Balanced Signings and the Chromatic Number of Oriented Matroids

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We consider the problem of reorienting an oriented matroid so that all its cocircuits are 'as balanced as possible in ratio'. It is well known that any oriented matroid having no coloops has a *totally cyclic reorientation*, a reorientation in which every signed cocircuit $B = \{B^+, B^-\}$ satisfies $B^+, B^- \neq \emptyset$. We show that, for some reorientation, every signed cocircuit satisfies

$$1/f(r) \leq |B^+|/|B^-| \leq f(r),$$

where $f(r) \leq 14\, r^2 \ln(r)$, and r is the rank of the oriented matroid.

In geometry, this problem corresponds to bounding the discrepancies (in ratio) that occur among the Radon partitions of a dependent set of vectors. For graphs, this result corresponds to bounding the chromatic number of a connected graph by a function of its Betti number (corank) $|E| - |V| + 1$.

Key Words: Oriented matroid, pseudosphere configuration, hyperplane arrangements, flow, circular chromatic number, discrepancy, cocircuit, wiring diagram

1. INTRODUCTION

This paper regards an optimization problem which is an oriented matroid analogue of the graph chromatic number. There are several ways in which a 'chromatic number' might be defined for more general matroids. One such formulation, introduced by Goddyn, Tarsi and Zhang [8], depends only on the sign patterns of (signed) circuits (or cocircuits). The result is a natural invariant of an oriented matroid. In fact, any oriented matroid is representable as a pseudosphere complex, a regular cell decomposition of the sphere, where the cocircuits correspond to the zero-dimensional cells, see Figure 1 for an example in rank 3. Accordingly, the invariant can be viewed as a 'discrepancy in ratio' of a hyperplane arrangement, and thus should be of interest to geometers. The main theorem answers a question raised in [8].
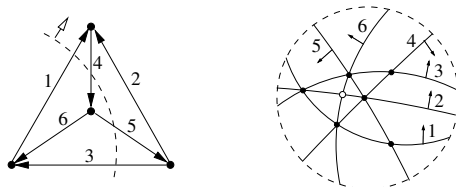

FIG. 1. An orientation of K_4 is represented as rank 3 pseudosphere arrangement. The dotted circle is the sphere equator. Each graph edge corresponds to a hypersphere, which is drawn as a circular arc with its "positive" side indicated with an arrow. Each vertex of the arrangement corresponds to one of the 7 cocircuits (directed cuts) of K_4. Indicated on both diagrams is the signed cocircuit $\{\{1, 5\}, \{3, 4\}\}$.

We first state the result and some consequences, using a minimal set of definitions. Detailed definitions appear in Section 2. It is convenient to view an oriented matroid $\mathcal{O}$ to be a matroid in which every circuit C (and cocircuit B) has been partitioned $C = C^+ \cup C^-$, (and $B = B^+ \cup B^-$) subject to a standard orthogonality condition. Each such partition is an unordered pair $\{C^+, C^-\}$, where one of the parts may be empty. For $I \subseteq E(\mathcal{O})$, the *reorientation* $\mathcal{O}_I$ of $\mathcal{O}$ is the new oriented matroid obtained from $\mathcal{O}$ by repartitioning each circuit C (and cocircuit B) according to the

rules

$$\{C^+, C^-\} \mapsto \{C^+ \triangle (I \cap C), C^- \triangle (I \cap C)\} \quad \text{and}$$
$$\{B^+, B^-\} \mapsto \{B^+ \triangle (I \cap B), B^- \triangle (I \cap B)\}. \tag{1}$$

where "$\triangle$" is the symmetric difference operator.

THEOREM 1.1. *Let $\mathcal{O}$ be an oriented matroid of rank at most r. There exists a reorientation $\mathcal{O}_I$ in which every cocircuit B of size at least two satisfies*

$$|B^+|, |B^-| \geq |B|/f(r)$$

where $f(3) \leq 17$, and $f(r) \leq 14r^2 \ln r$ for $r \geq 3$.

For $\mathbb{R}$-represented matroids, our result specializes to a new bound on the discrepancies 'in ratio' that occur among the *Radon partitions* of minimally dependent sets of real vectors of small corank. To be more precise:

COROLLARY 1.2. *Let $(\boldsymbol{v}_e : e \in E)$ be a list of nonzero vectors in $\mathbb{R}^r$. Then it is possible to replace some of these vectors by their negatives such that, for any minimal sublist $\{\boldsymbol{v}_e \mid e \in C\}$, $C \subseteq E$ of linearly dependent vectors, every nontrivial real solution to $\sum_{e \in C} \alpha_e \boldsymbol{v}_e = 0$ has at least $|C|/f(|E| - r)$ coefficients α_e of each sign, where $f(s) \leq 14s^2 \ln s$.*

We may restate this result in the dual. The *support* of a vector $\boldsymbol{t} = (t_e) \in \mathbb{R}^E$ is $\mathrm{supp}(\boldsymbol{t}) = \{e \in E \mid t_e \neq 0\}$. The *rowspace* of a matrix A is the set of vectors of the form yA, where y is a row vector.

COROLLARY 1.3. *Let $A \in \mathbb{R}^{R \times E}$ be a real matrix of rank r, where R and E are index sets. Then it is possible to multiply some columns of A by -1 so that, for every vector $\boldsymbol{t}$ in the rowspace of the resulting matrix, if $|\mathrm{supp}(\boldsymbol{t})| \geq 2$ and $\mathrm{supp}(\boldsymbol{t})$ is minimal among the nonzero vectors in the rowspace, then $\boldsymbol{t}$ contains at least $|\mathrm{supp}(\boldsymbol{t})|/f(r)$ positive and negative entries each, where $f(r) \leq 14r^2 \ln r$.*

We remark that, if $(\boldsymbol{v}_e : e \in E)$ are the columns of a real matrix A, then the sets $C \subseteq E$, and $\mathrm{supp}(\boldsymbol{t}) \subseteq E$ referred to in Corollaries 1.2 and 1.3 are, respectively, the circuits and cocircuits of the oriented matroid represented by the matrix A.

If each column $\boldsymbol{v}_e$ of A is the difference of two unit vectors, then we are in the setting of graph theory: Here $A \in \mathbb{R}^{V \times E}$ is the $\{0, \pm 1\}$-valued vertex-edge incidence matrix of a directed graph $\vec{G} = (V, E)$. Multiplying $\boldsymbol{v}_e$ by -1 corresponds to reorienting the edge e in $\vec{G}$.

A formula of Minty [11] relates the graph chromatic number $\chi(G)$ to ratios of the form $|C|/|C^+|$ seen among reorientations of a G. Here Corollary 1.2 further specializes to the observation that $\chi(G) \leq f(\mathrm{rk}^*(G))$ for

any loopless graph $G = (V, E)$, where $\mathrm{rk}^*(G) = |E| - |V| + 1$ is the *corank*, or the *first Betti number* of G. For graphs, the upper bound on f can be improved to $\chi(G) \leq f(s) \leq \lfloor \sqrt{2s} \rfloor + 2$. (This follows without much difficulty from Dirac's density result [4] for colour critical graphs.) This bound on $\chi(G)$ is (essentially) attained by complete graphs. The above discussion indicates that the best function f for our result satisfies

$$\left\lfloor \sqrt{2s} \right\rfloor + 2 \leq f(s) \leq 14s^2 \ln s.$$

We do not attempt to further optimize the function $f(s)$ of Theorem 1.1.

2. BACKGROUND AND DEFINITIONS

We shall use terminology from matroids, oriented matroids and real geometry. For sake of convenience, we tersely review the relevant definitions and connections, although the reader is expected to have some basic knowledge of graphs, matroids and linear algebra. The reader is referred to [10] for details on matroid theory, and to [2] for information on oriented matroids and their geometry. The experienced reader may prefer to skip to the fourth subsection, although a couple of our oriented-matroid terms are non-standard.

2.1.   Matroids

A *matroid* M is a *ground set* $E = E(M)$ together with a collection of subsets called *independent sets*. Independent sets are closed under taking subsets, and they satisfy a well-known *exchange axiom*. A maximal independent set is a *basis* of M. Minimal dependent sets in M are *circuits*. A *loop* is a circuit of size one. Two elements are *parallel* if they form a circuit. The *parallel class* $[e]$ is the equivalence class of elements parallel to $e \in E$. A matroid is *simple* if it has no loops or parallel elements. The *girth* of M is the least cardinality of one of its circuits. The *rank*, $\mathrm{rk}(X)$, of a set $X \subseteq E$ is the maximum cardinality of an independent subset of X. We write $\mathrm{rk}(M) = \mathrm{rk}(E(M))$ for the rank of the matroid. A *k-flat* is a maximal subset X of rank k. Equivalently X is a flat if no circuit of M contains exactly one element of $E - X$. The intersection of two flats is a flat. A *hyperplane* of M is an $(r(M) - 1)$-flat. A *connected component* of M is a maximal subset of E in which any two elements are contained in some circuit of M. A matroid with one component is *connected*.

The complements of the bases of M form the bases of the *dual matroid* M^*. We have $E(M) = E(M^*)$. The prefix "*co*" refers to sets or properties of the dual matroid. In particular, a set $X \subseteq E$ has *corank k*, is a *coloop*, a *cocircuit*, or a *coparallel class* in M if (resp.) X has rank k, is a loop, a circuit, or a parallel class in M^*. A matroid M is *cosimple* if M^* is

simple. A cocircuit of M is characterized as a minimal subset of E which has nonempty intersection with every basis of M. Alternatively cocircuits of M are precisely the complements of hyperplanes of M. A circuit and a cocircuit can never intersect in exactly one element.

By *deleting* (or *contracting*) an element e of M, we obtain a new matroid $M \backslash e$ (or M/e) on the ground set $E(M) - \{e\}$. The circuits of $M \backslash e$ are precisely the circuits of M which avoid e. The cocircuits of M/e are the cocircuits of M which avoid e. If S and T are disjoint subsets of $E(M)$, then $M \backslash S/T$ denotes a matroid obtained by successively deleting the elements in S and contracting those in T. The matroid $M \backslash S/T$ is well defined and is called a *minor* of M.

The *cycle matroid* $M(G)$ of a connected graph (or directed graph) G has ground set $E(G)$. The bases, circuits, and cocircuits of $M(G)$ are, respectively, (the edge sets of) the spanning trees, simple cycles and minimal edge cuts of G. A matroid of the form $M(G)$ is said to be *graphic*, and its dual is *cographic*. We have that $M(G)$ is connected if and only if G is a 2-connected graph. A matroid M is *represented* by a matrix A (over some field) if there is a bijective correspondence between $E(M)$ and columns of A, such that the independent sets of M correspond precisely to linearly independent sets of columns of A. Here we may write $M = M[A]$. The cocircuits of $M[A]$ are the supports of non-zero vectors in the rowspace of A having minimal support. A matroid is $\mathbb{R}$-*representable* if it can be represented by a real matrix. If M can be represented over any field, then M is *regular*. A regular matroid can be represented by a *totally unimodular matrix* A, a real matrix whose subdeterminants all belong to $\{0, \pm 1\}$. The $\{0, \pm 1\}$-valued incidence matrix of a directed graph is totally unimodular, so graphic (and cographic) matroids are regular.

2.2. Oriented Matroids

Among the several equivalent formulations of 'oriented matroid', the following, which is due to Bland and Las Vergnas [3] (cf. [2, Theorem 3.4.3]), is best suited to our purpose. A *signing* of a set X is an unordered partition $\vec{X} = \{X^+, X^-\}$ of $X = X^+ \cup X^-$, where either part may be empty. A pair $(\vec{C}, \vec{B})$ of signed sets is *orthogonal* if

$$(C^+ \cap B^+) \cup (C^- \cap B^-) = \emptyset \iff (C^+ \cap B^-) \cup (C^- \cap B^+) = \emptyset. \quad (2)$$

This terminology reflects the fact that, for any two orthogonal vectors in Euclidean space, either their supports are disjoint, or there is both a positive and a negative summand in their scalar product.

Any orientation of a graph G naturally signs each circuit and cocircuit of its cycle matroid $M(G)$. Moreover, each signed circuit-cocircuit pair

$(\vec{C}, \vec{B})$ in $M(G)$ is orthogonal. Accordingly, we define an *oriented matroid* on the ground set E to be a triple $\mathcal{O} = (M, \vec{\mathcal{C}}, \vec{\mathcal{B}})$ where

1. M is a matroid with ground set E, circuits $\mathcal{C}$, and cocircuits $\mathcal{B}$.
2. $\vec{\mathcal{C}} = \{\vec{C} \mid C \in \mathcal{C}\}$ and $\vec{\mathcal{B}} = \{\vec{B} \mid B \in \mathcal{B}\}$ are signings of the circuits and cocircuits such that each pair in $\vec{\mathcal{C}} \times \vec{\mathcal{B}}$ is orthogonal.

The oriented matroid $\mathcal{O} = (M, \vec{\mathcal{C}}, \vec{\mathcal{B}})$ is called an *orientation* of M, and M is said to be *orientable*. Graphic matroids are orientable; we write $\mathcal{O}(\vec{G})$ for the oriented matroid associated with a directed graph $\vec{G}$. More generally, matroids which are $\mathbb{R}$-representable are orientable. Moreover, every real matrix representation $M = M[A]$ naturally induces an orientation $\mathcal{O}[A]$ of M. The signings of circuits $C = C^+ \cup C^-$ in $\mathcal{O}[A]$ are the *Radon partitions* of minimally dependent sets of columns $\{\boldsymbol{v}_e \mid e \in C\}$ of A. Specifically, the Radon partition of C is determined by the signs of the real coefficients α_e in a non-trivial solution to $\sum_{e \in C} \alpha_e \boldsymbol{v}_e = 0$. The signings of cocircuits of $\mathcal{O}$ are determined by the sign patterns of nonzero vectors in the row-space of A having minimal support. Not all matroids are orientable. For example, a *binary* matroid, i.e. a matroid representable by a matrix over GF(2), is orientable if and only if it is regular. Every orientation of a regular matroid can be represented by a totally unimodular matrix.

Let $\mathcal{O} = (M, \vec{\mathcal{C}}, \vec{\mathcal{B}})$ be an oriented matroid. For any $I \subseteq E$, the reorientation $\mathcal{O}_I$ defined by (1) is easily seen to be an oriented matroid. The operation $\mathcal{O} \mapsto \mathcal{O}_I$ is analogous to reversing a subset I of the edges of a directed graph. The orientations of M are partitioned into *reorientation classes*, where the class of $\mathcal{O}$ is defined to be $[\mathcal{O}] = \{\mathcal{O}_I \mid I \subseteq E\}$. For sake of brevity, we say that $[\mathcal{O}]$ is a *reclass* of M. An orientable matroid may have several reclasses. However all regular matroids (including graphs and cographs) have a unique reclass. For this reason, the reclass of an *unoriented* graph G is well defined by $[\mathcal{O}(G)] = [\mathcal{O}(\vec{G})]$ for some orientation $\vec{G}$ of G. For any connected component I of $\mathcal{O}$, we have $\mathcal{O}_I = \mathcal{O}$. Indeed each reclass of M contains precisely $2^{|E|-\omega}$ orientations of M, where ω is the number of connected components of M.

Matroidal notions such as connectedness, simplicity, flats, and the rank function $\mathrm{rk}(\cdot)$ naturally carry over to oriented matroids and to reclasses. Matroid minors also carry over naturally: for disjoint subsets $S, T \subseteq E(\mathcal{O})$, the *minor* $\mathcal{O}/S \backslash T$ is the orientation of the matroid minor $M/S \backslash T$ obtained by restricting the signings in $\vec{\mathcal{C}} \cup \vec{\mathcal{B}}$ to the circuits and cocircuits of $M/S \backslash T$. The *dual* of $\mathcal{O} = (M, \vec{\mathcal{C}}, \vec{\mathcal{B}})$ is defined by $\mathcal{O}^* = (M^*, \vec{\mathcal{B}}, \vec{\mathcal{C}})$. An oriented matroid $(M, \vec{\mathcal{C}}, \vec{\mathcal{B}})$ is uniquely determined by either of the pairs $(M, \vec{\mathcal{C}})$ or $(M, \vec{\mathcal{B}})$.

2.3. Geometry

There is a bijective correspondence between simple reclasses $[\mathcal{O}]$ and topological objects called pseudosphere complexes. This is due to Folkman/Lawrence [7] and Edmonds/Mandel [12]. We describe a pseudosphere complex in the case $\mathcal{O}$ is $\mathbb{R}$-representable, and outline the construction for general reclasses $[\mathcal{O}]$. We then specialize to rank 3 and the easy-to-visualize wiring diagrams.

Let $\mathcal{O} = \mathcal{O}[A]$ where each column $\boldsymbol{v}_e$ of A is a vector in $\mathbb{R}^r$. Let $S_0 = \{x \in \mathbb{R}^r \mid \|x\| = 1\}$ be the unit $(r-1)$-sphere. Each $e \in E(\mathcal{O})$ corresponds to an $(r-2)$-subsphere H_e, called a *pseudohypersphere*, and consisting of those vectors in S_0 which are orthogonal to $\boldsymbol{v}_e$. The *positive side* of H_e are the vectors in S_0 having a positive scalar product with $\boldsymbol{v}_e$. The *pseudosphere complex* $\mathcal{S} = \mathcal{S}[\mathcal{O}]$ is the family of subspheres of S_0 that can be obtained as intersections of pseudohyperspheres. Each subsphere in $\mathcal{S}$ is a k-sphere for some k, and is called a k-*pseudosphere*. Evidently, a k-flat F of $\mathcal{O}$ corresponds to the set of pseudohyperspheres $\{H_e \mid e \in F\}$ which contain a particular $(r-k-1)$-pseudosphere in $\mathcal{S}$. This correspondence between flats and pseudospheres is bijective. In particular, the hyperplanes of $\mathcal{O}$ correspond to 0-spheres in $\mathcal{S}$. (A 0-sphere consists of two "antipodal" points of S_0.) Accordingly, each 0-sphere $S \in \mathcal{S}$ corresponds to a cocircuit $B = \{e \in E \mid H_e \cap S = \emptyset\}$. The signing $B = B^+ \cup B^-$ is found by determining, for each $e \in B$, which of the two points of S lies on the positive side of H_e. A reorientation $\mathcal{O}_I$ corresponds to interchanging the positive and negative sides of H_e, for each $e \in I$. Thus the complex $\mathcal{S}$ is well defined by the reclass $[\mathcal{O}]$.

The pseudosphere complex of a general simple reclass $[\mathcal{O}]$ of rank r is similarly defined, except that pseudospheres are no longer constrained to lie on subspaces of $\mathbb{R}^r$. Instead, pseudospheres are topological subspheres of S_0 which are subject to certain axioms. The axioms ensure that any 0-pseudosphere which is disjoint from a hyperpseudosphere H_e has exactly one of its two points on the positive side of H_e, so the signing of each cocircuit well defined. A proof of the correspondence between reclasses and pseudosphere complexes can be found in [7].

Deleting an element e of $\mathcal{O}$ corresponds to deleting the pseudohypersphere H_e in the construction of $\mathcal{S}[\mathcal{O}]$. Contracting e in $\mathcal{O}$ corresponds to restricting the complex $\mathcal{S}[\mathcal{O}]$ to those pseudospheres contained in H_e. Here H_e plays the role of S_0 and the pseudohyperspheres of $\mathcal{S}[\mathcal{O}/e]$ are the $(r-2)$-pseudospheres $\{H_e \cap H_f \mid f \in E - \{e\}\}$. Thus contracting a flat F in $\mathcal{O}$ corresponds to restricting $\mathcal{S}$ to those pseudospheres contained in S_F, where S_F is the pseudosphere in $\mathcal{S}$ corresponding to F. Here, the cocircuits of $\mathcal{O}/F$ correspond bijectively to those 0-pseudospheres in $\mathcal{O}$ which are contained in S_F. We have $\mathrm{rk}(\mathcal{O}/F) = \mathrm{rk}(\mathcal{O}) - \mathrm{rk}(F)$.

In a rank-3 pseudosphere complex $\mathcal{S} = \mathcal{S}[\mathcal{O}]$, the pseudohyperspheres are simple closed curves on a 2-sphere S_0. Any two such curves "cross" at

a 0-pseudosphere in $\mathcal{S}$. The axioms ensure that S_2 contains an *equator*, a simple closed curve K in general position such that every 0-pseudosphere has one point on each side of K. By restricting the complex to one side of K, we obtain an affine representation of $\mathcal{S}$ called a *wiring diagram*. A wiring diagram is a set of smooth plane curves $\{C_e \mid e \in E\}$, each curve connecting a pair of opposite boundary points of a fixed disc D in the plane. Any two curves cross exactly once, and are otherwise disjoint. Each crossing point x is a *vertex* which corresponds to the cocircuit $\{e \in E \mid C_e \not\ni x\} \in \mathcal{B}(\mathcal{O})$. An oriented matroid in $[\mathcal{O}]$ is determined by designating, for each $e \in E$, one of the two components of $D - C_e$ as being the "positive side" of C_e. Each element e of a cocircuit B is signed according to whether the corresponding vertex lies on the positive or negative side of C_e. See Figure 1 for an example. If $\mathcal{O}$ is not simple, then all elements of one parallel class $[e]$ are associated to the same curve C_e, although each element in $[e]$ is independently oriented. We may record the cardinality of $[e]$ on the diagram, and use arrows to indicate the positive side of each element. Loops geometrically correspond to the full sphere S_0 or disc D. Loops are not contained in any cocircuit and do not matter in our context. Because of the choice in selecting the equator K, several wiring diagrams may correspond to the same oriented matroid. The curves C_e may be drawn as straight lines if and only if $\mathcal{O}$ is $\mathbb{R}$-representable.

2.4. Oriented Flow Number

We define the *imbalance* or *log-discrepancy* of a signed set $\vec{X} = \{X^+, X^-\}$, where $X = X^+ \cup X^- \subseteq E$, by

$$2 \leq \mathrm{imbal}(\vec{X}) = \frac{|X|}{\min\{|X^+|, |X^-|\}} \leq \infty.$$

Here, the value ∞ indicates that one of X^+, X^- is empty.

Minty [11] considered the graph invariant

$$\chi_\mathrm{o}(G) := \min_{\vec{G}} \max_{\vec{C}} \mathrm{imbal}(\vec{C}),$$

where $\vec{G}$ varies over the set of orientations of G, and $\vec{C}$ varies over the set of signed circuits in $\vec{G}$. He showed that the graph chromatic number is given by $\chi(G) = \lceil \chi_\mathrm{o}(G) \rceil$. The invariant $\chi_\mathrm{o}(G)$, now called the *circular chromatic number*, has several equivalent definitions and has seen a flurry of recent interest (see [17] for a survey). Within graph theory, this invariant is more usually denoted by χ_c or χ^*, but we use χ_o to emphasize the viewpoint of orientations.

The definition of χ_o is suitable for generalization to oriented matroids. In the matroid setting, we prefer to speak in terms of the dual parameter.

Thus we define the *oriented flow number* of an oriented matroid to be

$$\phi_{\mathrm{o}}(\mathcal{O}) := \min_{\mathcal{O}_I} \max_{\vec{B}} \operatorname{imbal}(\vec{B}),$$

where $\mathcal{O}_I$ varies over the set of reorientations of $\mathcal{O}$, and $\vec{B}$ varies over the signed cocircuits in $\mathcal{O}_I$. The oriented flow number of any reclass $[\mathcal{O}]$ is well defined by $\phi_{\mathrm{o}}([\mathcal{O}]) = \phi_{\mathrm{o}}(\mathcal{O})$. If a matroid M has a unique reclass $[\mathcal{O}]$ (e.g. if M is regular), then $\phi_{\mathrm{o}}(M)$ is well defined to equal $\phi_{\mathrm{o}}([\mathcal{O}])$. Thus $\chi_{\mathrm{o}}(G) = \phi_{\mathrm{o}}(M^*(G))$ where $M^*(G)$ denotes the (cographic) dual of the cycle matroid $M(G)$. In general, the oriented flow number of an orientable matroid M is *not* well defined.

EXAMPLE 2.1. The uniform matroid $U_{3,6}$ is orientable and has precisely four reclasses, $[\mathcal{O}_i]$, $1 \leq i \leq 4$ (see e.g. [5]), whose wiring diagrams are shown in Figure 2.
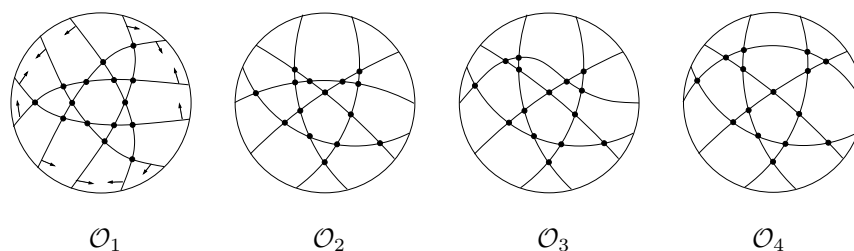

$$\mathcal{O}_1 \qquad\qquad \mathcal{O}_2 \qquad\qquad \mathcal{O}_3 \qquad\qquad \mathcal{O}_4$$

FIG. 2. The four reclasses of the uniform matroid $U_{3,6}$.

The first diagram shows an orientation with imbalance 2. We claim that the other three reclasses have no such orientation. Let P be an odd-sided polygonal cell in a diagram having imbalance 2. By considering adjacent vertices on P, one sees that P lies on the positive side of either all or none of its bounding curves. All three diagrams have two adjacent odd polygons, which leads to an easy contradiction. Since all cocircuits have size 4, the oriented flow number of $[\mathcal{O}_i]$ equals 2 or 4. Therefore $\phi_{\mathrm{o}}([\mathcal{O}_1]) = 2$ and $\phi_{\mathrm{o}}([\mathcal{O}_i]) = 4$ for $i = 2, 3, 4$.

In case M is graphic, $\phi_{\mathrm{o}}(M) = \phi_c(G)$ is the *circular flow number* of a graph G. This graph invariant, essentially introduced by Tutte [16], is also of contemporary interest [15, 18]. Seymour [14] showed that $\phi_c(G) \leq 6$ for any 2-edge connected graph G. More generally, there is an algebraic description of the oriented flow number $\phi_{\mathrm{o}}(M)$ of any regular matroid implicit in the work of Hoffman [9], and made explicit in [8]. In particular, if

A is a totally unimodular matrix representing the regular matroid M, then

$$\phi_{\mathrm{o}}(M) = \inf\{\alpha \in \mathbb{R} \mid \exists x \in \mathbb{R}^E,\ Ax = 0,\ 1 \le |x_e| \le \alpha - 1 \text{ for } e \in E \}.$$

If $\mathcal{O}$ has a coloop, then $\phi_{\mathrm{o}}(\mathcal{O}) = \infty$. The converse also holds, since every coloop-free oriented matroid has a *totally cyclic* reorientation, one in which every signed cocircuit $\vec{B}$ satisfies $B^+, B^- \ne \emptyset$. With this terminology, we may restate our main result.

THEOREM 2.1.   *For any coloop-free oriented matroid $\mathcal{O}$ of rank r,*

$$\phi_{\mathrm{o}}(\mathcal{O}) \le 14r^2 \ln r.$$

We may define the 'chromatic number' of $\mathcal{O}$ to be the invariant

$$\chi_{\mathrm{o}}(\mathcal{O}) = \phi_{\mathrm{o}}(\mathcal{O}^*).$$

Incidentally, $\chi_{\mathrm{o}}(\mathcal{O})$ is much easier than $\phi_{\mathrm{o}}(\mathcal{O})$ to bound by a function of the rank $r = \mathrm{rk}(\mathcal{O})$. Since no circuit of $\mathcal{O}$ has cardinality more than $r + 1$, no circuit of $\mathcal{O}$ has imbalance greater than $r + 1$ in a totally cyclic orientation of $\mathcal{O}^*$. It follows that $\chi_{\mathrm{o}}(\mathcal{O}) \le r + 1$ for any loopless oriented matroid $\mathcal{O}$ of rank r. This bound is best possible since it is achieved when $\mathcal{O}$ is (an orientation of) the complete graph $M(K_{r+1})$.

3. LOW RANK

The *odd cogirth* of an oriented matroid $\mathcal{O}$ is the least odd integer $2k + 1$ such that $\mathcal{O}$ has a cocircuit of cardinality $2k + 1$. We define the *odd cogirth bound number* of $\mathcal{O}$ to be the rational number

$$\mathrm{ocb}(\mathcal{O}) = 2 + 1/k,$$

where $2k + 1$ is the odd cogirth of $\mathcal{O}$. If $\mathcal{O}$ has no odd cocircuits then we define its odd cogirth to be ∞ and $\mathrm{ocb}(\mathcal{O}) = 2$. The imbalance of any signed cocircuit of size $2k + 1$ is at least $(2k + 1)/k$. Thus we have the following.

PROPOSITION 3.1.   *For any oriented matroid $\mathcal{O}$ we have $\phi_{\mathrm{o}}(\mathcal{O}) \ge \mathrm{ocb}(\mathcal{O})$.*

This bound is generally quite weak. It fails to be tight already for the orientations of the graphic matroid $M(K_4)$, and for the other orientations in Example 2.1. However, the bound is exact for oriented matroids of low rank.

LEMMA 3.2. *Every oriented matroid $\mathcal{O}$ of rank at most 2 satisfies $\phi_o(\mathcal{O}) = \text{ocb}(\mathcal{O})$. In particular, $\phi_o(\mathcal{O}) \leq 3$ if $\mathcal{O}$ is coloop-free.*

Proof.   If $\text{rk}(\mathcal{O}) = 1$, then $\mathcal{O}$ is an orientation of a single parallel class. Orienting this cocircuit as equitably as possible gives $\phi_o(\mathcal{O}) = \text{ocb}(\mathcal{O})$.

If the rank equals two, then we may affinely represent $\mathcal{O}$ as a list of points $P = (p_e \mid e \in E)$ on the real number line. Let us label the elements with $e_1, \ldots, e_n$ so that $p_{e_1} \leq \ldots \leq p_{e_m}$. For each $p \in P$ there is a corresponding cocircuit $B_p = \{e \in E \mid p_e \neq p\}$. We sign B_p according to

$$B_p^+ = \{e_i \mid p_{e_i} < p \text{ and } i \text{ is odd}\} \cup \{e_i \mid p_{e_i} > p \text{ and } i \text{ is even}\}.$$

It is easy to verify that this gives an orientation of $\mathcal{O}$, and that every cocircuit B_e satisfies $|B_e^+| - |B_e^-| \in \{0, \pm 1\}$. It follows that $\text{imbal}(B_e)$ equals 2 if $|B_e|$ is even, and equals $2 + 1/k$ if $|B_e| = 2k + 1$. Therefore $\phi_o(\mathcal{O}) = \text{ocb}(\mathcal{O})$. ∎

4. RANDOM RESIGNINGS AND RANK 3

We shall make use of the Chernoff bound from probability theory (see e.g. [1]).

LEMMA 4.1. *If $X_1, \ldots, X_m = \pm 1$ are i.i.d. random variables with probability $1/2$, then for $\lambda > 0$ we have*

$$\text{Prob}\left(\left|\sum_i X_i\right| > \lambda\sqrt{m}\right) < 2e^{-\lambda^2/2}.$$

∎

Here is some convenient terminology. Let A be a subset of a cocircuit B, and let $\vec{B} = \{B^+, B^-\}$ be a signing of B. For $s \geq 2$, we say that A is *s-unbalanced in $\vec{B}$* if either one of $A \cap B^+$ and $A \cap B^-$ is empty or

$$\frac{|A|}{\min\{|A \cap B^+|, |A \cap B^-|\}} > s.$$

If $\vec{B}$ is a signed cocircuit in an oriented matroid $\mathcal{O}$, and B is s-unbalanced in $\vec{B}$, then we say that B is *s-unbalanced in $\mathcal{O}$*. Therefore

$$\phi_o(\mathcal{O}) = \min_{I \subseteq E} \inf\{s \in \mathbb{R} : \text{no cocircuit is } s\text{-unbalanced in } \mathcal{O}_I\}. \qquad (3)$$

A *random resigning* of a subset $R \subseteq E(\mathcal{O})$ is a reorientation $\mathcal{O}_I$ where I is uniformly selected among the subsets of R.

LEMMA 4.2.   *Let A be a nonempty subset of a cocircuit B in $\mathcal{O}$, and let R satisfy $A \subseteq R \subseteq E$. Let $\vec{B}$ be the signing of B in a random resigning of R. Then for $s \geq 2$ the probability that A is s-unbalanced in $\vec{B}$ is less than*

$$2 \exp\left(-\left(1 - \frac{2}{s}\right)^2 \frac{|A|}{2} \right).$$

Proof.   We define random variables $\{X_e \mid e \in A\}$ by

$$X_e = \begin{cases} 1 & \text{if } e \in B^+ \\ -1 & \text{if } e \in B^-. \end{cases}$$

Then A is s-unbalanced in $\vec{B}$ if and only if $|\sum_{e \in A} X_e| > (1 - \frac{2}{s})|A|$. Since $A \subseteq R$, the random variables are i.i.d. among the resignings of R. The result follows by applying the Chernoff bound with $\lambda = \left(1 - \frac{2}{s}\right)\sqrt{|A|}$. ∎

We shall now prove a bound on $\phi_o(\mathcal{O})$ in case $\mathcal{O}$ has rank 3.

Figure 3 shows three oriented wiring diagrams which are all simple, except for one element of multiplicity two or three. More precisely, each example is the case $n = 4$ of a family of wiring diagrams on $n + 3$ elements. Orientations are given by the arrows shown. We say that these orientations are *alternating* with respect to the equators shown. The reader will notice that the orientation described in the proof of Lemma 3.2 is alternating in a similar sense. Alternating orientations of matroids of rank ≤ 3 tend to result in good upper bounds on $\phi_o(\mathcal{O})$.
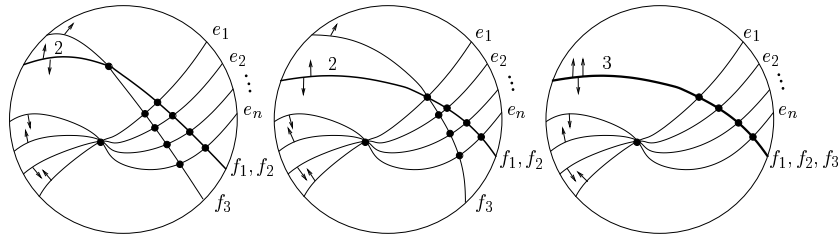

FIG. 3.   Three wiring diagrams with alternating orientations.

LEMMA 4.3.   *Let $\mathcal{O}$ be a coloop-free oriented matroid of rank* 3. *Then* $\phi_o(\mathcal{O}) \leq 17$. *Moreover, if $\mathcal{O}$ is simple and $n = |E(\mathcal{O})|$ is large, then*

$$\phi_o(\mathcal{O}) \leq \max\left\{\operatorname{ocb}(\mathcal{O}),\, 2 + O\left((n/\ln n)^{-1/3}\right)\right\}.$$

In particular, if $n \geq 159$ ($n \geq 427$), then $\phi_o(\mathcal{O}) \leq 4$ (resp. $\phi_o(\mathcal{O}) \leq 3$).

Proof.   We may assume $\mathcal{O}$ has no loops. Suppose that $S = \{f_1, f_2\}$ are parallel elements in $\mathcal{O}$ such that $\mathcal{O} \backslash S$ is coloop-free. By orienting f_1 and f_2 oppositely and using induction, one easily sees $\phi_o(\mathcal{O}) \leq \phi_o(\mathcal{O} \backslash S) \leq 17$. Thus, we may assume that $\mathcal{O} \backslash S$ has a coloop for all such pairs and, hence, any two parallel elements of $\mathcal{O}$ are contained in a cocircuit of size 3. If $\mathcal{O}$ is not simple, then $\mathcal{O}$ has $n + 3$ elements for some $n \geq 0$, and $\mathcal{O}$ corresponds to one of the three wiring diagrams as drawn in Figure 3 (the cocircuit is $\{f_1, f_2, f_3\}$). The alternating reorientations shown there, imply that $\phi_o(\mathcal{O}) \leq 3$ if $\mathcal{O}$ has no coloops. Thus we may assume $\mathcal{O}$ is simple.

Let $k = \min\{|B| \mid B \in \mathcal{B}(\mathcal{O})\}$ be the *cogirth* of $\mathcal{O}$. Let B_0 be a cocircuit of size k in $\mathcal{O}$. We may choose the equator of a wiring diagram to be near the vertex corresponding to B_0, so that the left half of the disk is as shown in Figure 4 (the diagram is undetermined under the white box). We consider an alternating reorientation $\mathcal{O}'$ of $\mathcal{O}$ as shown in the diagram.
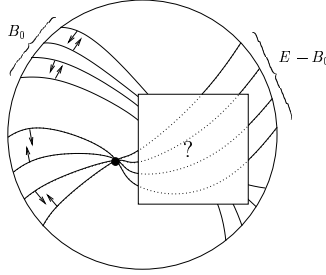

FIG. 4.   An alternating reorientation $\mathcal{O}'$ defined by a smallest cocircuit B_0.

By design, we have $\operatorname{imbal}_{\mathcal{O}'}(B_0) \leq \operatorname{ocb}(\mathcal{O}) \leq 3$. Every other cocircuit $B \in \mathcal{B}(\mathcal{O}') - \{B_0\}$ contains all but possibly one element of $E - B_0$. Since $\mathcal{O}'$ is alternating, we therefore have $|B^+|, |B^-| \geq \lfloor (n - k - 1)/2 \rfloor$. Since $|B| \leq n - 2$, this implies $\operatorname{imbal}_{\mathcal{O}'}(B) \leq g(n, k)$ where

$$g(n, k) = \frac{n - 2}{(n - k)/2 - 1} = \frac{2n - 4}{n - k - 2}.$$

Thus the reorientation $\mathcal{O}'$ shows that if $2 \leq k \leq n - 3$, then

$$\phi_{\mathrm{o}}(\mathcal{O}) \leq \max \{ \operatorname{ocb}(\mathcal{O}),\, g(n, k) \}. \tag{4}$$

We note that $\operatorname{ocb}(\mathcal{O}) > g(n, k)$ only if $k \leq \sqrt{n - 2}$.

Let $\mathcal{O}_I$ be a random resigning of $E(\mathcal{O})$. Clearly, $\mathcal{O}$ has at most $\binom{n}{2}$ cocircuits, all of which have size at least k. By Lemma 4.2, the probability that some cocircuit of $\mathcal{O}_I$ is s-unbalanced is less than

$$2\binom{n}{2} \exp\left(-\left(1 - \frac{2}{s}\right)^2 \frac{k}{2} \right). \tag{5}$$

Let $f(n, k)$ denote the least real number $s > 2$ such that this expression is less than or equal to 1. One can verify that $f(n, k)$ is well defined for $k > 2 \ln \left(2\binom{n}{2}\right)$. When $s \geq f(n, k)$, at least one of the random reorientations $\mathcal{O}_I$ has no s-unbalanced cocircuits. Therefore by (3), if $4 \ln n \leq k \leq n - 2$, then

$$\phi_{\mathrm{o}}(\mathcal{O}) \leq f(n, k). \tag{6}$$

For fixed $n \geq 5$, we find that $g(n, k)$ is increasing as k increases from 2 to $n - 3$, whereas $f(n, k)$ decreases with k, for $4 \ln n \leq k \leq n - 2$. The bounds in (4) and (6) are illustrated in Figure 5 for $n = 30$. In the plot of $\operatorname{ocb}(\mathcal{O})$ versus k in this figure reflects the fact $\operatorname{ocb}(\mathcal{O}) = 2k/(k - 1)$ if the cogirth k is odd, but that one can only deduce $\operatorname{ocb}(\mathcal{O}) \in \{2\} \cup \{2 + 1/i \mid i \geq k/2\}$ if k is even.
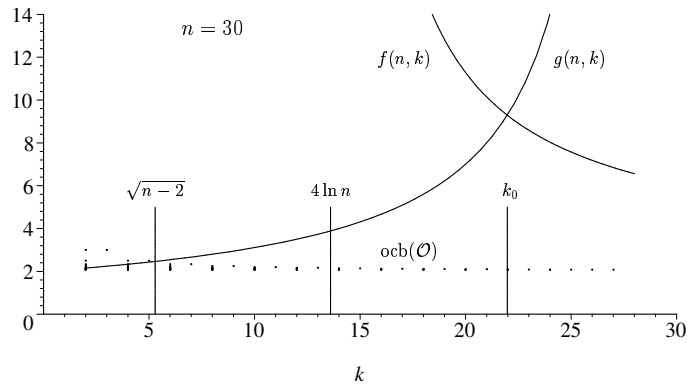

FIG. 5. $f(n, k)$, $g(n, k)$, and $\operatorname{ocb}(\mathcal{O})$ versus k, when $n = 30$.

Using a computer algebra system, we find that $g(n,k) = f(n,k)$ when

$$k = k_0 = \sqrt[3]{2(n-2)^2 \ln(n(n-1))}.$$

We have shown $\phi_o(\mathcal{O}) \le \max\{\mathrm{ocb}(\mathcal{O}), h(n)\}$ where

$$h(n) = g(n, k_0) = \frac{2n-4}{n-2-\sqrt[3]{2(n-2)^2 \ln(n(n-1))}} = 2 + O\left(\left(\frac{n}{\ln n}\right)^{-1/3}\right).$$

Using the trivial bound $\phi_o(\mathcal{O}) \le \max\{|B| \mid B \in \mathcal{B}(\mathcal{O})\} \le n-2$ and the easily verified fact that $f(n, k_0) \le 17$ when $n \ge 19$, we have proven $\phi_o(\mathcal{O}) \le 17$.

We further find that $\phi_o(\mathcal{O}) \le 4$ (resp. 3) provided that $n \ge 166$ (resp. 712). However, we can improve the above argument in case we are trying to prove $\phi_o(\mathcal{O}) \le s$ for some $s \le 4$. When $n \ge 162$ one finds that $k_0 < (n-1)/2$, which roughly corresponds, by (5), to $s \le 4$. Since $\mathcal{O}$ is simple, it contains at most one cocircuit of size less than $(n-1)/2$, and at most

$$\binom{n}{2} - \binom{n-k}{2} \le \frac{3n(n-2)}{8}$$

cocircuits of size at least $(n-1)/2$. For $s \ge \mathrm{ocb}(\mathcal{O})$, we may assume $k \ge (n-2)(s-2)/s$ since otherwise $\phi_o(\mathcal{O}) \le s$ by (4). Therefore, in case $k \le n/2 - 1$, we may replace (5) with the smaller quantity

$$2\exp\left(-\left(1-\frac{2}{s}\right)^2 \frac{(n-2)(s-2)}{2s}\right) + 2\left(\frac{3n(n-2)}{8}\right)\exp\left(-\left(1-\frac{2}{s}\right)^2 \frac{n-1}{4}\right).$$

It is straight forward to verify that this expression is at most 1 when $n \ge 427$ (for $s = 3$), and when $n \ge 159$ (for $s = 4$). ∎

With extra work, the minor restriction that $\mathcal{O}$ is simple can be omitted from the statement of Lemma 4.3. The bound $\phi_o(\mathcal{O}) \le 17$ is probably far from optimal.

CONJECTURE 4.4. *Every coloop-free oriented matroid $\mathcal{O}$ of rank 3 satisfies $\phi_o(\mathcal{O}) \le 4$.*

5. DENSE FLATS

There are additional challenges when considering oriented matroids of rank greater than three. It is not clear how to define an "alternating

orientation", so we shall resort to using random orientations. However a matroid of rank 4 may have many small cocircuits, so we can not argue as in the proof of Lemma 4.3 to bound the probability of having an unbalanced cocircuit. We give here a way to address this problem. The following example helps to illustrate the forthcoming strategy.

Let G be graph obtained from the complete graph K_{10} by replacing each vertex $v_i \in V(K_{10})$ with a copy, $K(i)$, of K_{1000}. For each $v_i v_j \in E(K_{10})$ there is an edge in G joining an arbitrary vertex of $K(i)$ to an arbitrary vertex of $K(j)$. Each of the subgraphs $K(i)$ contains $\binom{1000}{2}$ edges which form a flat, say F_i, in the cycle matroid $M(G)$. Because the graph G has cogirth only nine and a large number of cocircuits (about 2^{10000}), a naïve application of Lemma 4.2 would result in a very poor bound on $\phi_o(M(G_{20}))$. In general, a random orientation can not be shown to balance each of a large number of small cocircuits.

The solution is to select one of the flats, say $F_1 = E(K(1))$, and to consider separately the cocircuits of $M(G)$ which intersect with F_1, and those that are disjoint from F_1. Any cocircuit having an edge from F_1 has large cardinality, at least 999, so there is good probability that they are all fairly well balanced in a random orientation of G. The cocircuits of $M(G)$ which are disjoint from F_1 are precisely the cocircuits of the contracted graph G/F_1. We select another flat, say $F_2 = E(K(v_2)) \subseteq E(G/F_1)$, and partition the cocircuits of G/F_1 into those which contain an edge of F_2, and those which are disjoint from F_2. Again, cocircuits of the first type are large and easy to balance in a random orientation. After 10 steps we are left with the graph $K_{10} = G/(F_1 \cup \ldots \cup F_{10})$. Although K_{10} has small cocircuits, there are relatively few of them, so a probabilistic argument will again be successful. To apply this type of argument to an arbitrary oriented matroid $\mathcal{O}$, we must first define a suitable set of elements in $\mathcal{O}$ which can play the role of F_i in this example.

Let $\mathcal{O}$ be an oriented matroid with $E = E(\mathcal{O})$. A flat F is *dense in $\mathcal{O}$* if

$$\frac{|F|}{|E|} \geq \frac{\mathrm{rk}(F) + 1}{\mathrm{rk}(E) + 1}.$$

A dense flat F is *minimal* if no proper subflat of F is dense in $\mathcal{O}$. Since E is dense and $\emptyset$ is not dense, $\mathcal{O}$ has a minimal dense flat and all minimal dense flats are nonempty. A cocircuit B is *F-intersecting* if $B \cap F \neq \emptyset$. Let $\mathcal{B}_F$ denote the set of F-intersecting cocircuits in $\mathcal{O}$. We first show that a substantial portion of any F-intersecting cocircuit lies within F.

LEMMA 5.1. *Let F be a minimal dense flat in $\mathcal{O}$, and let $B \in \mathcal{B}_F$. Then*

$$|B \cap F| > \frac{|E|}{\mathrm{rk}(E) + 1}.$$

Proof. Suppose not. Since the complement $(E - B)$ of the cocircuit B is a matroid hyperplane and the intersection of two flats is a flat again, $B \cap F \neq \emptyset$ implies that $F' := F \cap (E - B)$ is a proper subflat of F in $\mathcal{O}$ satisfying

$$\frac{|F'|}{|E|} = \frac{|F|}{|E|} - \frac{|B \cap F|}{|E|} \geq \frac{\mathrm{rk}(F) + 1}{\mathrm{rk}(E) + 1} - \frac{1}{\mathrm{rk}(E) + 1} \geq \frac{\mathrm{rk}(F') + 1}{\mathrm{rk}(E) + 1}.$$

This contradicts that F is minimally dense in $\mathcal{O}$. ∎

LEMMA 5.2. *Let $\mathcal{O}$ be an oriented matroid of rank $r \geq 3$ and size n. Let $R \subseteq E(\mathcal{O})$ and $F \subseteq R$ be a minimal dense flat in $\mathcal{O}$. Suppose that $t \in \mathbb{R}$ satisfies $19r^2 \ln r \leq t \leq n$. Then the probability that some F-intersecting cocircuit is t-unbalanced in a random resigning of R is less than n^{-2}. If "3" is replaced by 4 or 5, then "19" may be replaced by 14 or 12 respectively.*

Proof. Let $B \in \mathcal{B}_F$. In any reorientation of $\mathcal{O}$, if $B^+ \neq \emptyset$, we have by Lemma 5.1,

$$\frac{|B|}{|B^+|} \leq \frac{n}{|B^+|} \leq \varrho \frac{|B \cap F|}{|B^+ \cap F|}$$

where $\varrho = r + 1$. The same inequality holds if we replace B^+ by B^-. Thus, in a random resigning of R, the probability that B is t-unbalanced is at most the probability that $B \cap F$ is $\frac{t}{\varrho}$-unbalanced in B. By Lemmas 4.2 and 5.1 this probability is at most

$$2 \exp\left(-\left(1 - \frac{2\varrho}{t}\right)^2 \frac{|B \cap F|}{2}\right) \leq 2 \exp\left(-\left(1 - \frac{2\varrho}{t}\right)^2 \frac{n}{2\varrho}\right) =: P.$$

We aim to show

$$n^2 |\mathcal{B}_F| P < 1. \tag{7}$$

We estimate $|\mathcal{B}_F| \leq \binom{n}{r-1} \leq \frac{n^{\varrho - 2}}{2}$ (for $r \geq 3$). Thus (7) holds provided that $n^\varrho \cdot \frac{P}{2} < 1$. Equivalently, we aim to show

$$\varrho \ln n - \left(1 - \frac{2\varrho}{t}\right)^2 \frac{n}{2\varrho} < 0,$$

$$2\frac{\ln n}{n} < \left(\frac{1}{\varrho} - \frac{2}{t}\right)^2. \tag{8}$$

For any integer $r_0 \geq 3$, let $f(r_0)$ be the smallest positive number such that every integer $r \geq r_0$ satisfies

$$f(r_0)\, r^2 \ln r \leq e^{-1/2}\, r^{f(r_0)\, r_0^2/(2(r_0+1)^2)}.$$

It is straight forward to verify that $f(3) \leq 19$, $f(4) \leq 14$, $f(5) \leq 12$, and that $f(r) \to 4$ slowly. Writing $t_0 = f(r_0)\, r^2 \ln r$, and $\varrho = r + 1$, we have for $r \geq r_0 \geq 3$,

$$\ln t_0 \leq \frac{f(r_0)\, r_0^2}{2(r_0+1)^2} \ln r - \frac{1}{2} \leq \frac{f(r_0)\, r^2}{2(r+1)^2} \ln r - \frac{2}{\varrho} = \frac{t_0}{2\varrho^2} - \frac{2}{\varrho}.$$

Together with $t_0 \leq t \leq n$, this completes the proof since the left hand side of (8) is at most $2\frac{\ln t_0}{t_0}$, whereas the right hand side is at least $\frac{1}{\varrho^2} - \frac{4}{\varrho\, t_0}$. ∎

6. PROOF OF MAIN THEOREM

Let $\mathcal{O}$ be an oriented matroid. To prove the bound on $\phi_o(\mathcal{O})$, we shall construct an ordered partition $(F_0, F_1, \ldots, F_p)$ of $E(\mathcal{O})$ as follows. Let F_0 be a minimal dense flat in $\mathcal{O}_0 = \mathcal{O}$ and suppose we have constructed $F_0, F_1, \ldots, F_k$, for some $k \geq 0$. If $\bigcup_{i=0}^{k} F_i = E(\mathcal{O})$, then we set $p = k$ and output the list. Otherwise let F_{k+1} be a minimal dense flat in the contracted oriented matroid (minor)

$$\mathcal{O}_{k+1} := \mathcal{O} \Big/ \left(\bigcup_{i=0}^{k} F_i \right) = \mathcal{O}_k / F_k \ .$$

(See Section 2.3 for the geometric interpretation of contracting a flat.)

Since each F_k is nonempty and $\mathcal{O}$ is finite, this procedure must terminate. Any sequence $(F_0, F_1, \ldots, F_p)$ constructed in this way is called a *dense flat sequence* of $\mathcal{O}$.

Let $(F_0, F_1, \ldots, F_p)$ be a dense flat sequence for $\mathcal{O}$. A cocircuit B of $\mathcal{O}$ is *of type k* if k is the least index for which $B \cap F_k \neq \emptyset$. Thus the cocircuits in $\mathcal{O}_k$ are precisely the cocircuits of type $\geq k$ in $\mathcal{O}$. Let $\mathcal{B}_k$ be the set of cocircuits of type k in $\mathcal{O}$, let $n_k = |\mathcal{O}_k|$ and let $r_k = r(\mathcal{O}_k) = r(\mathcal{O}) - \sum_{i=0}^{k-1} r(F_i)$. Figure 6 may help the reader with these definitions and the proof that follows.

THEOREM 6.1. *For any oriented matroid $\mathcal{O}$ of rank r and without coloops,*
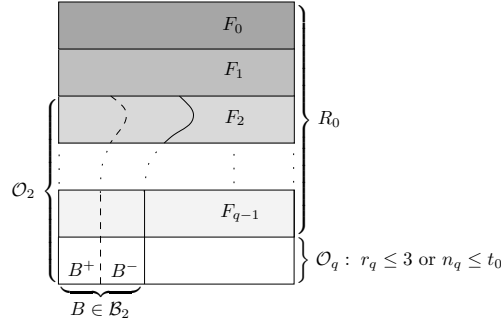
$$\phi_o(\mathcal{O}) < 14r^2 \ln r.$$

FIG. 6. A dense flat sequence and a cocircuit B of type 2

Proof.   We show that $\mathcal{O}$ has a reorientation in which no cocircuit is t_0-unbalanced, where $t_0 = 14r^2 \ln r$. Let $(F_0, F_1, \ldots, F_p)$ be a dense flat sequence for $\mathcal{O}$, with $\mathcal{O}_k$, $\mathcal{B}_k$, n_k, r_k defined as above. Let q be the least integer for which either $r_q \le 3$ or $n_q \le t_0$. If $r_q \le 3$, then by Lemma 4.3, $\mathcal{O}_q$ may be reoriented so that every cocircuit in $\mathcal{O}_q$ has imbalance at most 17. If $n_q \le t_0$, then in any totally cyclic orientation of $\mathcal{O}_q$, we have that every cocircuit has imbalance at most $t_0 - 2$. In any case we find a reorientation $\mathcal{O}'$ of $\mathcal{O}$ in which no cocircuit of type at least q is $\max\{17, t_0 - 2\}$-unbalanced.

We now randomly resign the all elements "outside of $\mathcal{O}_q$", namely $R_0 := \cup_{i=0}^{q-1} F_i$, in $\mathcal{O}'$ to obtain $\mathcal{O}''$. We aim to show that with probability greater than zero, no cocircuit is t_0-unbalanced in $\mathcal{O}''$. By choice of q we have $r_k \ge 4$ and $n_k > t_0$, for $0 \le k \le q-1$. So applying Lemma 5.2 to $\mathcal{O}_k$ and F_k with $R = R_k := \cup_{i=k}^{q-1} F_i$, we find that the probability that a cocircuit in $\mathcal{B}_k$ is t_0-unbalanced in $\mathcal{O}_k$ is at most n_k^{-2}. Note, that the random resigning of R_0 induces a random resigning of R_k. By definition each cocircuit from $\mathcal{B}_k$ is contained in $E(\mathcal{O}_k)$. Since $\big(\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_{q-1}, \mathcal{B}(\mathcal{O}_q)\big)$ is a partition of the cocircuits of $\mathcal{O}''$, the probability that some cocircuit in $\mathcal{O}''$ is $\max\{t_0, 17\}$-unbalanced is at most

$$\sum_{k=0}^{q-1} n_k^{-2} \le \sum_{i=t_0+1}^{\infty} i^{-2} < 1.$$

Thus $\phi_\mathrm{o}(\mathcal{O}) \le \max\{t_0, 17\} = t_0$. ▮

7. SMALL EXAMPLES

TABLE 1.

Number of cosimple reclasses with rank 3, size n and oriented flow number s.
In parentheses are listed the number of these which are uniform.

$s\backslash n$	5	6	7	8
2	0 (0)	1 (1)	1 (0)	18 (3)
5/2	0 (0)	0 (0)	137 (11)	631 (0)
3	1 (1)	7 (0)	57 (0)	5369 (132)
4	0 (0)	9 (3)	11 (0)	11 (0)
Total	1 (1)	17 (4)	206 (11)	6029 (135)

Using an implementation by Lukas Finschi [5] of a simple reclass gener-
ator described in [6], together with a program of Timothy Mott [13], for
calculating $\chi_o(\mathcal{O})$, we have found the following results.

For various values of n and s we list in Table 1 the number of coloop-
free cosimple reclasses $[\mathcal{O}]$ of rank 3 and size n for which $\phi_o([\mathcal{O}]) = s$.
In parentheses, we record the number of these which are reclasses of the
uniform matroid $U_{n,3}$. If $\mathcal{O}$ is not cosimple, we may contract any coparallel
element to obtain a rank 2 oriented matroid with the same oriented flow
number, which equals $\mathrm{ocb}(\mathcal{O})$ by Lemma 3.2.

For rank 4, we find that, there are 143 reclasses $[\mathcal{O}]$ of size 7. The number
of these which have $\phi_o([\mathcal{O}]) = s$ is tabulated as follows, with parentheses
indicating the number of which are uniform.

$$s = 2 : 1(1) \qquad s = 3 : 12(0) \qquad s = 4 : 130(10).$$

A natural question is whether the lower bound given by the Betti number
of complete graphs is the worst case or not. We consider this question rather
hard, and so we make no conjectures here. However, we do not know of
any construction of oriented matroids which would show that $\phi_o(\mathcal{O}) >
\Theta(\sqrt{\mathrm{rk}(\mathcal{O})})$.

Acknowledgment

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