*) $\qquad \varphi : V(H) \to V(G), \qquad \psi : E(H) \to E(G)$

such that ψ maps the edges incident with each vertex v in H
bijectively onto the edges incident with $\varphi(v)$ in G.



Remark. The edge $\psi(uv)$ has always ends $\varphi(u), \varphi(v)$, and hence only

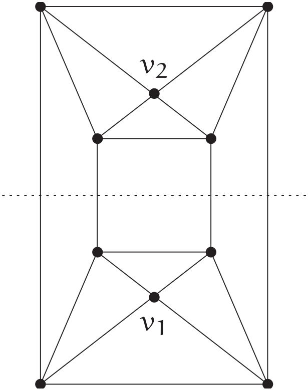
$\varphi : V(H) \to V(G)$, the *vertex projection*,

is enough to be specified for simple graphs.

Planar covers

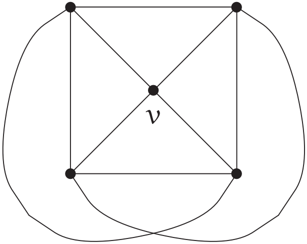
- We speak about a *planar cover* if H is a finite planar graph.

H



$\longrightarrow$

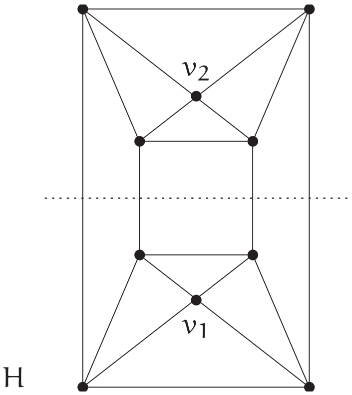
$\varphi(v_1) = \varphi(v_2) = v$



$G = K_5$

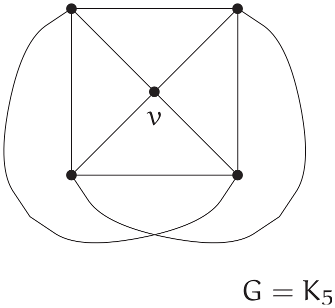
Planar covers

- We speak about a *planar cover* if H is a finite planar graph.



$\longrightarrow$

$\varphi(v_1) = \varphi(v_2) = v$

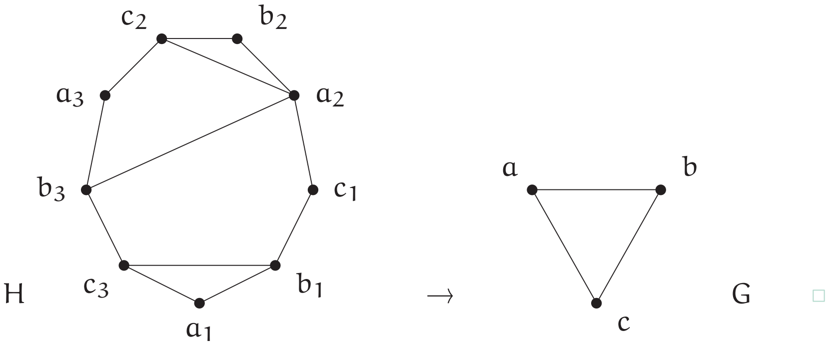


- Graph embedded in the *projective plane* has a double planar cover, via the universal covering map from the sphere onto the projective plane.

Planar emulators

- $\varphi : V(H) \to V(G)$, an *emulator* vs. a cover:

  ... map the edges inc. with v in H surjectively onto the edges inc. with $\varphi(v)$ in G.

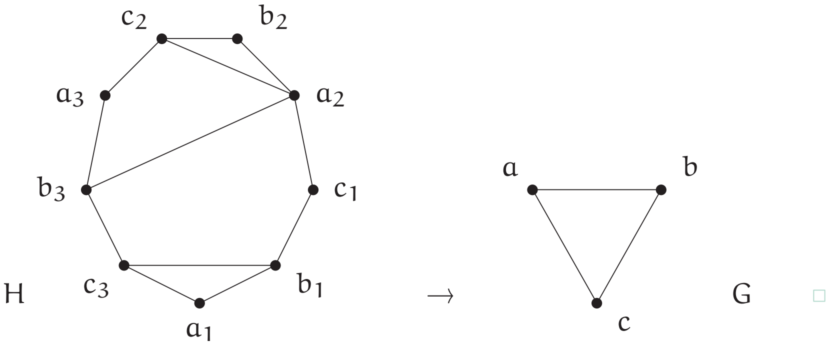


- Can a planar emulator be "more than" a planar cover?

Planar emulators

- $\varphi : V(H) \to V(G)$, an *emulator* vs. a cover:

  ... map the edges inc. with v in H surjectively
  onto the edges inc. with $\varphi(v)$ in G.



- Can a planar emulator be "more than" a planar cover?
- Not many remarkable results until 2008... Interesting at all?

2 Interest in planar covers

- Raised by Negami [1986] in relation to enumeration of projective embeddings of 3-connected graphs.

2 Interest in planar covers

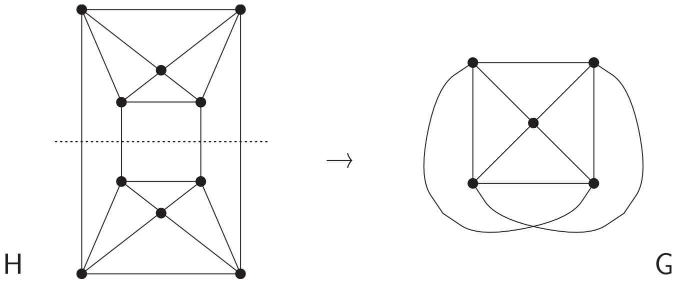
- Raised by Negami [1986] in relation to enumeration of projective embeddings of 3-connected graphs.

- Independently, *planar emulators* considered by Fellows in his CS-oriented thesis [1985] ("embedding graphs in graphs").

2 Interest in planar covers

- Raised by Negami [1986] in relation to enumeration of projective embeddings of 3-connected graphs.

- Independently, *planar emulators* considered by Fellows in his CS-oriented thesis [1985] ("embedding graphs in graphs").



Theorem 1 (Negami, 1986) *A connected graph has a double planar cover if and only if it embeds in the projective plane.*

Negami's planar cover conjecture

- A cover $\varphi : V(H) \to V(G)$ is *regular*

  if there is a subgroup $A \subseteq Aut(H)$ such that $\varphi(u) = \varphi(v)$ for $u, v \in V(H)$ if, and only if $\tau(u) = v$ for some $\tau \in A$.

Theorem 2 (Negami, 1988**)** *A connected graph has a finite regular planar cover if and only if it embeds in the projective plane.*

Negami's planar cover conjecture

- A cover $\varphi : V(H) \to V(G)$ is *regular*

if there is a subgroup $A \subseteq Aut(H)$ such that $\varphi(u) = \varphi(v)$
for $u, v \in V(H)$ if, and only if $\tau(u) = v$ for some $\tau \in A$.

Theorem 2 (Negami, 1988) *A connected graph has a finite regular planar cover if and only if it embeds in the projective plane.*

And now an immediate generalization reads. . .

Conjecture 3 (Negami, 1988)

A connected graph has a finite ~~regular~~ planar cover
if and only if
it embeds in the projective plane.

Fellows' planar emulator conjecture

Fact. A planar cover is also a planar emulator.

Why a planar emulator should be "more than" a planar cover?

Fellows' planar emulator conjecture

Fact. A planar cover is also a planar emulator.

Why a planar emulator should be "more than" a planar cover?

- We only "use more edges" – this takes us farther away from planarity!

Fellows' planar emulator conjecture

Fact. A planar cover is also a planar emulator.

Why a planar emulator should be "more than" a planar cover?

- We only "use more edges" – this takes us farther away from planarity!
- Until the end of 2008, most people considered planar emulators just as a strange redefinition of covers. . .

Fellows' planar emulator conjecture

Fact. A planar cover is also a planar emulator.

Why a planar emulator should be "more than" a planar cover?

- We only "use more edges" – this takes us farther away from planarity!
- Until the end of 2008, most people considered planar emulators just as a strange redefinition of covers...

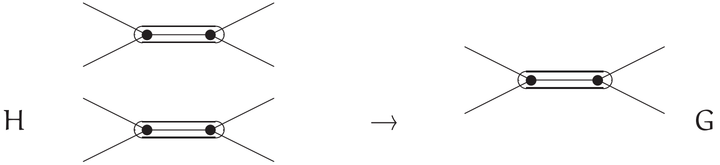
Conjecture 4 (Fellows, 1989)

A connected graph has a finite planar emulator
if and only if
it has a finite planar cover.

Conjecture 5 (Kitakubo, 1991) *A connected graph has a finite planar emulator if and only if it embeds in the projective plane.*

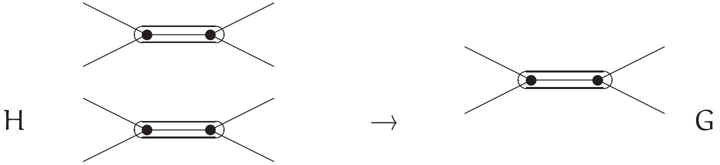
3 Some useful properties

- If G has a planar cover, then so does every minor of G.

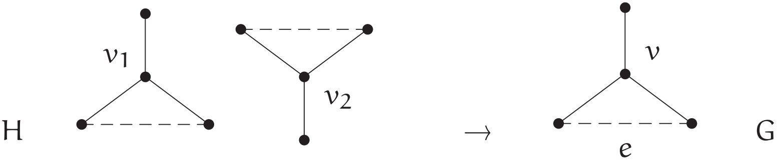


3 Some useful properties

- If G has a planar cover, then so does every minor of G.

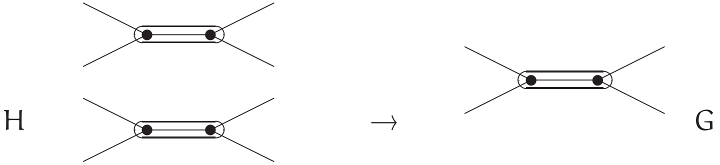


Consider e between two neighbours of a cubic vertex.
If $G - e$ has a planar cover, then so does G.

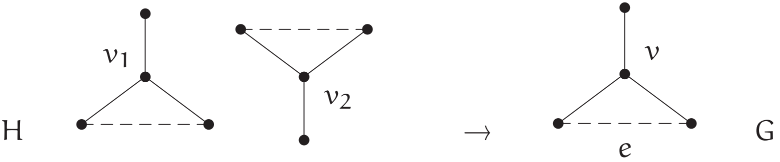


3 Some useful properties

- If G has a planar cover, then so does every minor of G.



Consider e between two neighbours of a cubic vertex.
If $G - e$ has a planar cover, then so does G.



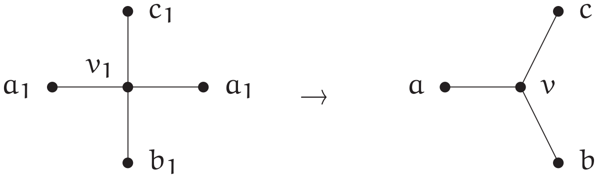
- Therefore, if G has a planar cover, and G' is obtained from G by *$Y\Delta$-transformations*, then G' has a planar cover, too.

Extending to emulators

- If G has a *planar emulator*, then so does every minor of G.

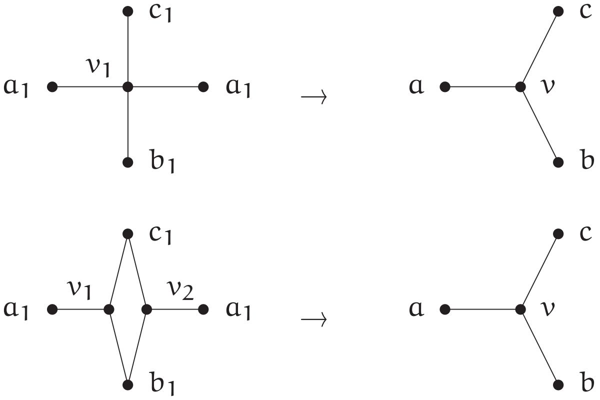
Extending to emulators

- If G has a *planar emulator*, then so does every minor of G.
- If G has a planar emulator, and v is a cubic vertex of G, then some planar emulator H of G has all vertices in $\varphi^{-1}(v)$ also cubic.



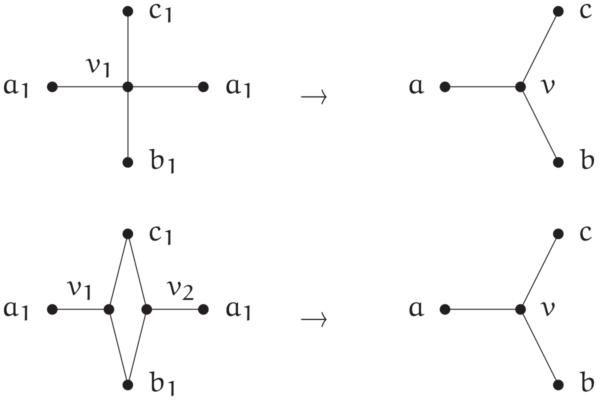
Extending to emulators

- If G has a *planar emulator*, then so does every minor of G.
- If G has a planar emulator, and v is a cubic vertex of G, then some planar emulator H of G has all vertices in $\varphi^{-1}(v)$ also cubic.



Extending to emulators

- If G has a *planar emulator*, then so does every minor of G.
- If G has a planar emulator, and v is a cubic vertex of G, then some planar emulator H of G has all vertices in $\varphi^{-1}(v)$ also cubic.



Therefore, if G has a planar emulator, and G' is obtained from G by *YΔ-transformations*, then G' has a planar emulator, too.

4 Approaching the conjectures

A connected graph has a finite planar cover / emulator if and only if it embeds in the projective plane.

We recall the above basic properties...

- Assume a *projective graph* G. Then G has a double planar cover / emulator.

4 Approaching the conjectures

A connected graph has a finite planar cover / emulator if and only if it embeds in the projective plane.

We recall the above basic properties. . .

- Assume a *projective graph* G. Then G has a double planar cover / emulator.
- Conversely, assume connected G is not projective.
  Then G contains some F of the *forbidden minors* for the projective plane.
  We just have to show that this F has no finite planar cover / emulator.

4 Approaching the conjectures

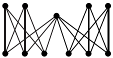
A connected graph has a finite planar cover / emulator if and only if it embeds in the projective plane.

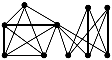
We recall the above basic properties...

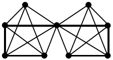
- Assume a *projective graph* G. Then G has a double planar cover / emulator.

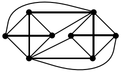
- Conversely, assume connected G is not projective.
  Then G contains some F of the *forbidden minors* for the projective plane.
  We just have to show that this F has no finite planar cover / emulator.

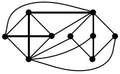
- Furthermore, it is enough to consider only those F which are $Y\Delta$-transforms of some forbidden minor in G.

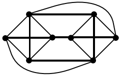

$K_{3,3} \cdot K_{3,3}$

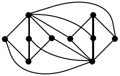

$K_5 \cdot K_{3,3}$

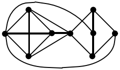

$K_5 \cdot K_5$

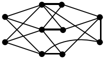

$\mathcal{B}_3$

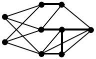

$\mathcal{C}_2$

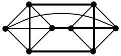

$\mathcal{C}_7$

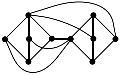

$\mathcal{D}_1$

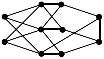

$\mathcal{D}_4$

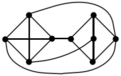

$\mathcal{D}_9$

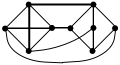

$\mathcal{D}_{12}$

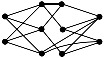

$\mathcal{D}_{17}$

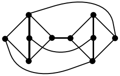

$\mathcal{E}_6$

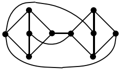

$\mathcal{E}_{11}$

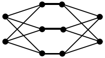

$\mathcal{E}_{19}$

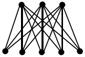

$\mathcal{E}_{20}$

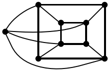

$\mathcal{E}_{27}$

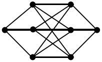

$\mathcal{F}_4$

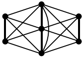

$\mathcal{F}_6$

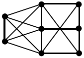

$\mathcal{G}_1$

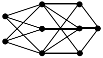

$K_{3,5}$

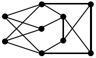

$K_{4,5} - 4K_2$

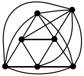

$K_{4,4} - e$

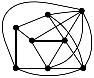

$K_7 - C_4$

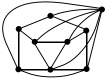

$\mathcal{D}_3$

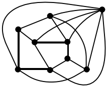

$\mathcal{E}_5$

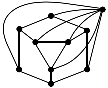

$\mathcal{F}_1$

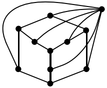

$K_{1,2,2,2}$


$\mathcal{B}_7$


$\mathcal{C}_3$


$\mathcal{C}_4$


$\mathcal{D}_2$


$\mathcal{E}_2$

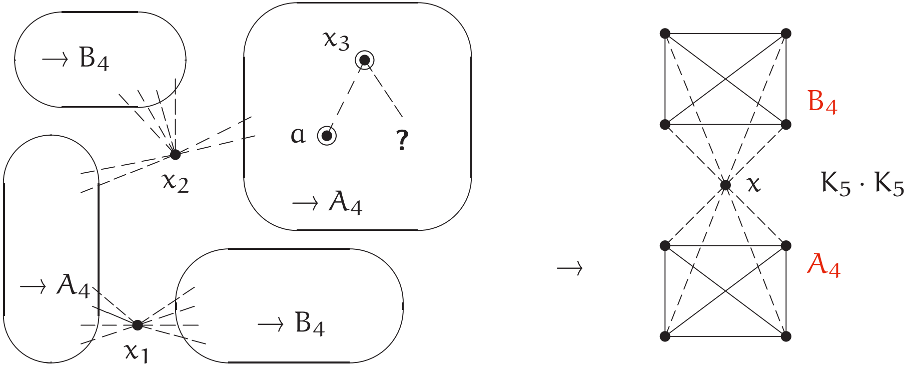
Disjoint $\Bbbk$-graphs

Theorem 6 (Negami / Archdeacon 1988, Fellows 1989)
Neither of the graphs $K_{3,3} \cdot K_{3,3}$, $K_5 \cdot K_{3,3}$, $K_5 \cdot K_5$, $\mathcal{B}_3$, $\mathcal{C}_2$, $\mathcal{C}_7$, $\mathcal{D}_1$, $\mathcal{D}_4$, $\mathcal{D}_9$, $\mathcal{D}_{12}$, $\mathcal{D}_{17}$, $\mathcal{E}_6$, $\mathcal{E}_{11}$, $\mathcal{E}_{19}$, $\mathcal{E}_{20}$, $\mathcal{E}_{27}$, $\mathcal{F}_4$, $\mathcal{F}_6$, $\mathcal{G}_1$ *have a finite planar emulator.*

Disjoint k-graphs

Theorem 6 (Negami / Archdeacon 1988, Fellows 1989)
Neither of the graphs $K_{3,3} \cdot K_{3,3}$, $K_5 \cdot K_{3,3}$, $K_5 \cdot K_5$, $\mathcal{B}_3$, $\mathcal{C}_2$, $\mathcal{C}_7$, $\mathcal{D}_1$, $\mathcal{D}_4$, $\mathcal{D}_9$, $\mathcal{D}_{12}$, $\mathcal{D}_{17}$, $\mathcal{E}_6$, $\mathcal{E}_{11}$, $\mathcal{E}_{19}$, $\mathcal{E}_{20}$, $\mathcal{E}_{27}$, $\mathcal{F}_4$, $\mathcal{F}_6$, $\mathcal{G}_1$ *have a finite planar emulator.*

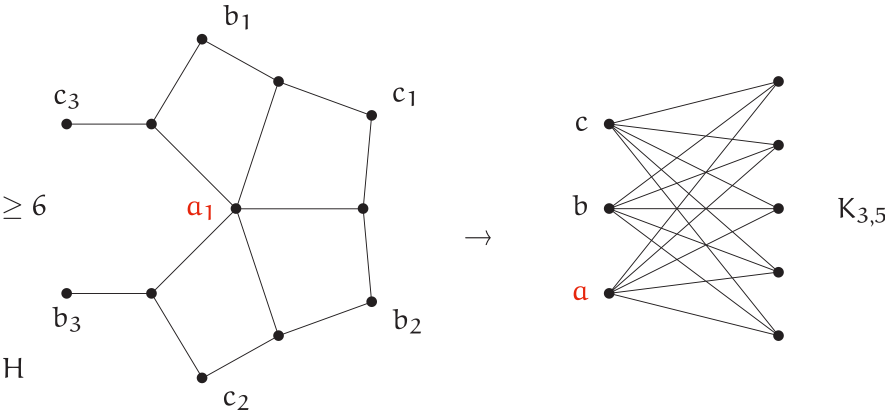
Proof sketch. We choose the $K_5 \cdot K_5$ case for an illustration. . .



Discharging technique

Theorem 7 (?? 1988 – 1993) *The graph $K_{3,5}$ has no finite planar emulator.*

Proof sketch. Assuming H is a finite planar cover of $K_{3,5}$, we shall derive a contradiction to Euler's formula (or, easy *discharging*)...



Further results (and a big surprise)

Long-term development around Negami's conjecture led to. . .

Theorem 8 (since 1998)
If $K_{1,2,2,2}$ had no finite planar cover, then Negami's conjecture would be proved.

Further results (and a big surprise)

Long-term development around Negami's conjecture led to...

Theorem 8 (since 1998)
If $K_{1,2,2,2}$ had no finite planar cover, then Negami's conjecture would be proved.

... and then...

Further results (and a big surprise)

Long-term development around Negami's conjecture led to...

Theorem 8 (since 1998)
If $K_{1,2,2,2}$ had no finite planar cover, then Negami's conjecture would be proved.

... and then... Suddenly, Fellows' conjecture falls down...

Fact. The graph $K_{4,5}-4K_2$ has no finite planar cover.

Theorem 9 (Rieck and Yamashita 2008)
The graphs $K_{1,2,2,2}$ and $K_{4,5}-4K_2$ do have finite planar emulators!!!

Further results (and a big surprise)

Long-term development around Negami's conjecture led to...

Theorem 8 (since 1998)
If $K_{1,2,2,2}$ had no finite planar cover, then Negami's conjecture would be proved.

... and then... Suddenly, Fellows' conjecture falls down...

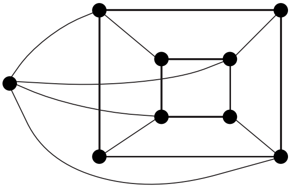
Fact. The graph $K_{4,5} - 4K_2$ has no finite planar cover.

Theorem 9 (Rieck and Yamashita 2008)
The graphs $K_{1,2,2,2}$ and $K_{4,5} - 4K_2$ do have finite planar emulators!!!

- Now we know that the class of graphs having finite *planar emulators*
  - is different from the class of graphs having finite *planar covers*,
  - and different from the class of *projective planar* graphs, too.
- So, let us study this class...!

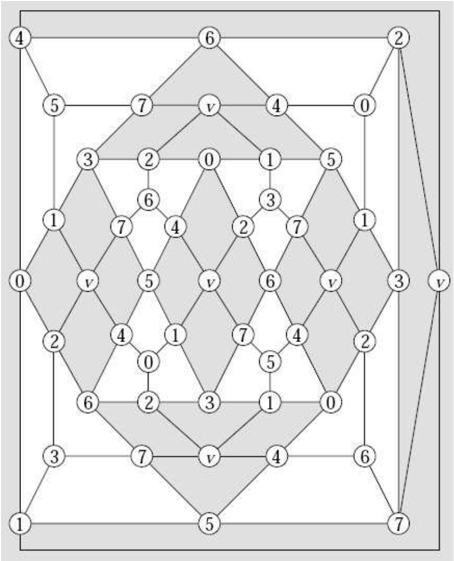
5 Constructing new planar emulators

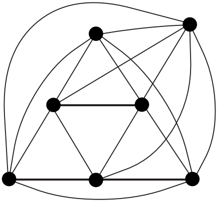
Rieck and Yamashita, 2008



$K_{4,5} - 4K_2$

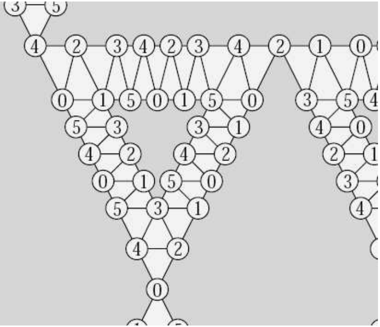
$\longleftarrow$

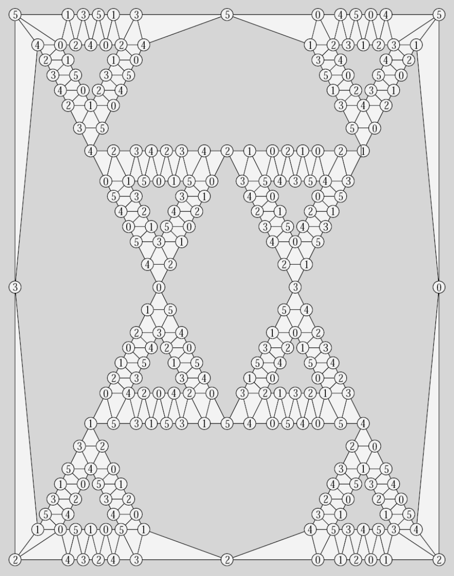




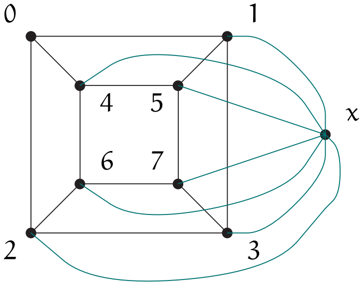
←

$K_{1,2,2,2}$

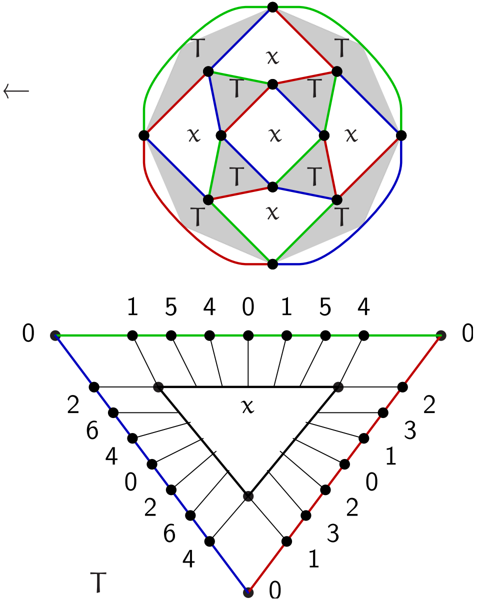


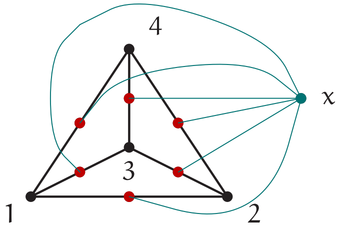


Chimani and PH, 2009

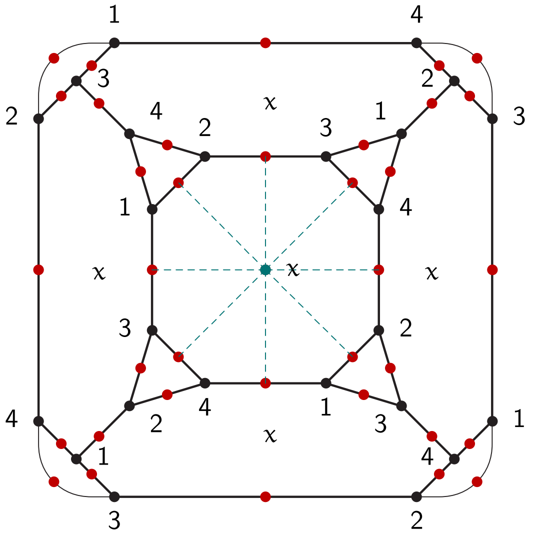


$\mathcal{C}_4$





$\mathcal{E}_2$



6 Conclusion

Repeating the previous message. . .

- Now we know that the class of graphs having finite *planar emulators*
  - is different from the class of graphs having finite *planar covers*,
  - and different from the class of *projective planar* graphs, too.

6 Conclusion

Repeating the previous message...

- Now we know that the class of graphs having finite *planar emulators*
  - is different from the class of graphs having finite *planar covers*,
  - and different from the class of *projective planar* graphs, too.

- Many other nontrivial planar emulators can be derived from the ones of Chimani and PH, particularly a small one for $K_{1,2,2,2}$.

6 Conclusion

Repeating the previous message. . .

- Now we know that the class of graphs having finite *planar emulators*
  - is different from the class of graphs having finite *planar covers*,
  - and different from the class of *projective planar* graphs, too.

- Many other nontrivial planar emulators can be derived from the ones of Chimani and PH, particularly a small one for $K_{1,2,2,2}$.
- Are there finite planar emulators of, say, $K_{4,4}-e$ and K_7-C_4?
- Is there an infinite (nontrivial) family of non-projective graphs having finite planar emulators?

6 Conclusion

Repeating the previous message. . .

- Now we know that the class of graphs having finite *planar emulators*
  - is different from the class of graphs having finite *planar covers*,
  - and different from the class of *projective planar* graphs, too.

- Many other nontrivial planar emulators can be derived from the ones of Chimani and PH, particularly a small one for $K_{1,2,2,2}$.
- Are there finite planar emulators of, say, $K_{4,4}-e$ and K_7-C_4?
- Is there an infinite (nontrivial) family of non-projective graphs having finite planar emulators?
- Finally, the class of graphs having finite *planar emulators* definitely deserves further study.
  - the subject of ongoing computer-aided research with M. Derka.