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Tree Decompositions – Why Matroids are Useful

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Abstract

Tree-like decompositions, and the notions of tree-width and branch-width were successfully formalized in the deep Graph Minor project by Robertson and Seymour. Since then they have found many interesting applications in combinatorics, logic, and computer science. Following this success, researchers have thoroughly consider branch-width of matroids, and extended many of the interesting graph results to matroid theory.

The recent approximation algorithm for clique-width of a graph by Oum and Seymour, and subsequent (logic-oriented) results by Courcelle and Oum clearly show how matroidal extensions of graphic results can be useful back in graph theory and logic. The main goal of our presentation is to introduce matroids, and to show the traditional tree-width of graphs can be defined in a "vertex-free" fashion, and hence extended to matroids.

Part of the results in this talk are based on joint works with Detlef Seese, AIFB, University Karlsruhe, and with Geoff Whittle, Victoria University of Wellington.

1 Motivation

The Graph Minor Project
[Robertson and Seymour, 80's – 90's], [others later...]

- Formalized the notions of *tree-width* and *branch-width* (similar notions).
- Proved *Wagner's conjecture* – WQO property of graph minors.
  (Among the partial steps: WQO of graphs of bounded tree-width, excluded grid theorem, description of graphs excluding a complete minor.)
- Testing for an arbitrary fixed graph minor in cubic time.

Tree-like Graphs and Logic

- [Seese, 1975] Undecidability of *MSO theories of large grids*.
- [Courcelle, 1988] Decidability of an *MSO theory* of graphs: The class of all (finite) graphs of bounded tree-width has a decidable MS_2 theory.
  (Also by [Arnborg, Lagergren, and Seese, 1991] via interpretation.)
- [Seese, 1991] Decidability of the MS_2 theory implies bounded tree-width.

Seese's conjecture: Any class of countable structures with a decidable MSO theory has an *MSO interpretation in a class of trees*. (C.f. Rabin's S2S.)

Matroidal Extensions

- [Geelen, Gerards, Robertson, Whittle, and ..., late 90's – future] Extending the *ideas of graph minors* to matroids (over finite fields). (For example: WQO, excluded grid theorem, other technical results...)
- [PH, 2002] *Decidability for matroids*: The class of all $GF(q)$-representable matroids of bounded branch-width has a decidable MSO theory.
- [Whittle and PH, 2003, (Geelen)] Matroidal (vertex-free, i.e. mapping only edges) definition of *tree-width*.
- [Seese and PH, 2004] Decidability of the MSO theory implies a bounded branch-width. (Hence an interpretation in trees.)

Clique-Width and Rank-Width

- [Courcelle et al, 1993] The definition of a *clique-width* – constructing a graph using a bounded number of labels.
- [C., Makowsky, Rotics, 2000] Decidability of the MS_1 theory for gr. cl.-w.
- [Oum and Seymour, 2003] An approximation of a graph clique-width via *rank-width*, which actually comes from *matroidal branch-width*.
- [Courcelle and Oum, 2004] Decidability of the C_2MS_1 theory implies a bounded clique-width. (Hence an interpretation in trees.)

We can see above an interesting blend of (deep) ideas from

– structural graph and matroid theories: tree-like decompositions,
– logic: models and decidability, interpretability, and
– computational (parametrized) complexity,

proving often very deep and nice results...

Algorithmic Consequences

- Graph Minor project
  $\Rightarrow$ every minor-closed graph property can be tested in cubic time!

  (For example, also the problem of a "knotless embedding".)

- Decidability of MS_2 + Bodlaender's algorithm
  $\Rightarrow$ every MS_2-definable property can be computed in linear time on graphs of bounded tree-width.

  (For example, 3-colourability, hamiltonicity, clique, etc.)

- Similar results for MS_1 properties on graphs of bounded clique-width,

- and for MSO properties on $GF(q)$-r. matroids of bounded branch-width.

2 Basics of Matroids

A matroid is a pair $M = (E, \mathcal{B})$ where

- $E = E(M)$ is the *ground set* of M (elements of M),
- $\mathcal{B} \subseteq 2^E$ is a collection of *bases* of M,
- the bases satisfy the "exchange axiom"
  $\forall B_1, B_2 \in \mathcal{B}$ and $\forall x \in B_1 - B_2$,
  $\exists y \in B_2 - B_1$ s.t. $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$.

Otherwise, a *matroid* is a pair $M = (E, \mathcal{I})$ where

- $\mathcal{I} \subseteq 2^E$ is the collection of *independent sets* (subsets of bases) of M.

The definition was inspired by an abstract view of *independence* in linear algebra and in combinatorics [Whitney, Birkhoff, Tutte,. . .].

Notice exponential amount of information carried by a matroid.

Literature: J. Oxley, Matroid Theory, Oxford University Press 1992,1997.

Some **elementary matroid terms** are

- *independent set* $\approx$ a subset of some basis,
  dependent set $\approx$ not independent,
- *circuit* $\approx$ a minimal dependent set of elements,
  triangle $\approx$ a circuit on 3 elements,
- *hyperplane* $\approx$ a maximal set containing no basis,
  cocircuit $\approx$ the complement of a hyperplane,
- *rank function* $\approx$ "*dimension*" of X,
  $\mathrm{r}_M(X)$ = maximal size of an M-independent subset $I_X \subseteq X$.

(Notation is taken from linear algebra and from graph theory...)

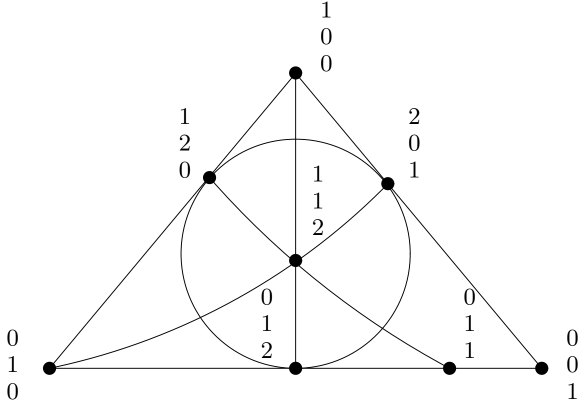
Axiomatic descriptions of matroids via independent sets, circuits, hyperplanes, or rank function are possible, and often used.

Vector matroid — a straightforward motivation:

- Elements are vectors over $\mathbb{F}$,
- independence is usual linear independence,
- the vectors are considered as columns of a matrix $\boldsymbol{A} \in \mathbb{F}^{r\times n}$.
  ($\boldsymbol{A}$ is called a *representation* of the matroid $M(\boldsymbol{A})$ over $\mathbb{F}$.)

Not all matroids are vector matroids.

An example of a rank-3 vector matroid with 8 elements over $GF(3)$:

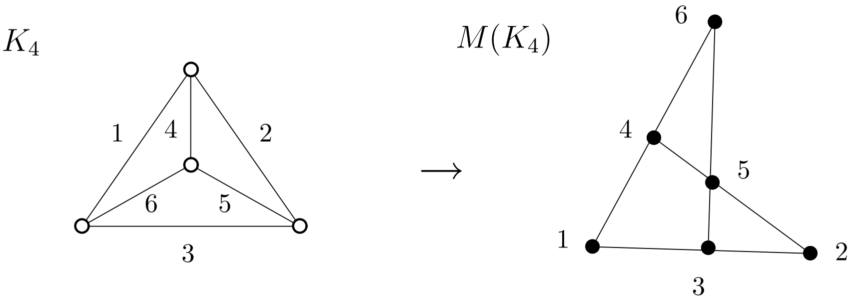


Graphic matroid $M(G)$ — the combinatorial link:

- Elements are the edges of a graph,
- independence $\sim$ acyclic edge subsets,
- bases $\sim$ spanning (maximal) forests,
- circuits $\sim$ graph cycles,
- the rank function $\mathrm{r}_M(X)$ = the number of vertices minus the number of components induced by X.

Only few matroids are graphic, but all *graphic ones are vector matroids* over any field.

Example:



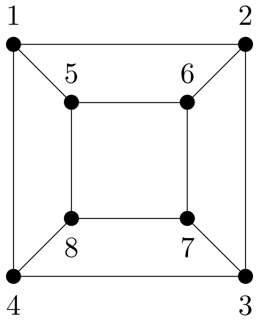
3 Tree-Width and Branch-Width

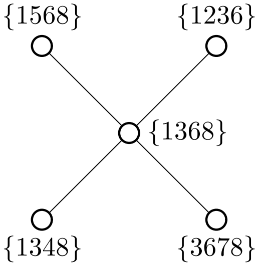
[Robertson and Seymour]

A **tree decomposition** of a graph G is a pair (T, β), where T is a tree and $\beta : V(T) \to 2^{V(G)}$ is a mapping (to "bags") satisfying the following:

- For each edge $e = uv \in E(G)$ exists $x \in V(T)$ such that $\{u, v\} \subseteq \beta(x)$.
- (IP) If $x \in V(T)$, and if $y, z \in V(T)$ are two vertices in distinct components of $T - x$, then $\beta(y) \cap \beta(z) \subseteq \beta(x)$.

Tree-width = min. of max. "bag" sizes $|\beta(x)| - 1$ over all decompositions.

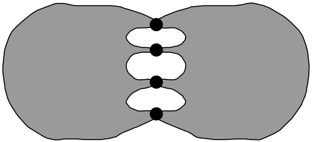


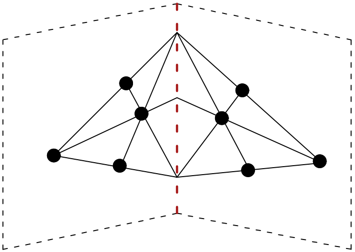


Matroid Connectivity – an alternative view of graph connectivity

Connectivity function $\lambda_G(X)$ = number of vertices in G
incident both with edges of X and of $E(G) - X$.

A 4-separation in a graph:





A 3-separation in a matroid:

Matroid connectivity $\underline{\lambda_M(X) = \mathrm{r}_M(X) + \mathrm{r}_M(E - X) - \mathrm{r}(M) + 1}$

(geometrically the "rank of spans intersection" $\langle X \rangle \cap \langle E - X \rangle$ plus 1).

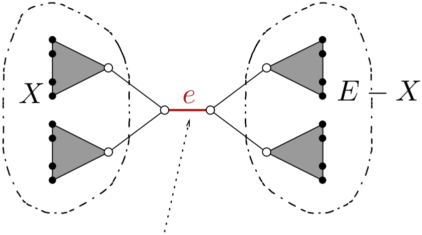
A *k-separation* $(X, E - X)$: $\lambda(X) \le k$ and $|X|, |E - X| \ge k$.

Then, **high connectivity $\approx$ no small separations**.

The (often better) alternative to a tree decomposition: [Robertson and Seymour again]

Graphs or matroids (or arb. sym. submod. λ) $\longrightarrow$ a **branch decomposition.**

- Decomposed to a *cubic tree* (degrees ≤ 3), and
- edges / elements mapped one-to-one to the tree leaves
  (with no reference to graph vertices).
- Tree edges have the *width* as follows:



width(e) $=$ $\lambda(X)$ where X is *"displayed" by* e in the tree.

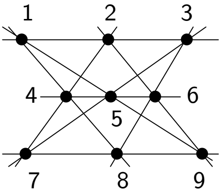
(Using graph connectivity $\lambda_G()$, or matroid connectivity $\lambda_M()$, resp.)

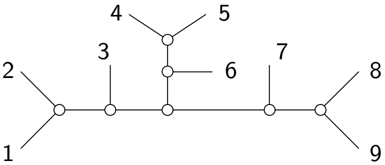
Branch-width $=$ min. of max. edge widths over all decompositions.

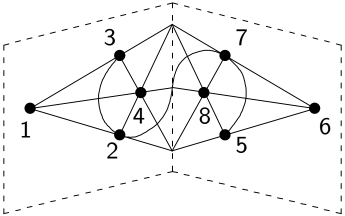
(Branch-width is within a constant factor of tree-width.)

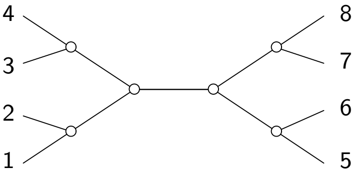
Branch decompositions of matroids

both of width 3:









4 Logic of Matroids to Clique-Width

MSOL – monadic second-order logic:
propositional $+$ quantification over elements and sets.

MSOL $+$ class of structures $\implies$ **MSO theory** (of the structures).

Matroidal MSO Theory

A *matroid in logic* – the ground set $E = E(M)$ with all subsets 2^E,
– and a predicate *indep* on 2^E, s.t. *indep*(F) iff $F \subseteq E$ is independent.

The *MSO theory of matroids* – language of MSOL applied to such matroids.

Some basic expressions:

- *basis*$(B) \equiv$ *indep*$(B) \wedge \forall D(B \not\subseteq D \vee B = D \vee \neg$ *indep*$(D))$
  A basis is a maximal independent set.
- *circuit*$(C) \equiv \neg$ *indep*$(C) \wedge \forall D(D \not\subseteq C \vee D = C \vee$ *indep*$(D))$
  A circuit C is dependent, but all proper subsets of C are independent.

Expressive Power of Matroidal MSO

– [PH] equivalent to graph MS$_2$, quantification over vertices and edges (assuming 3-connectivity for technical reasons).

Clique-Width of a graph G

$=$ the minimal k s.t. there is a k-expression constructing G.

A *k-expression* over vertex-labeled graphs with k labels:

- Creation of a new vertex of label i,
- disjoint union,
- addition of all edges between the vertices of label i and j, and
- relabeling all vertices of label i to j, where $1 \leq i, j \leq k$.

(The underlying tree of this definition is the parse tree of the expression.)

Some interesting **facts**:

- The clique-width of a clique is 2, the clique-width of a large grid is large.
- [Courcelle, Makowsky, Rotics, 2000] The MS_1 theory (quantification over vertices only) of all graphs of bounded clique-width is decidable.
- [Oum, Seymour, 2003] A polynomial approximation algorithm for clique-width of a graph (between k and 8^k) – so no need for an expression on the input now.
  Algorithm uses a factor-3 approximation of *rank-width* – a symmetric submodular function, equal to matroid branch-width for bipartite graphs.

Decidability of Theories

Decidability of a theory $\mathcal{T} \approx$
for every formula ψ one can (algorithmically) decide whether $\mathcal{T} \models \psi$.

For matroids (analogously to an MS_2 theory of graphs [Seese, 1991])...

Theorem 4.1. [Seese and PH, 2004] *Let $\mathcal{N}$ be a class of matroids that are representable by matrices over $\mathbb{F}$. If the MSO theory of $\mathcal{N}$ is decidable, then the class $\mathcal{N}$ has bounded branch-width.*

For a (weaker) MS_1 theory of graphs...

Theorem 4.2. [Courcelle and Oum, 2004] *Assume that a class $\mathcal{K}$ of adjacency graphs has a decidable C_2MS_1 theory. Then $\mathcal{K}$ has bounded clique-width.*

A sketch of logic *interpretation*:

All graphs of $\mathcal{K}$ $\longrightarrow$ bipartite graphs $\longrightarrow$ "bipartite" binary matroids (vectors over $GF(2)$) $\longrightarrow$ (large grids) Theorem 4.1.

$\Longrightarrow$ Bounded branch-width of the "bipartite" matroids $=$
$=$ the rank-width of the bipartite graphs,
$\Longrightarrow$ a bound on the clique-width of $\mathcal{K}$.

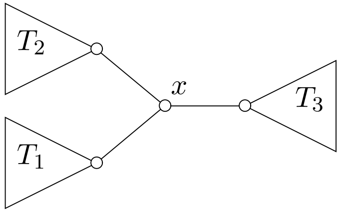
5 Matroid Tree Decompositions

(First suggestion by [Geelen, unpublished], modified [PH and Whittle, 2003].)

A **tree decomposition of a matroid** M is (T, τ), where

- T a tree, and $\tau : E(M) \to V(T)$ an arbitrary mapping (nothing like the "bags"!),
- the *width of a node* x in T is as follows:
  Let $T_1, \dots, T_d$ be the connected components of $T - x$ (branches), then

$$\text{width of } x = \sum_{i=1}^{d} \mathrm{r}_M\left(E(M) - \tau^{-1}(V(T_i))\right) - (d-1) \cdot \mathrm{r}(M)\,.$$

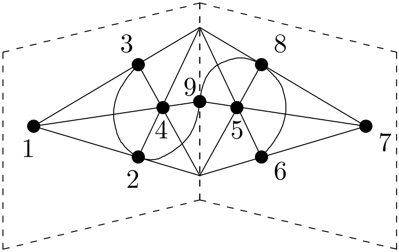
$\tau : E(M) \to$ 

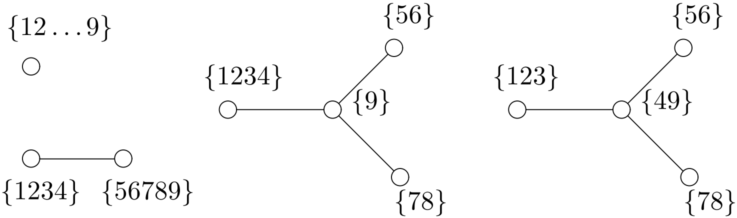
Tree-width of M = min. of max. node widths over all decompositions.

(This parameter equals usual tree-width on graphic matroids!)

Understanding the width of a node I – *projective subspaces*:

A "bag" at a node x is seen as the affine closure of the elements at x plus the guts of the separations induced by the (single) branches of x in the tree.





widths: 4, 3 3 4

Understanding the width of a node II – *rank "defects"*:

We sum together the rank "defects" happening after deleting the elements of each (single) branch of the node x in the tree.

$$\text{width of } x = \sum_{i=1}^{d} \mathrm{r}_M \left(E - \tau^{-1}(V_i)\right) - (d-1) \cdot \mathrm{r}(M)$$

$$= \mathrm{r}(M) - \sum_{i=1}^{d} \left[\mathrm{r}(M) - \mathrm{r}_M \left(E - \tau^{-1}(V_i)\right)\right]$$

Our results:

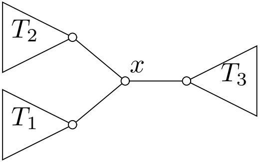
Theorem 5.1. (Whittle and PH, 2003) *Let M be a matroid of tree-width k and branch-width b. Then $b - 1 \le k \le \max(2b-2, 1)$.*

Theorem 5.2. (Whittle and PH, 2003) *Let G be a graph with at least one edge, and let $M = M(G)$ be the cycle matroid of G. Then the tree-width of G equals the tree-width of M.*

Vertex-Free Graph Tree-Width

A **V-F tree decomposition** of a graph G is (T, τ), where

- T a tree, and $\tau : E(G) \to V(T)$ an arbitrary mapping of edges,
- the *width of a node* x in T is as follows:

$\tau : E(G) \to$ 

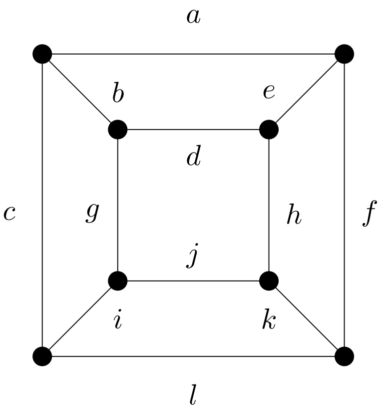
Let $T_1, \ldots, T_d$ be the connected components of $T - x$ (branches) and $F_i = \tau^{-1}(V(T_i))$, then (likewise the rank “defect” view)

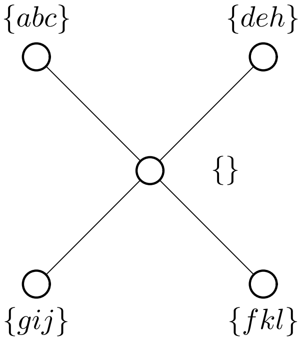
$$\begin{aligned} \text{width of } x &= |V(G)| - \sum_{i=1}^{d} c(G - F_i) + (d-1)c(G) \\ &= |V(G)| - c(G) - \sum_{i=1}^{d} \Big[c(G - F_i) - c(G) \Big] , \end{aligned}$$

where $c(H)$ denotes the number of components of H.

Tree-width of M = min. of max. node widths over all decompositions.

An example of a V-F tree decomposition of width 3:





Thank You!

And Merry Christmas!