Approximating the Crossing Number for Graphs close to "Planarity"

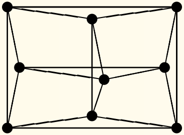
Petr Hliněný

Faculty of Informatics, Masaryk University
Botanická 68a, 602 00 Brno, Czech Rep.

e-mail: `hlineny@fi.muni.cz`
`http://www.fi.muni.cz/~hlineny`

joint work with **Gelasio Salazar**

Universidad Autónoma de San Luis Potosí, Mexico



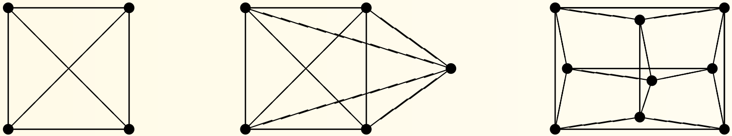
Overview

1 Drawings and the Crossing Number

Definition. *Drawing of a graph G*:

– The vertices of G are distinct points,
  and every edge $e = uv \in E(G)$ is a simple curve joining u to v.

– No edge passes through another vertex,
  and no three edges intersect in a common point.



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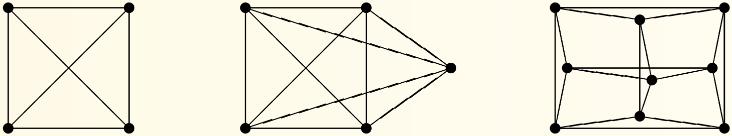

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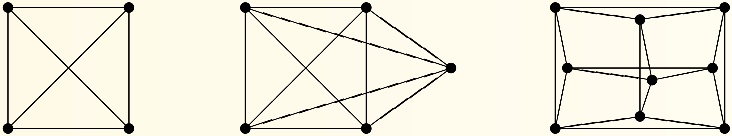

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Warning. There are slight variations of the definition of crossing number, some giving different numbers! (Like counting odd-crossing pairs of edges.)

Computational complexity

Remark. It is (practically) very hard to determine crossing number.

Observation. The problem $\textsc{CrossingNumber}(\leq k)$ is in NP:
Guess a drawing of G, then replace crossings with vertices, and test planarity.

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- [**PH**, 2004] $\textsc{CrossingNumber}$ is NP-hard even on simple 3-connected cubic graphs, hence also in the minor-monotone setting.
- [**Kawarabayashi and Reed**, 2007] $\textsc{CrossingNumber}(\leq k)$ is linear FPT with parameter k, i.e. solvable in time $O(f(k) \cdot n)$.

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- Any other idea of a "nontrivial" graph class with an efficient $\textsc{Crossing-Number}$ solution?

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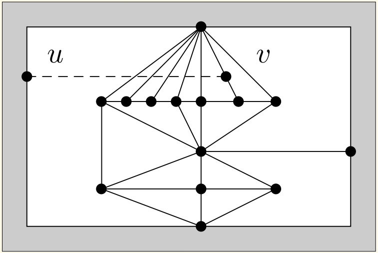
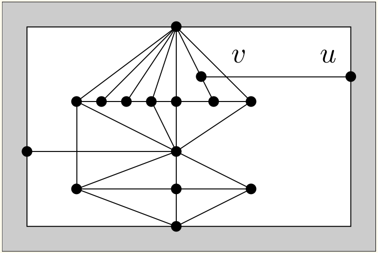
How good in theory is this solution?

- [**Farr** 2005; indep. PH and GS] A solution to one-edge bridging minimization (left) can be arbitrarily far from the crossing number (right).





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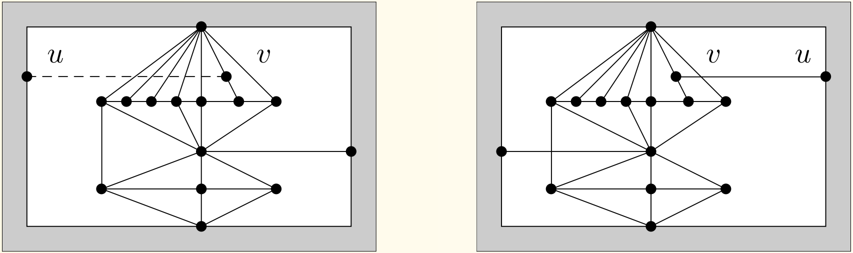

Theorem 2. [**PH and GS**, 2006]
The bridging minimization problem on G and uv has a solution with at most

$$\boldsymbol{\Delta(G) \cdot \mathrm{cr}(G + uv)}$$

crossings; hence it approximates up to factor $\Delta(G)$.

Proof idea: Whitney flips between two planar subdrawings, ≤ 2 flips per crossing and each one makes $\leq \Delta(G)/2$ new crossings.

Graphs on small surfaces; projective and toroidal

- Recent results drawing graphs with linear number of crossings:

  [**Böröczky, Pach and Tóth**] surface embedded graphs,

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A natural approach. (See also [Pach and Tóth])

Cut the surface along short noncontractible loops ($\rightarrow$ face-width or dual edge-width), then re-insert edges to resulting planar subgraph(s).

Such loops can be computed quickly [Cabello and Mohar] $O(n\sqrt{n})$ time, [unpublished improvements. . .] $O(n \log n)$ time.

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- **Ongoing work**: Extending good lower bounds to higher (orientable, at least) surfaces.
- **Problem**: Can one get rid of dependence on $\Delta(G)$?

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Crossing number (with no upper bound on the number of crossings) shows a "very global" behavior, which makes parametrized approaches harder. . .

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- Can we prove that $\textsc{CrossingNumber}$ is NP-hard for graphs of bounded clique-width?

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