Trees, Grids, and MSO Decidability: from Graphs to Matroids

Petr Hliněný[1]* and Detlef Seese[2]

[1] Dept. of Computer Science, VŠB – Technical University of Ostrava, 17. listopadu 15, 70833 Ostrava, Czech Republic
petr.hlineny@vsb.cz

[2] Institute AIFB, University Karlsruhe (TH), D-76128 Karlsruhe, Germany
seese@aifb.uni-karlsruhe.de

Abstract. Monadic second order (MSO) logic proved to be a useful tool in many areas of application, reaching from decidability and complexity to picture processing, correctness of programs and parallel processes. To characterize the structural borderline between decidability and undecidability is a classical research problem here. This problem is related to questions in computational complexity, especially to the model checking problem, for which many tools developed in the area of decidability proved to be useful. For more than two decades it was conjectured in [76] that decidability of monadic theories of countable structures implies that the theory can be reduced via interpretability to a theory of trees.

It is one of the main goals of this article to prove a variant of this conjecture for matroids representable over a finite field. (Matroids can be viewed as a wide generalization of graphs, and they seem to capture some second order properties in a more suitable way than graphs themselves, c.f. the recent development in matroid structure theory [39, 41].) More exactly we prove, for every finite field $\mathbb{F}$, that any class of $\mathbb{F}$-representable matroids with a decidable MSO theory must have uniformly bounded branch-width. Moreover we show that bounding the branch-width of all matroids in general is not sufficient to obtain a decidable MSO theory.

Our paper gives a (rather detailed) introduction into these different subjects, and shows that a blend of ideas and methods from logic together with structural matroid theory can lead to new tools and algorithms, and can shed light into some old open problems.

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* Other affiliation: Institute of Mathematics and Computer Science, Matej Bel University, Banská Bystrica, Slovakia. Partial support by grant VEGA 1/1002/04.

1 Introduction

Trying to understand the complexity of decision problems from a descriptional and parametric point of view leads to the impression that on one side, almost all uniform approaches for low complexity, i.e. descriptions for classes of problems solvable in polynomial or even linear time, are related to a similarity of the input structures to trees, measured e.g. by the tree-width, branch-width or clique-width, and on the other side, high complexity is related to containment or definability of large grids inside the input structures.

Of course this is not an exact mathematical correlation, since there are examples of NP-hard problems for trees (e.g. bandwidth) as well as linear time solvable problems for large grids (e.g. planarity). The problem to study the trade-off between the expressive power of the language, the structure of the input objects and the complexity of the algorithmic solution becomes a bit more tractable in case one fixes the language. One of the languages for which many suitable and promising results are known is monadic second order logic (MSO logic), which extends first order logic by allowing quantification over monadic predicates. This logic is famous for its high expressive power in combination with a manageable model theory (see e.g. [42] and [35]), and it has found many applications in different areas, as decidability, model checking, data bases, and computational complexity.

Of special importance in this area are classes of graphs (or other structures) of bounded tree-width, branch-width, or clique-width, since for these classes MSO logic poses besides the good model theory also very good algorithmic properties. For instance, for MSO logic and several of its extensions one can show that all problems expressible in it can be solved in polynomial or even in linear time if they are restricted to classes of structures of bounded tree-width (see e.g. [3] or [31]), or of bounded clique-width (see [30]).

There are two basic principles in the proofs of those results.

The first one reduces the original problem P to a related problem P' for binary trees (e.g. via interpretability [3], or via transductions [20]), and then it solves the equivalent problem P' for binary trees, which are related to the original input structure, via standard equivalences of MSO-formulas to tree automata. This reduction opens a way to solve the problems via usual dynamic programming techniques, starting the computation in the leaves of the tree which represent the input structure, and following then all the branches till the root is reached, performimg only local computations.

The second one uses the idea underlying the Feferman-Vaught theorem [36] and [78] to reduce the original structure to an equivalent but more simple one, i.e. the problem is solved via decomposition to smaller structures (see [30] and [56]). Basically all ideas used here have their historical origin in investigations of decidability of theories.

On the other hand, if classes of structures of unbounded tree-width are regarded, then it is often easy to find a reduction of the square tiling problem (see [37]) to the original problem P and thus showing that it is NP-hard. Here one often uses the existence of large grid substructures in the input structures to encode the tiling area, while monadic predicates are used to code the tiles. Also this technique is a basic technique developed and used for proofs of the undecidability of theories, e.g. of planar graphs [46, 38] (see also [71, 72]).

Looking more closely to algorithmic complexity of decision problems as well as to the decidability of theories, one can observe many similarities. Large grids can often be used to show high complexity of decision problems as well as undecidability of theories and the similarity to trees of the regarded input structures or the structure of the models often leads to efficient solution algorithms of the regarded problems or to decision procedures of the corresponding theories. A certain explanation of the structural gap between graphs containing grids on one side and graphs of bounded tree-width on the other is given by the landmark result of Robertson and Seymour – that graphs without large grids as minors have a universally bounded tree-width [65]. This result is a part of their famous Graph Minor project [64] which, besides many deep theoretical results, also revolutionized the area of algorithm design in computer science.

It is interesting to observe that, for the decidability/undecidability question, the structural borderline between easy (i.e. decidable) and difficult (i.e. undecidable) appears in a more clear way than for the P vs. NP problem. This is not surprising since the decidability of a theory is a very strong assumption – there has to be an algorithm solving satisfiability for all formulas of the language restricted to all structures of the class of models. Moreover, to show that a theory is undecidable it is often not very difficult to find a formula in the language defining an arbitrarily given tiling problem inside the models containing suitable large grids. It was one of the fundamental observations since the start of investigation of decidability of MSO theories in the 60's (see e.g. [13]); that all decidable MSO theories found could be reduced via interpretability to Rabin's landmark result on the decidability of $S2S$ [61], the MSO theory of two successor functions, in other words the MSO theory of the infinite binary tree. This was observed in [72] (see also [70, 71]), and led in [76] (see also [77]) to *the conjecture* that all decidable MSO theories of arbitrary classes of countable structures are interpretable into a certain class of trees via interpretability. Several successful attempts to prove special cases of this conjecture were made in [27, 33] (see also [19, 22, 26]), but the full general case is still open (cf. the last section).

One of the goals of this article is to show that the conjecture holds true for matroids representable over finite fields.

Matroids are of special interest here because they present a strong combinatorial generalization of graphs. Nowadays, one can witness in the matroid community a great effort to extend the above mentioned Robertson-Seymour's Graph Minor project as far as possible to matroids, followed by important new structural results about representable matroids, e.g. [39–41]. Building on those structural advances, and on our recent results [47, 48] , we work on extending the above mentioned decidability results for MSO theories from graphs to matroids as more general combinatorial structures. We note that these extensions are interesting not only to matroid theory – they can bring new advances also for graph theories, for example [33] (see also in Section 3).

This paper, which is the extended full version of [52], is organized as follows:

Since the paper is intended for general computer-science and logic audiences, we provide basic definitions and facts concerning matroid structure and branch-width from combinatorics, and decidability and interpretability of theories from mathematical logic, in the next three sections. We then bring up the MSO logic of matroids in Section 5, and present some related recent results there; like we show [47] that the MSO theory of the class of all matroids of bounded branch-width over a finite field is decidable.

We present our main result in Section 6 (Theorem 6.2), which extends the results from [76]: We prove that, for every finite field, a class of matroids representable over it and with a decidable MSO theory must have uniformly bounded branch-width. These results for matroids asre of special interest since, on one side, matroids naturally include the set concept in their structure, and allow in spite of this decidable MSO theories with high expressive power. On the other side, matroids have a rich structure theory, allowing to code graphs in a natural way (and the conjecture is still open for arbitrary graphs). In contrast to matroids, the other class of structures including the set concept in a natural way – Boolean algebras, lead immediately to undecidable MSO theories if their size is not bounded. Yet in Section 7 we exhibit undecidability of the MSO theory for matroid classes of bounded branch-width, but not restricted by representability (specifically, for the spikes of branch-width three, Theorem 7.2).

Finally, regarding current work in progress (e.g. [53]), we present some informal thoughts and questions about decidable MSO theories of matroids in general (not restricted by the representability assumption). In particular, we ask about other, structurally different, matroidal obstructions to MSO decidability than traditional grids (c.f. Corollary 7.8). We relate those interesting questions to the conjecture about interpretability of all decidable MSO theories in trees. In connection with the mentioned work [33] of Courcelle and Oum on the C_2MS theory of graphs of bounded clique-width, we remark on an example of a theory which is undecidable in C_2MS but decidable in pure MSO logic. (So proving the conjecture for C_2MS does not imply a proof of the complete conjecture for MSO.)

2 Basics of Matroids

We assume that the reader is familiar with graph theory. Since our paper generalizes graph results to matroids, which are far less known than graphs, we include a brief introduction to necessary matroidal concepts here. We refer to Oxley [57] for our matroid terminology.

A *matroid* is a pair $M = (E, \mathcal{B})$ where $E = E(M)$ is the ground set of M (elements of M), and $\mathcal{B} \subseteq 2^E$ is a nonempty collection of *bases* of M. Moreover, matroid bases satisfy the "exchange axiom"; if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$, then there is $y \in B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$. We consider only finite matroids. Subsets of bases are called *independent sets*, and the remaining sets are *dependent*. Minimal dependent sets are called *circuits*. All bases have the same cardinality called the *rank* $\mathrm{r}(M)$ of the matroid. The *rank function* $\mathrm{r}_M(X)$ in M is the maximal cardinality of an independent subset of a set $X \subseteq E(M)$.

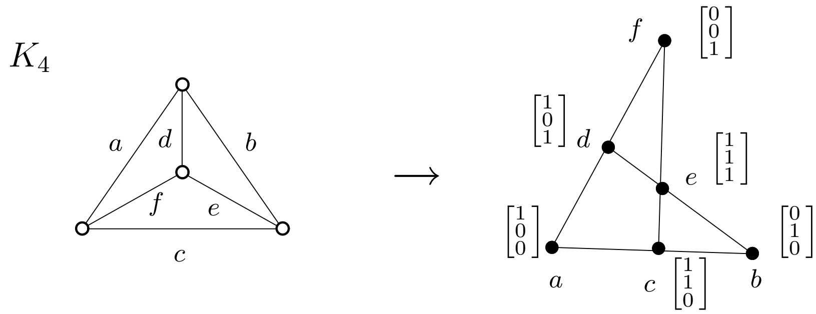

Fig. 1. An example of a vector representation of the cycle matroid $M(K_4)$. The matroid elements are depicted by dots, and their (linear) dependency is shown using lines.

If G is a (multi)graph, then its *cycle matroid* on the ground set $E(G)$ is denoted by $M(G)$. The independent sets of $M(G)$ are acyclic subsets (forests) in G, and the circuits of $M(G)$ are the cycles in G. Another example of a matroid is a finite set of vectors with usual linear dependency. If $\boldsymbol{A}$ is a matrix, then the matroid formed by the column vectors of $\boldsymbol{A}$ is called the *vector matroid* of $\boldsymbol{A}$, and denoted by $M(\boldsymbol{A})$. The matrix $\boldsymbol{A}$ is a *representation* of a matroid $M \simeq M(\boldsymbol{A})$. We say that the matroid $M(\boldsymbol{A})$ is $\mathbb{F}$*-represented* if $\boldsymbol{A}$ is a matrix over a field $\mathbb{F}$. (Fig. 1.) A *graphic matroid*, i.e. a cycle matroid of some multigraph, is representable over any field.

An interesting question about matroids arises in connection with computational complexity: What is the input size of an n-element matroid? Truth saying, it is $\Theta(2^n)$ since a matroid carries information about all subsets of its ground set, but acceptance of that would ruin any reasonable algorithmic complexity measures. That is why matroids are considered with particular representations of polynomial size, like the above mentioned graphic or vector matroids over finite fields, or as an abstract matroid given by an oracle (answering queries about the rank function). In general it is hard (exponential) to tell whether an abstract matroid is representable by a matrix, but one can test whether a matroid is graphic in polynomial time, both in [79].

The *dual* matroid M^* of M is defined on the same ground set E, and the bases of M^* are the set-complements of the bases of M. A set X is *coindependent* in M if it is independent in M^*. An element e of M is called a *loop* (a *coloop*), if $\{e\}$ is dependent in M (in M^*). The matroid $M \setminus e$ obtained by *deleting* a non-coloop element e is defined as $(E - \{e\}, \mathcal{B}^-)$ where $\mathcal{B}^- = \{B : B \in \mathcal{B},\, e \notin B\}$. The matroid M/e obtained by *contracting* a non-loop element e is defined using duality $M/e = (M^* \setminus e)^*$. (This corresponds to contracting an edge in a graph.) A *minor* of a matroid is obtained by a sequence of deletions and contractions of elements. Since these operations naturally commute, a minor M' of a matroid M can be uniquely expressed as $M' = M \setminus D/C$ where D are the coindependent deleted elements and C are the independent contracted elements. The following claim is folklore in matroid theory:

Lemma 2.1. *Let $N = M \setminus D/C$. Then a set $X \subseteq E(N)$ is dependent in N if and only if there is a dependent set $Y \subseteq E(M)$ in M such that $Y - X \subseteq C$.*

The notion of a matroid minor directly extends minors in graphs. (In fact it was matroid theory which inspired this notion.) Unlike for graphs, which are well-quasi-ordered with respect to minors (the Graph Minor project [66]), no such result is true for all matroids, but important restricted cases of the WQO property are known true, e.g. [39]. Another consequence of the Graph Minor project is that one can test for an arbitrary fixed minor in a graph in cubic time. Again, no such general result extends to all matroids, not even to matroids representable by rational matrices [51].

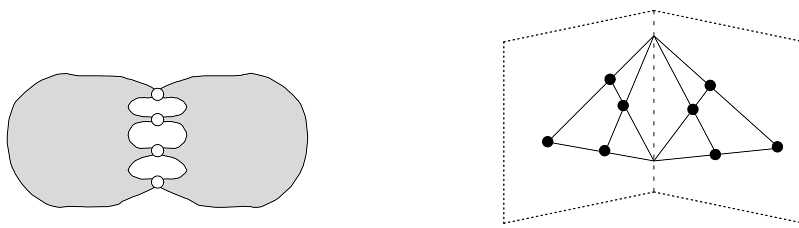

Fig. 2. An illustration to a 4-separation in a graph, and to a 3-separation in a matroid.

Another important concept is matroid connectivity, which is close, but somehow different, to traditional graph connectivity. The *connectivity function* λ_M of a matroid M is defined for all subsets $A \subseteq E$ by

$$\lambda_M(A) = r_M(A) + r_M(E - A) - r(M) + 1\,.$$

Here $r(M) = r_M(E)$. A subset $A \subseteq E$ is *k-separating* if $\lambda_M(A) \leq k$. A partition $(A, E - A)$ is called a *k-separation* if A is k-separating and both $|A|, |E - A| \geq k$. Geometrically, the spans of the two sides of a k-separation intersect in a subspace of rank less than k. See in Fig. 2. In a corresponding graph view, the connectivity function $\lambda_G(F)$ of an edge subset $F \subseteq E(G)$ equals the number of vertices of G incident both with F and with $E(G) - F$. (Then $\lambda_G(F) = \lambda_{M(G)}(F)$ provided both sides of the separation are connected in G.)

3 Tree-Width and Branch-Width

Again, we assume that the reader is familiar with *tree-width* of graphs. (Though we shall mostly work with branch-width instead.) Just for a quick reference we review a few important results concerning graph tree-width here.

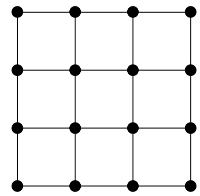

Fig. 3. An illustration to a 4×4 grid.

Let Q_n denote the $n \times n$-*grid graph*, i.e. the graph on $V(Q_n) = \{1, 2, \dots, n\}^2$ and $E(Q_n) = \big\{\{(i,j)(i',j')\} : 1 \le i,j,i',j' \le n, \{|i-i'|, |j-j'|\} = \{0,1\}\big\}$. We say that a class $\mathcal{G}$ of graphs has *bounded tree-width* if there is a constant k such that any graph $G \in \mathcal{G}$ has tree-width at most k. A basic structural result on tree-width is given in [65]:

Theorem 3.1. (Robertson, Seymour) *A graph class $\mathcal{G}$ has bounded tree-width if and only if there exists a constant m such that no graph $G \in \mathcal{G}$ has a minor isomorphic to Q_m.*

On the algorithmic side, the best known result is a linear time "FPT" algorithm for graph tree-width [11]:

Theorem 3.2. (Bodlaender) *For every fixed $t > 0$, it can be decided in linear time whether a given graph G has tree-width at most t or not. Moreover, an optimal tree decomposition for G can be constructed as well in the "yes" case.*

Unfortunately, this is not a truly polynomial algorithm since the running time depends on t exponentially. In general the problem to determine tree-width of a given graph is NP-complete [1].

Besides tree-width, Robertson and Seymour also introduced [65] a similar, but less known, parameter called *branch-width*, and they proved that branch-width is within a constant factor of tree-width on graphs. We think it is unfortunate that branch-width is not used as much as tree-width since branch-width is often technically easier to handle and more suitable for applications, both on theoretical and algorithmic sides.

In matroid theory the situation is quite different – the notion of branch-width took over tree-width completely, since a branch decomposition routinely extends from graphs to matroids, while a tree decomposition (in the traditional sense) is impossible to define. However, we just remark that it is possible to define a matroid tree-width parameter [54] which is within a constant factor of branch-width and exactly equal to graph tree-width on graphs, but that is not a straightforward extension of traditional graph tree-width.

To demonstrate the close relation between graph and matroid "width" parameters, we provide a common definition of a branch-width for any symmetric connectivity function:

Assume that λ is a symmetric function on the subsets of a ground set E. (Here $\lambda \equiv \lambda_G$ is the connectivity function of a graph, or $\lambda \equiv \lambda_M$ of a matroid.) A *branch decomposition* of λ is a pair (T, τ) where T is a sub-cubic tree ($\Delta(T) \leq 3$), and τ is a bijection of E into the leaves of T. For e being an edge of T, the *width* of e in (T, τ) equals $\lambda(A) = \lambda(E - A)$, where $A \subseteq E$ are the elements mapped by τ to leaves of one of the two connected components of $T - e$. (We say that e *displays* the separation $(A, E - A)$ of E.) The width of the branch decomposition (T, τ) is maximum of the widths of all edges of T, and *branch-width* of λ is the minimal width over all branch decompositions of λ.

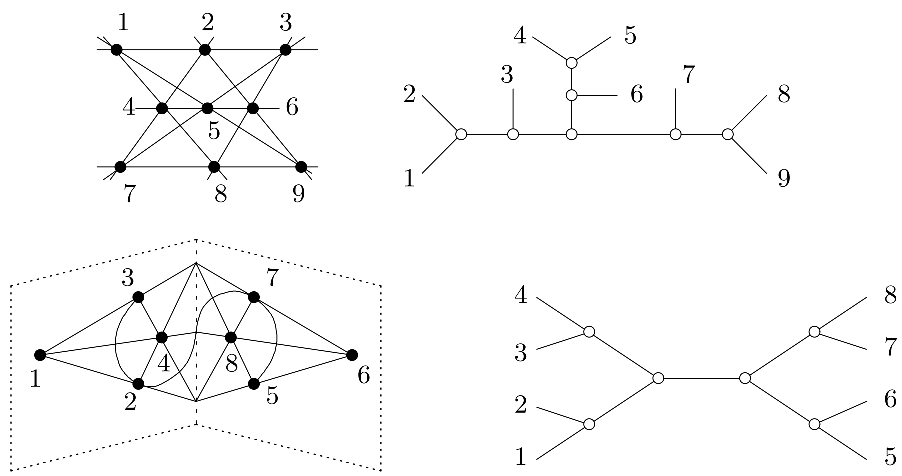

Fig. 4. Two examples of width-3 branch decompositions of the Pappus matroid (top left, in rank 3) and of the binary affine cube (bottom left, in rank 4). The lines in matroid pictures show dependencies among elements.

Recall the definitions of the graph and matroid connectivity functions λ_G and λ_M on the ground sets $E(G)$ and $E(M)$, respectively, from Section 2. Then branch-width of $\lambda \equiv \lambda_G$ is called *branch-width of a graph* G, and that of $\lambda \equiv \lambda_M$ is called *branch-width of a matroid* M. (See examples in Fig. 4.) Considering branch-width on matroids, the following recent result [41] analogous to Theorem 3.1 is crucial for our paper:

Theorem 3.3. (Geelen, Gerards, Whittle) *For every finite field* $\mathbb{F}$*; a class* $\mathcal{N}$ *of* $\mathbb{F}$*-representable matroids has bounded branch-width if and only if there exists a constant* m *such that no matroid* $N \in \mathcal{N}$ *has a minor isomorphic to* $M(Q_m)$*.*

An algorithm analogous to that in Theorem 3.2 is given in [50].

Theorem 3.4. (Hliněný) *For every fixed* $t > 0$ *and a finite field* $\mathbb{F}$*, it can be decided in cubic time whether a given matroid* M *represented by a matrix over* $\mathbb{F}$ *has branch-width at most* t *or not. Moreover, a near-optimal branch decomposition for* M *can be constructed in the "yes" case.*

Lastly in this section we note another interesting measure of "tree-likeness" of graphs - called *clique-width* [28]. The clique-width of a graph G is defined as the minimum k such that there is a k-expression constructing G; where a k-expression is an expression using the following four operations over vertex-labeled graphs with k labels: Creation of a new vertex of label i, disjoint union, addition of all edges between the vertices of label i and j, and relabeling all vertices of label i to j, where $1 \leq i, j \leq k$. (The underlying tree of this definition is the parse tree of the expression.)

A surprising connection between graph clique-width and matroid branch-width – the notion of rank-width, is exhibited in the approximation algorithm for clique-width [58] of Oum and Seymour, and in the subsequent structural work [59]. This clearly shows that matroid branch-width is not only of interest in matroid theory, since its investigation led to exciting new results in the area of graph theory and in computational complexity as well. Furthermore we mention here some of the recent results of Courcelle and Oum [33]: They showed that, from our main result – Theorem 6.2, it follows that the decidability of monadic theories of arbitrary classes of countable graphs in a monadic logic allowing counting modulo 2 implies that all models of such a theory have bounded clique-width, hence proving such a weaker version of the nearly 25 year old Conjecture 4.12 of the next section in case of this monadic second order logic with counting modulo 2 instead of pure monadic second order logic.

4 Decidability of Theories

In this section we will review some basic notions on the decidability of theories from mathematical logic, and give some motivation to the main result we prove by adding a detailed historical survey on results related to the main problem for which we present a partial solution in the area of matroids. This survey includes some material from articles written in German, which were never published in English (see [68–72]) or appeared only in a preprint form as [74, 75].

In this section we allow also infinite structures. We will use the following notion of a theory. Let $\mathcal{K}$ be a class of structures and let L be a suitable logic for $\mathcal{K}$. A sentence is a set of well-formed L-formulas without free variables. The set of all L-sentences true in $\mathcal{K}$ is denoted as L-theory of $\mathcal{K}$. We use $\mathrm{Th}_L(\mathcal{K})$ as a short notation for this theory. Hence, a theory can be viewed as the set of all properties, expressible in L, which all structures of $\mathcal{K}$ posses. In case that $\mathcal{K} = \{G\}$ we write $\mathrm{Th}_L(G)$ instead of $\mathrm{Th}_L(\mathcal{K})$. Using this definition we obtain $\mathrm{Th}_L(\mathcal{K}) = \bigcap\{\mathrm{Th}_L(G) : G \in \mathcal{K}\}$. We write $\mathrm{Th}(\mathcal{K})$, $\mathrm{Th}_{MSO}(\mathcal{K})$ if L is first order logic, or monadic second order logic (abbreviated as MSO logic), respectively.

For graphs there are actually two variants of MSO logic, commonly denoted by MSO_1 and MSO_2. In MSO_1, set variables only denote sets of vertices. In MSO_2, set variables can also denote sets of edges of the considered graph. In other words the difference between both logics is that in MSO_1 the domain of the graph consists of the vertices only and the relation is just the usual adjacency between vertices, while in MSO_2 the domain is two-sorted and contains vertices as well as edges and the relation is the incidence relation. To distinguish these two different classes of structures we will speak about adjacency and incidence graphs, respectively. The expressive power of both logics was studied by Courcelle in [19].

The weak monadic second order logic (WMSO logic) results from MSO by restricting the interpretation of the set variables to finite sets only. The corresponding theory is denoted as $\mathrm{Th}_{WMSO}(\mathcal{K})$.

A theory is said to be *decidable* if there is an algorithm deciding, for an arbitrary sentence $\varphi \in L$, whether $\varphi \in \mathrm{Th}_L(\mathcal{K})$ or not, i.e. whether φ is true in all structures of $\mathcal{K}$. Otherwise this theory is said to be *undecidable.* More information concerning the terminology from logic needed in this section can be found in classical textbooks as [34]. A good introduction into the decidability of theories can be found in [62] (see also [42] for a survey on monadic theories).

One of the strongest results on decidability is the following theorem of Rabin [61].

Theorem 4.1. (Rabin) *Let S2S be the MSO theory of the following structure* $(\{0,1\}^*, sc_1, sc_2)$*, where* $\{0,1\}^*$ *denotes the set of all finite sequences over the alphabet* $\{0,1\}$ *and* sc_i *for* $i \in \{0,1\}$ *denotes the function* $\{(x, xi) : x \in \{0,1\}^*\}$*. Then S2S is decidable.*

The decidability of many MSO theories can be reduced to this result by the classical method of model interpretability, introduced in [60], which is often the best tool of choice to prove the decidability of theories. To describe the idea of the method assume that two classes of structures $\mathcal{K}$ and $\mathcal{K}'$ are given, and that L and L', respectively, are corresponding languages for the structures of these classes. The basic idea of the interpretability of theory $\mathrm{Th}_L(\mathcal{K})$ into $\mathrm{Th}_{L'}(\mathcal{K}')$ is to transform formulas of L into formulas of L', by translating the nonlogical symbols of L by formulas of L', in such a way that truth is preserved in a certain way. Here we assume that the logics underlying both languages are the same. Otherwise, one has to translate also the logical symbols.

We explain this translation in a simple case of relational structures. First one chooses an L'-formula $\alpha(x)$ intended to define in each L'-structure $G \in \mathcal{K}'$ a set of individuals $G[\alpha] := \{a : a \in dom(G) \text{ and } G \models \alpha(a)\}$, where $dom(G)$ denotes the domain (set of individuals) of G. Then one chooses for each s-ary relational sign R from L an L'-formula $\beta_R(x_1, \ldots, x_s)$, with the intended meaning to define a corresponding relation $G[\beta_R] := \{(a_1, \ldots, a_s) : a_1, \ldots, a_s \in dom(G) \text{ and } G \models \beta_R(a_1, \ldots, a_s)\}$. All these formulas build the formulas of the interpretation $I = \big(\alpha(x), \beta_R(x_1, \ldots, x_s), \ldots\big)$.

With the help of these formulas one can define for each L'-structure G a structure $G^I := \big(G[\alpha], G[\beta_R], \ldots\big)$, which is just the structure defined by the chosen formulas in G. Sometimes G^I is also denoted as $I(G)$ and I is called an (L, L')-interpretation of G^I in G. In case that both L and L' are MSO languages, this interpretation is also denoted as MSO-interpretation. Using these formulas there is also a natural way to translate each L-formula φ into an L'-formula φ^I. This is done by induction on the structure of formulas. The atomic formulas are simply substituted by the corresponding chosen formulas with the corresponding substituted variables. Then one may proceed via induction as follows:

$$(\neg\chi)^I := \neg(\chi^I), \quad (\chi_1 \wedge \chi_2)^I := (\chi_1)^I \wedge (\chi_2)^I,$$
$$\big(\exists x\, \chi(x)\big)^I := \exists x \,\big(\alpha(x) \wedge \chi^I(x)\big)\,,$$
$$(x \in X)^I := x \in X, \quad \big(\exists X\, \chi(X)\big)^I := \exists X\, \chi^I(X).$$

The resulting translation is called an interpretation with respect to L and L'. Its concept could be briefly illustrated with a picture:

$$\begin{array}{ccc} \varphi \in L & I & \varphi^I \in L' \\ H \in \mathcal{K} & \xrightarrow{\qquad\qquad} & G \in \mathcal{K}' \\ & & \\ G^I \simeq H & \xleftarrow[\qquad\qquad]{I} & G \end{array}$$

Fig. 5. The concept of an (L, L')-interpretation I.

For theories, interpretability is now defined as follows. Let $\mathcal{K}$ and $\mathcal{K}'$ be classes of structures and L and L' be corresponding languages. Theory $\mathrm{Th}_L(\mathcal{K})$ is said to be interpretable in $\mathrm{Th}_{L'}(\mathcal{K}')$ if there is an (L, L')-interpretation I translating each L-formula φ into an L'-formula φ^I, and each L'-structure $G \in \mathcal{K}'$ into an L-structure G^I as above, such that the following two conditions are satisfied:

(i) For every structure $H \in \mathcal{K}$, there is a structure $G \in \mathcal{K}'$ such that $G^I \cong H$,
(ii) for every $G \in \mathcal{K}'$, the structure G^I is isomorphic to some structure of $\mathcal{K}$.

It is easy to see that interpretability is transitive. The key result for interpretability of theories is the following theorem [60]:

Theorem 4.2. (Rabin) *Let $\mathcal{K}$ and $\mathcal{K}'$ be classes of structures, and L and L' be suitable languages. If $\mathrm{Th}_L(\mathcal{K})$ is interpretable in $\mathrm{Th}_{L'}(\mathcal{K}')$, then undecidability of $\mathrm{Th}_L(\mathcal{K})$ implies undecidability of $\mathrm{Th}_{L'}(\mathcal{K}')$.*

This interpretability technique is a natural and very powerful tool to show the decidability or the undecidability of other theories (see e.g. [60, 61, 63, 6, 7]). Analysing the structure of decidable and undecidable MSO theories with this tool, one gets the following results.

Theorem 4.3. (Rabin, Shelah, Stupp) *The MSO_1-theory of the class of all trees is decidable.*

The monadic theory of countable trees can easily be shown to be interpretable into S2S [61]. The uncountable case follows from a result of Shelah and Stupp [67].

The MSO-theories of many other classes of graphs were shown to be decidable via reduction (using interpretability) to the MSO_1-theory of trees (see e.g. [44] and [70]). Most of them had bounded tree-width.

Theorem 4.4. (Courcelle, Arnborg, Lagergren, Seese) *For each positive integer m, the MSO_2 theory of the class of all incidence graphs of tree-width $< m$ is interpretable in the MSO_1-theory of all trees, and hence it is decidable.*

This result was explicitly stated in [17] and proved via MSO transductions, a notion equivalent to interpretability. In [2] the result is contained implicitly by giving an explicit interpretation into the class of binary trees. The result was generalized to clique-width by Courcelle [25] (see also [28, 32, 9, 10]):

Theorem 4.5. (Courcelle) *For each positive integer m, the MSO_1 theory of the class of all adjacency graphs of clique-width $< m$ is interpretable in the theory of all trees, and hence it is decidable.*

Clique-width is of special interest in our context since there is a close relation between interpretability and clique-width (see [20, 29, 23, 33]):

Theorem 4.6. (Courcelle, Engelfriet) *A set $\mathcal{K}$ of adjacency graphs has bounded clique-width if and only if there is a class $\mathcal{T}$ of trees such that $\mathrm{Th}_{MSO_1}(\mathcal{K})$ is interpretable in $\mathrm{Th}_{MSO_1}(\mathcal{T})$.*

This result is important because it is a complete combinatorial characterization of interpretabality into trees. The relation of bounded tree-width to clique-width is given in the following result (see [4, 21, 8]):

Theorem 4.7. (Barthelmann) *Let G be a graph of bounded clique-width. The following statements are equivalent:*
(i) G has finite tree-width,
(ii) G does not contain $K_{n,n}$ as a subgraph for some positive integer n.

Till now no decidable MSO theories could be found which are not interpretable into a class of trees. A reason might appear in the ideas which led to the following result from [71] (see also [72, 76]):

Theorem 4.8. (Seese) *Let $\mathcal{K}$ be a class of graphs such that each planar graph H is a minor of some planar graph $G \in \mathcal{K}$. Then $\mathrm{Th}_{MSO_1}(\mathcal{K})$ is undecidable.*

The way to prove this result is to use the high expressive power of MSO logic to show that some large grids are definable inside the structures, and can be used to define the tiling or domino problems (see e.g. [12]), which are undecidable by [80].

From this result one gets via Theorem 3.1 the following result of [76]:

Theorem 4.9. (Seese) *If $\mathcal{K}$ is a class of planar graphs such that $\mathrm{Th}_{MSO_1}(\mathcal{K})$ is decidable, then there is an n such that each $G \in \mathcal{K}$ has tree-width $\leq n$.*

Hence there is a class $\mathcal{T}$ of trees such that $\mathrm{Th}_{MSO_1}(\mathcal{K})$ is interpretable in $\mathrm{Th}_{MSO_1}(\mathcal{T})$. Courcelle extended this result in [18, 19, 21, 24, 26, 27] (see also [76, 77]) to many other structures:

Theorem 4.10. (Courcelle) *If $\mathcal{K}$ is a class of graphs of bounded degree, bounded genus, graphs without a fixed graph H as minor, uniformly k-sparse graphs, interval graphs, line graphs or a class of partial orders of dimension 2, such that $\mathrm{Th}_{MSO_1}(\mathcal{K})$ is decidable, then there is a class $\mathcal{T}$ of trees such that $\mathrm{Th}_{MSO_1}(\mathcal{K})$ is interpretable in $\mathrm{Th}_{MSO_1}(\mathcal{T})$.*

This results holds also for WMSO logic.

Looking to other classes of structures one can observe that there are almost no interesting decidable MSO theories. A basic result in this area was proved by [38] and in a slightly more general form by [72]. A special case for groups was proved by [5]: A structure $(A, \circ)$ is a groupoid if A is a nonempty set and $\circ$ is an arbitrary binary operation defined on A. A cancellative groupoid is a groupoid satisfying both left and right cancellation laws $\forall x \forall y \forall z (x \circ y = x \circ z \Rightarrow y = z)$ and $\forall x \forall y \forall z (y \circ x = z \circ x \Rightarrow y = z)$.

Theorem 4.11. (Garfunkel, Schmerl) *If a class of structures K with one binary operation $\circ$ contains, for each natural number n, a cancellative groupoid of size $\geq n$ as substructure, then $\mathrm{Th}_{MSO_{\forall 1}}(\mathcal{K})$, and hence $\mathrm{Th}_{MSO}(\mathcal{K})$ are undecidable. The same holds for the weak MSO theories.*

Here $\mathrm{Th}_{MSO_{\forall 1}}(\mathcal{K}) := \{\varphi : \varphi \in \mathrm{Th}_{MSO_{\forall 1}}(\mathcal{K})$ and φ has the form $\forall X \psi(X)$, where $\psi(X)$ does not contain quantifiers over sets $\}$.

As a corollary one gets that all (weak) MSO theories of classes of groups, Abelian groups, rings, fields and vector spaces are undecidable if the sizes of the models can not be bounded by a positive integer. This and related ideas led in [76] and [77] to the following conjecture:

Conjecture 4.12. (Seese) Assume that $\mathcal{K}$ is a class of countable structures (of arbitrary finite signature) with $\mathrm{Th}_{MSO}(\mathcal{K})$ decidable. Then there exists a class $\mathcal{T}$ of trees such that $\mathrm{Th}_{MSO}(\mathcal{K})$ is interpretable into $\mathrm{Th}_{MSO_1}(\mathcal{T})$.

For classes $\mathcal{K}$ of countable adjacency graphs, this is equivalent to the statement that the decidability of $\mathrm{Th}_{MSO_1}(\mathcal{K})$ implies that $\mathcal{K}$ has bounded clique-width (via Theorem 4.7). In a stronger form of the general conjecture one could even ask for a class $\mathcal{T}$ having the additional property that $\mathrm{Th}_{MSO_1}(\mathcal{T})$ is decidable). Courcelle proved in [27] that the conjecture for classes of graphs is equivalent to analogous conjectures for bipartite graphs, directed graphs, comparability graphs and partial orders.

When the expressive power of MSO logic on graphs is made higher by allowing quantification on sets of edges in addition to quantification on sets of vertices, i.e. by considering MSO_2 instead of MSO_1, or formally considering an incidence relation instead of the usual adjacency relation, then the conjecture can be proved [76] (see also [77]).

Theorem 4.13. (Seese) *Let $\mathcal{K}$ be a class of graphs considered in the logic MSO_2 with sets of edges, i.e. formalized using incidence relations. If each planar graph H is a minor of some graph $G \in K$, then $\mathrm{Th}_{MSO_2}(\mathcal{K})$ is undecidable. Hence* *if the tree-width of $\mathcal{K}$ is unbounded, then* $\mathrm{Th}_{MSO_2}(\mathcal{K})$ *is undecidable. The same holds for the corresponding weak monadic theory.*

Corollary 4.14. (Seese) *Let $\mathcal{K}$ be a class of graphs considered in the logic MSO_2 with sets of edges. Then the decidability of* $\mathrm{Th}_{MSO_2}(\mathcal{K})$ *implies that the tree-width of $\mathcal{K}$ is bounded, and there is a class $\mathcal{T}$ of trees such that* $\mathrm{Th}_{MSO_2}(\mathcal{K})$ *is interpretable in* $\mathrm{Th}_{MSO_1}(\mathcal{T})$*. The same holds for the corresponding weak monadic theory.*

Using the following result of Lapoire [55] it is even possible to prove the strong conjecture in this case.

Theorem 4.15. (Lapoire) *For every finite graph of tree width k, a tree decomposition of width k is MSO_1-definable inside the graph.*

Using this definable tree-decompositions one can construct easily a class of trees $\mathcal{T}$ with a decidable MSO theory such that the original theory is interpretable in it. Via the definability one finds an interpretation of $\mathrm{Th}_{MSO_2}(\mathcal{T})$ in $\mathrm{Th}_{MSO_2}(\mathcal{K})$ and hence $\mathrm{Th}_{MSO_2}(\mathcal{T})$ is decidable. Moreover it is easy to show that $\mathrm{Th}_{MSO_2}(\mathcal{T})$ is interpretable in $\mathrm{Th}_{MSO_1}(\mathcal{T})$. This gives:

Corollary 4.16. *Assume that* $\mathrm{Th}_{MSO_2}(\mathcal{K})$ *is decidable for a class $\mathcal{K}$ of finite graphs. Then the tree-width of $\mathcal{K}$ is bounded. Moreover, there is a class $\mathcal{T}$ of trees such that* $\mathrm{Th}_{MSO_1}(\mathcal{T})$ *is decidable and* $\mathrm{Th}_{MSO_2}(\mathcal{K})$ *is interpretable in* $\mathrm{Th}_{MSO_1}(\mathcal{T})$*.*

Related results have been proved by Courcelle for many other classes of graphs. Of special interest is the following equivalence (see [27]).

Theorem 4.17. (Courcelle) *Conjecture 4.12 is valid for graphs iff it is valid for bipartite graphs, iff it is valid for directed graphs, iff it is valid for comparability graphs, iff it is valid for partial orders.*

One of the latest results with respect to Conjecture 4.12 is the result from [33], that the conjecture holds for counting monadic second order logic on adjacency graphs. Counting monadic second order logic (CMS) was introduced by Courcelle in [18]. It results as an extension of MSO-logic by allowing counting modulo k, i.e. one adds a predicate $Card_k(X)$, expressing that the cardinality of X is a multiple of k for an arbitrary integer $k > 0$. The logic C_2MS results by allowing only counting modulo 2, i.e. only the additional predicate $Card_2(X)$ is allowed. $Card_2(X)$ will be donted here as $Even(X)$.

Theorem 4.18. (Courcelle, Oum) *Assume that a class $\mathcal{K}$ of adjacency graphs has a decidable C_2MS theory. Then there is a class $\mathcal{T}$ of trees such that the C_2MS theory of $\mathcal{K}$ is interpretable in the MSO_1-theory of $\mathcal{T}$, or equivalently $\mathcal{K}$ has bounded clique-width.*

With respect to Conjecture 4.12 this is at the moment the strongest result known. Moreover as we have noted in Section 3, it is closely related to our results here; since as it is shown in [33], it can be easily deduced from our main Theorem 6.2. However, one can find a class $\mathcal{T}$ of trees with an undecidable C_2MS theory and a decidable MSO-theory (see in last section and [53]).

Regarding Conjecture 4.12 one can ask, why is it stated for classes of countable structures only? One of the essential results which indicate that for uncountable structures the situation could be more difficult is the following theorem from [43].

Theorem 4.19. (Gurevic, Magidor, Shelah) *Decidability of the MSO-theory of ω_2, the ordering of the second uncountable ordinal, depends from axioms of set theory.*

Nevertheless, so far even for uncountable structures, there could be found no counterexample to the conjecture. But for some classes of uncountable structures it is open whether there is an interpretation into a class of trees. One of the prominent examples is the following result of Büchi and Siefkes [14, 16, 15].

Theorem 4.20. (Büchi, Siefkes) *The MSO-theory of ω_1, the ordering of the first uncountable ordinal is decidable.*

A short proof following the line of the Feferman-Vaught theorem (see [36]) can be found in [42] (see also [78] and [56]). Beside this result there are many other classes of structures with a decidable MSO-theory. We will mention here only finite or countable linear orderings [61], countable ordered trees [74, 75] and countable well-founded trees [73] and [74]. But there are many others. Among undecidable MSO-theories of partial orderings one can find the classes of all partial orderings (obvious via interpretability of all graphs), of all linear orderings and the ordering of the real line [78]. A good survey on decidability of MSO-theories with a detailed section on linear orderings can be found in [42].

5 MSO Theory of Matroids

Unlike MSO theories of trees, graphs, algebraic and other structures, MSO theory of matroids has not been considered before. We present it here in the setting introduced in [47]. Working with matroids in logic is a bit tricky since one has to use a second order predicate to fully describe a matroid. (That is due to a simple counting argument considering the numbers of non-isomorphic matroids on n elements.)

¿From a logic point of view, a matroid M on a finite ground set E is the collection of all subsets 2^E together with a unary predicate *indep* such that *indep*(F) if and only if $F \subseteq E$ is independent in M. (One may equivalently consider a matroid with a unary predicate for bases or for circuits, see a discussion in [47].) We shortly write MSO_M to say that the language of *MSO logic is applied* *to (independence) matroids.* If $\mathcal{N}$ is a class of independence matroids, then the MSO_M theory of $\mathcal{N}$ is denoted by $\mathrm{Th}_{MSO_M}(\mathcal{N})$.

To give readers a better feeling for the expressive power of MSO_M on a matroid, we write down a few basic matroid predicates now.

- We write $\mathit{basis}(B) \equiv \mathit{indep}(B) \wedge \forall D\big(B \not\subseteq D \vee B = D \vee \neg\, \mathit{indep}(D)\big)$ to express the fact that a basis is a maximal independent set.
- Similarly, we write $\mathit{circuit}(C) \equiv \neg\, \mathit{indep}(C) \wedge \forall D\big(D \not\subseteq C \vee D = C \vee \mathit{indep}(D)\big)$, saying that C is dependent, but all proper subsets of C are independent.
- A cocircuit is a dual circuit in a matroid (i.e. a bond in a graph). We write $\mathit{cocircuit}(C) \equiv \forall B\big[\, \mathit{basis}(B) \rightarrow \exists x(x \in B \wedge x \in C)\big] \wedge \forall X\big[X \not\subseteq C \vee X = C \vee \exists B\big(\, \mathit{basis}(B) \wedge \forall x(x \notin B \vee x \notin X)\big)\big]$ saying that a cocircuit C intersects every basis, but each proper subset of C is disjoint from some basis.

More examples could be found in [48].

It can be shown that the language of MSO_M is (at least) as powerful as that of MSO_2 on (incidence) graphs. Specifically, we have proved [47] that any MSO_2 sentence ϕ about a 3-connected simple graph G can be translated into an equivalent MSO_M sentence about the cycle matroid $M(G)$. (The need to require 3-connectivity follows not from logic, but from the simple fact that non-isomorphic graphs may have isomorphic cycle matroids unless they are simple 3-connected. In particular, possible loops in a graph are pairwise indistinguishable in its cycle matroid.) Let $G \uplus H$ denote the graph obtained from disjoint copies of G and H by adding all edges between them. The following extended statement is also proved in [47]:

Theorem 5.1. (Hliněný) *Let G be a loopless multigraph, and let M be the cycle matroid of $G \uplus K_3$. Then any MSO_2 about the incidence graph G can be expressed as a sentence about the matroid M in MSO_M.*

In other words, the MSO_2 theory of all loopless multigraphs is interpretable in the MSO_M theory of a certain subclass of 3-connected graphic matroids.

The next result we are going to mention speaks about (restricted) recognizability of MSO_M-definable matroid properties via tree automata. To formulate this, we have to introduce briefly the concept of *parse trees* for representable matroids of bounded branch-width, which has been first defined in [47]. For a finite field $\mathbb{F}$, an integer $t \geq 1$, and an arbitrary $\mathbb{F}$-represented matroid M of branch-width at most $t+1$; a t-boundaried parse tree $\bar{T}$ over $\mathbb{F}$ is a rooted ordered binary tree, whose leaves are labeled with elements of M, and the inner nodes are labeled with symbols of a certain finite alphabet (depending on $\mathbb{F}$ and t). Saying roughly, symbols of the alphabet are "small configurations" in the projective geometry over $\mathbb{F}$. The parse tree $\bar{T}$ uniquely determines an $\mathbb{F}$-representation (up to projective transformations) of the matroid $P(\bar{T}) \simeq M$. See [47] for more details and the result:

Theorem 5.2. (Hliněný) *Let $\mathbb{F}$ be a finite field, $t \geq 1$, and let ϕ be a sentence in the language of MSO_M. Then there exists a finite tree automaton $\mathcal{A}_t^{\phi}$ such* *that the following is true: A t-boundaried parse tree $\bar{T}$ over $\mathbb{F}$ is accepted by $\mathcal{A}_t^\phi$ if and only if $P(\bar{T}) \models \phi$. Moreover, the automaton $\mathcal{A}_t^\phi$ can be constructed (algorithmically) from given $\mathbb{F}$, t, and ϕ.*

In connection with Theorem 3.4 the theorem implies [47] polynomial algorithms for testing MSO-definable properties over $\mathbb{F}$-represented matroids of fixed branch-width. (A direct analogue of the situation with incidence graphs, Theorems 3.2 and 4.4.) Although the statement was originally formulated in the setting of computational complexity, it implicitly applies also to logic decidability, as we make explicit here.

Corollary 5.3. *Let $\mathbb{F}$ be a finite field, $t \geq 1$, and let $\mathcal{B}_t$ be the class of all matroids representable over $\mathbb{F}$ of branch-width at most $t+1$. Then the theory $\mathrm{Th}_{MSO_M}(\mathcal{B}_t)$ is decidable.*

Proof. Assume we are given an MSO_M-sentence ϕ. We construct the automaton $\mathcal{A}_t^\phi$ from Theorem 5.2. Moreover, there is an (easily constructible [47]) automaton $\mathcal{V}_t$ accepting valid t-boundaried parse trees over $\mathbb{F}$. Then $\mathcal{B}_t \not\models \phi$ if and only if there is a parse tree accepted by $\mathcal{V}_t$, but not accepted by $\mathcal{A}_t^\phi$. We thus, denoting by $-\mathcal{A}_t^\phi$ the complement of $\mathcal{A}_t^\phi$, construct the cartesian product automaton $\mathcal{A} = (-\mathcal{A}_t^\phi) \times \mathcal{V}_t$ accepting the intersection of the languages of $-\mathcal{A}_t^\phi$ and of $\mathcal{V}_t$. Then we check for emptiness of $\mathcal{A}$ using standard tools of automata theory. □

The corollary suggests that branch-width could be the right measure of MSO decidability for matroids representable over finite fields, as it is for graphs. (Recall that branch-width is always within a constant factor of tree-width, and so branch-width is bounded in a class iff tree-width is.) We prove that in the next section, and thus support Conjecture 4.12, in Theorem 6.2 and Corollary 6.7.

Remark. We add a short remark about a possibility of considering infinite (countable) matroids. Yes, one could simply extend the above definition of a matroid to infinite sets, and say, to restrict independent sets and circuits to a finite size only, to allow for their handling within WMSO logic. However, then matroid duality, and likely also most interesting structural connections to graph theory would be lost. Thus we choose to stay within the boundary of finite matroids in this paper.

6 Matroid Grids and Undecidability

We need the following special form of Theorem 4.8, which was proved first in a more general form in [71] (see also [76]).

Theorem 6.1. (Seese) *Let $\mathcal{K}$ be a class of adjacency graphs such that for every integer $k > 1$ there is a graph $G \in \mathcal{K}$ such that G has the $k \times k$ grid Q_k as an induced subgraph. Then the MSO_1 theory of $\mathcal{K}$ is undecidable.*

Here we remark that the troubles with MSO_1 logic of graphs – which lacks expressive power to handle arbitrary subgraphs or minors, do not occur at all for matroids since by Theorem 5.1 we have an expressive power equivalent to graph MSO_2 logic. That is why we can extend Theorem 4.13 in the strong form as follows:

Theorem 6.2. *Let $\mathbb{F}$ be a finite field, and let $\mathcal{N}$ be a class of matroids that are representable by matrices over $\mathbb{F}$. If the class $\mathcal{N}$ has unbounded branch-width, then the (monadic second-order) MSO_M theory $\mathrm{Th}_{MSO_M}(\mathcal{N})$ is undecidable.*

The key to the presented extension is given in Theorem 3.3, which basically states that the obstructions to small branch-width on matroids are the same as on graphs, namely large matroid grids. Unfortunately, the seemingly straightforward way to prove Theorem 6.2 — via the direct interpretation of graphs (Theorems 4.13 and 4.9) in the class of graphic minors of matroids in $\mathcal{N}$, is not so simple due to technical problems with low connectivity and with non-graphic matroids. That is why we give here a variant of this idea bypassing Theorems 4.13 and 4.9, and using an indirect interpretation of (graph) grids in matroid grid minors.

Remark. A restriction to $\mathbb{F}$-representable matroids in Theorem 6.2 is not really necessary; it comes more from the context of the related matroid structure research. According to [41], it is enough to assume that no member of $\mathcal{N}$ has a $U_{2,m}$- or $U^*_{2,m}$-minor (i.e. an m-point line or an m-point dual line) for some constant m.

We begin the proof of Theorem 6.2 with an interpretation of the MSO_M theory of all minors of the class $\mathcal{N}$. To achieve this goal, we use a little technical trick first. Let a *DC-equipped matroid* be a matroid M with two distinguished unary predicates D and C on $E(M)$ (with intended meaning as a pair of sets $D, C \subseteq E(M)$ defining a minor $N = M \setminus D/C$).

Lemma 6.3. *Let $\mathcal{N}$ be a class of matroids, and let $\mathcal{N}_{DC}$ denote the class of all DC-equipped matroids induced by members of $\mathcal{N}$. If $\mathrm{Th}_{MSO_M}(\mathcal{N})$ is decidable, then so is $\mathrm{Th}_{MSO_M}(\mathcal{N}_{DC})$.*

Proof. We may equivalently view the distinguished predicates D, C as free set variables in MSO_M. Let $\phi(D, C)$ be an MSO_M formula, and $N \in \mathcal{N}$. Then, by standard logic arguments, $N_{DC} \models \phi(D, C)$ for all DC-equipped matroids N_{DC} induced by N if and only if $N \models \forall D, C\, \phi(D, C)$. Hence $\mathcal{N}_{DC} \models \phi(D, C)$ if and only if $\mathcal{N} \models \forall D, C\, \phi(D, C)$. Since $\forall D, C\, \phi(D, C)$ is an MSO formula if ϕ is such, the statement follows. □

Lemma 6.4. *Let $\mathcal{N}$ be a class of matroids, and $\mathcal{N}_m$ be the class of all minors of members of $\mathcal{N}$. Then $\mathrm{Th}_{MSO_M}(\mathcal{N}_m)$ is interpretable in $\mathrm{Th}_{MSO_M}(\mathcal{N}_{DC})$.*

Proof. We again regard the distinguished predicates D, C of $\mathcal{N}_{DC}$ as free set variables in MSO_M. Let us consider a matroid $N_1 \in \mathcal{N}_m$ such that $N_1 = N \setminus$

D_1/C_1 for $N \in \mathcal{N}$. We are going to use a "natural" interpretation of N_1 in the DC-equipped matroid N_{DC} which results from N with a particular equipment $D = D_1$, $C = C_1$. (Notice that both theories use the same language of MSO logic, and the individuals of N_1 form a subset of the individuals of N.) Let ψ be an MSO_M formula. The translation ψ^I of ψ is obtained inductively:

- For each (bound) element variable x in ψ; it is replaced with

$$\exists x\, \theta(x) \quad \longrightarrow \quad \exists x\, \big(x \notin C \wedge x \notin D \wedge \theta(x)\big)\,.$$

- For each (bound) set variable X in ψ; it is replaced with

$$\exists X \theta(X) \quad \longrightarrow \quad \exists X \forall z\, \big((z \notin X \vee z \notin C) \wedge (z \notin X \vee z \notin D) \wedge \theta(X)\big)\,.$$

- Every occurrence of the *indep* predicate in ψ is rewritten as (cf. Lemma 2.1)

$$indep^I(X) \equiv \forall Y\, \big(\, indep(Y) \vee \exists z(z \in Y \wedge z \notin X \wedge z \notin C)\big)\,,$$

  saying that there is no dependent set Y such that $Y \subseteq X \cup C$.

Consider now the structure N^I defined by $indep^I$ in $N_{DC} \in \mathcal{N}_{DC}$. By Lemma 2.1, a set $X \subseteq E(N^I) = E(N_1)$ is independent in N^I if and only if X is independent in N_1, and hence N^I is a matroid isomorphic to $N_1 = N \setminus D/C \in \mathcal{N}_m$. Moreover, it is immediate from the construction of ψ^I that $N_1 \models \psi$ iff $N_{DC} \models \psi^I$. Thus, I is an interpretation of $\mathrm{Th}_{MSO_M}(\mathcal{N}_m)$ in $\mathrm{Th}_{MSO_M}(\mathcal{N}_{DC})$. $\square$

Next, we define, for a matroid M, a *4CC-graph of* M as the graph G on the vertex set $E(M)$, and edges of G connecting those pairs of elements $e, f \in E(M)$, such that there are a 4-element circuit C and a 4-element cocircuit C' in M containing both $e, f \in C \cap C'$. (This is *not* the usual way of interpretation in which the ground set of a matroid is formed by graph edges.) The importance of our definition is in that 4CC-graphs "preserve" large grids:

Lemma 6.5. *Let $m \geq 6$ be even, and $M = M(Q_m)$. Denote by G the 4CC-graph of M. Then G has an induced subgraph isomorphic to Q_{m-2}.*

Proof. Recall that circuits in a cycle matroid of a graph correspond to graph cycles, and cocircuits to graph bonds (minimal edge cuts). The only 4-element cycles in a grid clearly are the face-cycles in the natural planar drawing of Q_m. The only edge cuts with at most 4 edges in Q_m are formed by the sets of edges incident with a single vertex in Q_m, or possibly by edges that are "close to the corners".

Let $E' \subseteq E(Q_m)$ denote the edge set of the subgraph induced on the vertices (i, j) where $1 < i, j < m$. Let G' denotes the corresponding subgraph of G induced on E'. Choose $x \in E'$, and assume up to symmetry $x = \{(i, j), (i', j')\}$ where $i' = i+1$ and $j' = j$. According to the above arguments, the only neighbors of x in G' are in the set

$$E' \cap \Big\{\{(i, j-1), (i, j)\}, \{(i, j), (i, j+1)\}, \{(i', j'-1), (i', j')\}, \{(i', j'), (i', j'+1)\}\Big\}\,.$$

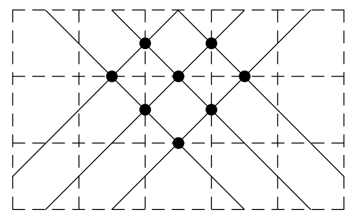

Fig. 6. An illustration to a 4CC-graph of a grid matroid (a fragment).

We now define "coordinates" for the elements $x \in E'$ as follows

$$x = \{(i,j),(i',j')\},\ i \le i', j \le j' : \qquad k_x = i + j, \qquad \ell_x = i + j' - 2j\,.$$

As one may easily check from the above description of neighbors, two elements $x, y \in E'$ are adjacent in G' if and only if $\{|k_x - k_y|, |\ell_x - \ell_y|\} = \{0,1\}$. Hence the elements $x \in E'$ such that $\frac{m}{2} + 1 < k_x, \ell_x < \frac{m}{2} + m - 1$ induce in G' a grid isomorphic to Q_{m-2}. □

Now we are to finish a chain of interpretations from Theorem 6.1 to a proof of our Theorem 6.2.

Lemma 6.6. *Let $\mathcal{M}$ be a matroid family, and let $\mathcal{F}_4$ denote the class of all adjacency graphs which are 4CC-graphs of the members of $\mathcal{M}$. Then the MSO_1 theory of $\mathcal{F}_4$ is interpretable in the theory* $\mathrm{Th}_{MSO_M}(\mathcal{M})$.

Proof. Let us take a graph $G \in \mathcal{F}_4$ which is a 4CC-graph of a matroid $M \in \mathcal{M}$. Now G is regarded as an adjacency graph structure, and so the individuals (the domain) of G are the vertices $V(G)$. These are to be interpreted in the ground set $E(M)$, the domain of M. Let ψ be an MSO_1 formula. The translation ψ^I in MSO_M of ψ is obtained simply by replacing every occurrence of the *adj* predicate in ψ with

$$adj^I(x,y) \equiv \exists C, C'$$
$$\big(|C| = |C'| = 4 \wedge circuit(C) \wedge cocircuit(C') \wedge x,y \in C \wedge x,y \in C'\big)\,,$$

where the matroid MSO_M predicates *circuit* and *cocircuit* are defined in Section 5, and $|X| = 4$ has an obvious interpretation in FO logic.

Consider the adjacency structure G^I defined by the predicate adj^I on the domain $E(M)$ of the matroid M. It is $G^I \simeq G$ by definition, for all pairs G, M as above. Moreover, adj^I is defined in MSO logic. Hence we have got an interpretation I of $\mathrm{Th}_{MSO_1}(\mathcal{F}_4)$ in $\mathrm{Th}_{MSO_M}(\mathcal{M})$. □

Proof of Theorem 6.2. Assume that a matroid class $\mathcal{N}$ does not have bounded branch-width, and denote by $\mathcal{N}_m$ the class of all matroids which are minors of some member of $\mathcal{N}$. By Theorem 3.3, for every integer $m > 1$, there is a matroid $N \in \mathcal{N}_m$ isomorphic to the cycle matroid of the grid $N \simeq M(Q_m)$. Now denote by $\mathcal{F}_4$ the class of all graphs which are 4CC-graphs of members of $\mathcal{N}_m$.

Then, using Lemma 6.5, there exist members of $\mathcal{F}_4$ having induced subgraphs isomorphic to the grid Q_k, for every integer $k > 1$.

Hence the class $\mathcal{K} = \mathcal{F}_4$ satisfies the assumptions of Theorem 6.1, and so the MSO_1 theory of $\mathcal{F}_4$ is undecidable. So is the theory $\mathrm{Th}_{MSO_M}(\mathcal{N}_m)$ using the interpretation in Lemma 6.6, and Theorem 4.2. We analogously apply the interpretation in Lemma 6.4 to $\mathrm{Th}_{MSO_M}(\mathcal{N}_m)$, and conclude that also $\mathrm{Th}_{MSO_M}(\mathcal{N}_{DC})$ is undecidable, where $\mathcal{N}_{DC}$ is the class of all DC-equipped matroids induced by $\mathcal{N}$ as above. Finally, Lemma 6.3 implies that the theory $\mathrm{Th}_{MSO_M}(\mathcal{N})$ is undecidable, as needed. □

Conversely, we may easily derive this conclusion:

Corollary 6.7. *Let $\mathbb{F}$ be a finite field, and let $\mathcal{N}$ be a class of matroids that are representable by matrices over $\mathbb{F}$. If the monadic second-order theory $\mathrm{Th}_{MSO_M}(\mathcal{N})$ is decidable, then the class $\mathcal{N}$ has bounded branch-width. Moreover, there is a class $\mathcal{T}$ of trees such that $\mathrm{Th}_{MSO_M}(\mathcal{N})$ is interpretable in $\mathrm{Th}_{MSO_1}(\mathcal{T})$.*

Sketch of proof. The first claim is just an easy reformulation of Theorem 6.2.

To show that the second part holds, we recall the definition of matroid parse trees from Section 5. Let t bound the branch-width of $\mathcal{N}$. We claim that it is enough to consider the class $\mathcal{T}$ of all $(t-1)$-boundaried parse trees of the matroids in $\mathcal{N}$. (Here we have to assume $\mathbb{F}$-representability of our matroids since parse trees could not be defined otherwise. See also in the next section.) Indeed, the claim follows from the construction leading to Theorem 5.2 by transforming the t-boundaried parse tree representations of these matroids into representations suitable for classical MSO-interpretability, as used e.g. in [2, 3]. □

Hence we have verified Conjecture 4.12 for (finite) matroids representable over finite fields.

7 Undecidability for Matroid Spikes

Although branch-width is the right measure of decidability of MSO theories of matroids representable over finite fields, the situation is quite different when considering all matroids. Here we show that matroidal obstructions to MSO decidability can be structurally very different from usual grids. For that we look at an interesting class of matroids called “spikes”, which have already shown to be a good source for hardness examples in computational complexity [49] of matroids.

We start with a formal definition of spikes. We recall that a circuit in a matroid is a minimal dependent set. Let $n \geq 3$ and S_0 be a matroid circuit on the ground set $e_0, e_1, \ldots, e_n$. Denote by S_1 an arbitrary simple matroid obtained from S_0 by adding n new elements f_i for $i \in [1, n]$ such that e_0, e_i, f_i are triangles (i.e. lie on a common line). Then the matroid $S = S_1 \setminus e_0$ obtained by deleting

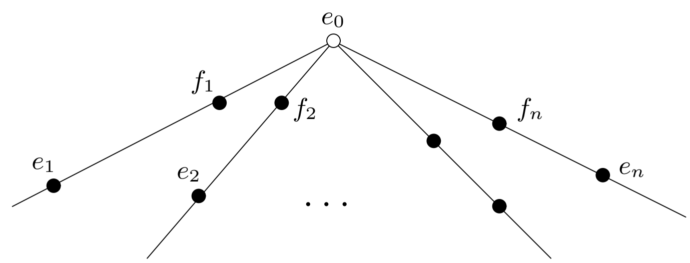

Fig. 7. An illustration to the definition of a rank-n spike. (A picture similar to the skeleton of an "$(n-1)$-dimensional umbrella".)

the central element e_0 is called a *rank-n spike*. The pairs $\{e_i, f_i\}$, $i \in [1, n]$ are called the *legs* of the spike. (Fig. 7.)

Spikes more or less explicitly appear in several research papers in structural matroid theory, first implicitly in [79] and recently, say, in [39, 49]. There seems to be no "usual definition" of a spike; the above definition was suggested by Whittle. The following simple properties of spikes are folklore in the matroid structure community.

Proposition 7.1. *Let S be a rank-n spike where $n \geq 3$. Then*

a) the union of any two legs forms a 4-element circuit in S,
b) every other circuit intersects all legs of S, and
c) branch-width of S is 3.

We are going to prove in this section that having a bounded branch-width is not a sufficient condition for a matroid class to have decidable MSO theory in general. It is even that branch-width three does not suffice.

Theorem 7.2. *Let $\mathcal{P}$ be a matroid class containing all the spikes (of any finite rank). Then the monadic second-order theory $\mathrm{Th}_{MSO_M}(\mathcal{P})$ is undecidable. (Though the branch-width of all spikes is bounded by 3.)*

This result is the best possible, since having branch-width ≤ 2 immediately implies representability over any field.

We need one technical claim about great variability of spikes for the proof.

Lemma 7.3. *Let $Z = \{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ be a set of $2n$ elements, $n > 4$.*

- *Let $\mathcal{L}$ denote the family of all the sets $\{e_i, f_i, e_j, f_j\} \subset Z$ for $1 \leq i < j \leq n$.*
- *Let $\mathcal{A}$ be an arbitrary set family which satisfies: $|\{e_i, f_i\} \triangle A| = 1$ for every $A \in \mathcal{A}$ and all $i = 1, \ldots, n$ (hence $|A| = n$), and $|A \triangle B| \geq 4$ for every distinct $A, B \in \mathcal{A}$.*
- *Let $\mathcal{D}$ denote the family of all the sets $D \subset Z$ such that $|D| = n + 1$ and D contains no subset from $\mathcal{A} \cup \mathcal{L}$.*

Then $\mathcal{A} \cup \mathcal{D} \cup \mathcal{L}$ is the collection of all circuits of some rank-n spike on the ground set Z.

Proof. A set family $\mathcal{C}$ is the collection of all circuits of a matroid if and only if the following two properties are satisfied (see for example in [57]):

- No two distinct sets in $\mathcal{C}$ are in inclusion.
- For any two distinct intersecting $X, Y \in \mathcal{C}$ and each $x \in X \cap Y$, the set $(X \cup Y) - \{x\}$ contains another set from $\mathcal{C}$.

The first condition is clearly true for $\mathcal{C} = \mathcal{A} \cup \mathcal{D} \cup \mathcal{L}$, and so we have to verify the second one.

For simplicity, we call the pairs $\{e_i, f_i\}$ the legs. Then every set in $\mathcal{A} \cup \mathcal{D}$ intersects all the legs, and every set in $\mathcal{D}$ contains some leg. Conversely, if a set Q, $|Q| = n + 1$, intersects all the legs, then either $Q \in \mathcal{D}$ or Q contains a subset from $\mathcal{A}$ by the assumptions. So if any set Q, $|Q| \geq n + 1$, intersects all the legs, then Q contains a subset from $\mathcal{A} \cup \mathcal{D}$. We solve all the possibilities as follows:

- If $X, Y \in \mathcal{L}$ and $x = e_i \in X \cap Y$, then $(X \cup Y) - \{e_i, f_i\} \in \mathcal{L}$.
- Suppose, up to symmetry, that $X \in \mathcal{L}$, $Y \in \mathcal{A} \cup \mathcal{D}$, and $x = e_i \in X \cap Y$. Then $(X \cup Y) - \{e_i\}$ intersects all the legs and has at least $n + 1$ elements.
- Suppose now that $X, Y \in \mathcal{D}$. Then either $x \in X \cap Y$ belongs to none of the legs contained in X and in Y, and so $(X \cup Y) - \{x\}$ contains a set from $\mathcal{L}$, or $(X \cup Y) - \{x\}$ intersects all the legs again.
- The case $X \in \mathcal{A}$ and $Y \in \mathcal{D}$, up to symmetry, can be reduced to the previous one since $X \not\subseteq Y$.
- Finally, suppose that $X, Y \in \mathcal{A}$. Then $|X \triangle Y| \geq 4$ and those four elements form a set from $\mathcal{L}$.

Hence the family $\mathcal{A} \cup \mathcal{D} \cup \mathcal{L}$ forms the collection of circuits of some matroid M on Z. It now easily follows from the definition of the sets $\mathcal{A}, \mathcal{D}, \mathcal{L}$ and Proposition 7.1 that M is isomorphic to a spike of rank n. $\square$

Lemma 7.3 is needed to argue that the following definition is correct. For a simple graph G on $n > 4$ vertices numbered $1, 2, \dots n$, we denote by $SR(G)$ the rank-n spike on the ground set $Z = \{e_1, \dots, e_n, f_1, \dots, f_n\}$ which is defined by Lemma 7.3 for $\mathcal{A} = \mathcal{A}_G$, where

$$\mathcal{A}_G = \big\{\{e_1, \dots, e_n\}\big\} \cup \big\{\{e_1, \dots, e_n\} \triangle \{e_i, f_i, e_j, f_j\} : ij \in E(G)\big\} \ .$$

We call $SR(G)$ the *spike representation* of G.

Lemma 7.4. *Let $M = SR(H)$ be the spike defined as above for a simple graph H such that no edge of H is incident with all other edges. Assume that C_0 is a circuit of M such that $|C_0| > 4$, and that no basis of M is contained in C_0. If every other circuit C in M contains a basis or satisfies $|C - C_0| \leq 2$, then $C_0 = \{e_1, \dots, e_n\}$.*

Proof. A circuit in a rank-r matroid having $r + 1$ elements (called a spanning circuit) must contain a basis. So one simply has to check that the required property holds for none of the circuits $C_{ij} = \{e_1, \dots, e_n\} \triangle \{e_i, f_i, e_j, f_j\}$ of $\mathcal{A}_G$ for $ij \in E(G)$. By the assumption, there are i', j' distinct from each i, j such that $i'j' \in E(G)$. However, then $C_{i'j'} \in \mathcal{A}_G$ is a circuit containing no basis, and $|C_{i'j'} - C_{ij}| = 4$. □

Let us, for an arbitrary matroid M, call a circuit C_0 satisfying all the assumptions in Lemma 7.4 the *base circuit* of M. (Not all matroids contain a base circuit, and some may contain more than one.) The lemma then says that $SR(H)$ has a unique base circuit.

Using the notion of a base circuit, we define a *base-circuit graph* $H = BG(M)$ of any matroid M as follows: The vertices of H are those $x \in E(M)$ such that $x \notin C_0 \subset E(M)$ for some base circuit C_0 of M, and the edges are those pairs $\{x, y\}$ such that there exist a base circuit C_0 and another circuit C in M for which $C - C_0 = \{x, y\}$. Notice that the graph H is empty if M has no base circuit, and H is well-defined even when M has more than one base circuit. Then the definition of a spike representation and Lemma 7.4 immediately imply:

Corollary 7.5. *Let a simple graph H be such that no edge of H is incident with all other edges. Then $BG(SR(H)) \simeq H$.* □

Now we move to the core result of this section – an interpretation of the MSO logic of adjacency graphs in the MSO logic of their spike representations, where the notion of base-circuit graphs provides the backward translation of structures (as in Figure 5).

Lemma 7.6. *Let $\mathcal{P}$ be a matroid family, and let $\mathcal{B}$ denote the class of all adjacency graphs which are base-circuit graphs of the members of $\mathcal{P}$. Then the MSO_1 theory of $\mathcal{B}$ is interpretable in the MSO_M theory of $\mathcal{P}$.*

Proof. We first express the predicate *base-circuit* in the MSO_M logic

$$\mathit{base\text{-}circuit}(C_0) \equiv |C_0| > 4 \land \forall B \left(B \subseteq C_0 \to \neg\, \mathit{basis}(B) \right) \land$$

$$\land \forall C \left[\neg\, \mathit{circuit}(C) \lor \exists B \left(\mathit{basis}(B) \land B \subseteq C \right) \lor |C - C_0| \leq 2 \right] .$$

(Here $|C_0| > 4$ and $|C - C_0| \leq 2$ have obvious FO interpretations.)

Let us now consider a matroid $M \in \mathcal{P}$ and the corresponding graph $H = BG(M)$. Notice that the individuals–vertices of H form a subset of the individuals–elements of M. The MSO translation ψ^I of a formula ψ in MSO_1 of H is obtained as follows:

- Each (bound) individual variable x in ψ is replaced with

$$\exists x\, \theta(x) \quad \longrightarrow \quad \exists x\, \exists C_0 \left(x \notin C_0 \land \mathit{base\text{-}circuit}(C_0) \land \theta(x) \right) .$$

- Set variables in ψ are replaced correspondingly.
- Every occurrence of the *adj* predicate in ψ is rewritten as

$$\mathit{adj}^I(x, y) \equiv \exists C, C_0 \left[\mathit{circuit}(C) \land \mathit{base\text{-}circuit}(C_0) \land \right.$$

$$\left. \land\, x, y \in C \land x, y \notin C_0 \land \forall z \big(z \notin C \lor z \in C_0 \big) \right] .$$

By definition, the structure M^I defined by the above interpretation I is isomorphic to the adjacency graph $H = BG(M)$. Moreover, it is clear that $H \models \psi$ iff $M \models \psi^I$. Thus, I is an interpretation of $\mathrm{Th}_{MSO_1}(\mathcal{B})$ in $\mathrm{Th}_{MSO_M}(\mathcal{P})$. □

Theorem 7.7. *Let $\mathcal{P}$ be a matroid family such that, for every planar graph F, there is a planar graph G containing F as a minor, and the spike representation $SR(G)$ is isomorphic to some member of $\mathcal{P}$. Then the monadic second-order theory $\mathrm{Th}_{MSO_M}(\mathcal{P})$ is undecidable.*

Proof. Let $\mathcal{B}$ denote the class of all adjacency graphs which are base-circuit graphs of the members of $\mathcal{P}$. Then, by Corollary 7.5 and the assumption, every planar graph F is isomorphic to a minor of some planar graph $G \in \mathcal{B}$. So we may apply Theorem 4.8 to show that $\mathrm{Th}_{MSO_1}(\mathcal{B})$ is undecidable. Hence $\mathrm{Th}_{MSO_M}(\mathcal{P})$ is undecidable by Lemma 7.6 via Theorem 4.2. □

Since the assumption of Theorem 7.7 is clearly satisfied for the class $\mathcal{P}$ of all spikes, this is actually a stronger version of Theorem 7.2. In particular, the class $\mathcal{P}$ may contain all matroids of branch-width three.

Moreover, we may now easily identify the other particular obstructions to decidability of the MSO_M theory of arbitrary (non-representable) matroids: We call *grid spikes* the spike representations $SR(Q_m)$ of graph grids.

Corollary 7.8. *Let $\mathcal{P}$ be a matroid family such that, for every $m > 0$, the grid spike $SR(Q_m)$ is isomorphic to a minor of some matroid in $\mathcal{P}$. Then the theory $\mathrm{Th}_{MSO_M}(\mathcal{P})$ is undecidable.*

Proof. Let $\mathcal{P}_m$ denote the class of all minors of members of $\mathcal{P}$. Since every planar graph is a minor of a sufficiently large grid, the assumptions of Theorem 7.7 are satisfied for $\mathcal{P}_m$. Hence the theory $\mathrm{Th}_{MSO_M}(\mathcal{P}_m)$ is undecidable. Now it is enough to recall Lemmas 6.3 and 6.4 to argue that also $\mathrm{Th}_{MSO_M}(\mathcal{P})$ is undecidable. □

8 Conclusions and remarks

Using our main result, Theorem 6.2, it is not difficult to deduce the above discussed recent Theorem 4.18 ([33]) of Courcelle and Oum. The basic idea to do this was developed in [33]: First arbitrary graphs were reduced to bipartite graphs using interpretability, and then bipartite graphs were reduced via a C_2MS interpretation to binary matroids, i.e. matroids represented by vectors over $GF(2)$. The point of this reduction is in that the starting graphs have bounded clique-width if and only if the resulting binary matroids have bounded branch-width. Now the ingenious step here is the observation that for binary matroids, having a representation by $\{0, 1\}$ vectors, linear independence can be reduced to a check of the parity of the coordinates, which can be easily described via the predicate $Even(X)$ in MSO-logic. So Conjecture 4.12 is proved for C_2MS-logic instead of MSO-logic. Unfortunately this result does not imply the whole conjecture, as the next theorem shows.

Theorem 8.1. *There exists a class* $\mathcal{T}$ *of trees with* $\mathrm{Th}_{C_2MS}(\mathcal{T})$ *undecidable but* $\mathrm{Th}_{MSO}(\mathcal{T})$ *decidable.*

This is interesting, since for binary trees the predicate $Even(X)$ is definable in MSO-logic, what follows from the observation of Courcelle [22], that $Even(X)$ is definable for each structure for which a linear ordering of the domain is definable.

The idea of the proof is to code the Halting set for a universal Turing machine into the structure of the members of a class of countable trees in such a way that the set can be recognized via a simple sentence in C_2MS-logic. This makes the C_2MS-theory of this class undecidable. On the other hand this set of trees can be constructed in such a way that it has a decidable MSO-theory. Here the idea is to reduce this structures via Ehrenfeucht-Fraisse-games to very simple periodic structures with a decidable theory (see [53] for details).

From the previous section we see that bounding the branch-width is not sufficient to get a decidable MSO_M theory of all matroids. The above presented results naturally lead to the following important questions.

Question 8.2. Regarding Corollary 7.8:

- What is a structural description of the classes of all matroids excluding big grids and big grid spikes as minors?
- What can be said about the decidability of the MSO_M for such classes?
- Alternatively, what other obstructions should be excluded to get a decidable MSO_M theory of a matroid class including also non-representable ones?
- Is such a theory interpretable in a class of trees (c.f. Conjecture 4.12)?

We do not have any good answer to these questions, not even a plausible conjecture, since we are only at the start of a new and exciting research area. Indeed we do not expect easy answers here since the questions combine the (difficult) area of structural matroid theory with fundamental problems of logic. However, a possible way to the right answers may be shown via a suitable combination of the notions of a matroid "monarchy" (suggested by Edmonds [2002, private communication]), and a "shadow" of a separation in a matroid. We try to briefly outline our ideas here.

A matroid family $\mathcal{M}$ is called a *monarchy* if, for each $r > 0$, there is a unique (up to isomorphism) maximal matroid of rank r in $\mathcal{M}$. For example, in the class of all simple graphs the monarch of rank r is the complete graph K_{r+1}. For simple $GF(q)$-representable matroids of rank r the monarch is the rank-r projective geometry over $GF(q)$. On the other hand, all simple matroids do not form a monarchy since already the n-element lines $U_{2,n}$ do not have a maximal one.

To sketch the notion of a shadow, let us consider a partition (L, R) (a *separation*) of the ground set of a matroid M. We define on the family of all subsets $2^L \cup 2^R$ an equivalence relation $\sim$ as follows. Let us shortly denote by $i(X, Y) = \mathrm{r}_M(X) + \mathrm{r}_M(Y) - \mathrm{r}_M(X \cup Y)$.

- $X \sim Y$ for $X, Y \subseteq L$ iff $i(X, R) = i(Y, R) = i(X \cup Y, R)$, and analogously for $X, Y \subseteq R$.

– $X \sim Y$ for $X \subseteq L$ and $Y \subseteq R$ iff $i(X,R) = i(X,Y) = i(Y,L)$.

Then it can be shown that the equivalence classes of $\sim$ determine the collection of flats of a certain matroid $M_{(L,R)}$ of rank $\lambda_M(L) = \lambda_M(R)$ that we call the *matroid shadow* of the separation (L,R) in M. (Geometrically, the flats of the shadow $M_{(L,R)}$ are determined by intersections of the spans $\langle F \rangle$ of all flats F of M with the guts – the subspace $\langle L \rangle \cap \langle R \rangle$, of the separation (L,R).)

For an integer t and a matroid monarchy $\mathcal{M}$, we say that the branch-width of a matroid N is *bounded by t and* $\mathcal{M}$ if there exist a branch decomposition (T,τ) of width $\leq t$ such that the shadows of all separations displayed in the tree T belong to the monarchy $\mathcal{M}$. (For $\mathcal{M}$ being the class of all $GF(q)$-representable matroids, and N also $GF(q)$-representable, this definition reduces to ordinary branch-width. On the other hand, there are spikes whose branch-width is not bounded by any monarchy.) We shortly say that the branch-width of a matroid class $\mathcal{N}$ is *bounded with respect to a monarchy* $\mathcal{M}$ if there is a t such that the branch-width of each matroid in $\mathcal{N}$ is bounded by t and $\mathcal{M}$.

A direct extension of the ideas from [47] leads to a possible definition of parse trees for all matroids of branch-width bounded by t and a monarchy $\mathcal{M}$, and to a corresponding extension of Theorem 5.2. Hence we can prove that the MSO_M of the classes of all matroids of branch-width bounded with respect to a monarchy $\mathcal{M}$ are decidable.

Unfortunately, one of the problems with the above sketched ideas is that the class of all free spikes (i.e. the spikes defined by Lemma 7.3 with $\mathcal{A} = \emptyset$) has unbounded branch-width with respect to any monarchy, but the MSO_M theory of the free spikes is decidable since it is quite trivial. Hence our notion of branch-width bounded with respect to a monarchy seems to be too restrictive if we want to find the right borderline between decidable and undecidable matroid MSO_M theories. We neverthless continue our research in this direction.

Altogether, with the above mentioned results of Courcelle and Oum from [33], our main result looks like a significant step toward a solution of Conjecture 4.12, but to substitute $\mathrm{C_2MS}$ by MSO in Theorem 4.18 and to prove it for countable structures of arbitrary finite signature is still open. To prove the conjecture for arbitrary structures it could be of interest to show that for each class $\mathcal{K}$ of countable structures there is a class $C(\mathcal{K})$ of simple graphs such that $\mathrm{Th}_{MSO}(\mathcal{K})$ is interpretable in $\mathrm{Th}_{MSO}\big(C(\mathcal{K})\big)$ and $\mathrm{Th}_{MSO}(\mathcal{K})$ is decidable if and only if $\mathrm{Th}_{MSO}\big(C(\mathcal{K})\big)$ is decidable, i.e. graphs are universal with respect to MSO logic and decidability (see [45] for a related notion of universality for elementary theories).

With respect to the general conjecture for graphs it could be of interest to show the following. Assume that the tree-width of a class $\mathcal{K}$ of countable graphs is unbounded. Show that then also $\mathrm{Th}_{MSO_{\forall 1}}(\mathcal{K})$ and $\mathrm{Th}_{MSO_{\exists 1}}(\mathcal{K})$, i.e. the theories of all formulas satisfied in $\mathcal{K}$ with only one universal or only one existential set quantifier in a prenex form, are undecidable. Such a result could give an essential strengthening of Theorem 4.8.

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References

1. S. Arnborg, D.G. Corneil, A. Proskurowski, *Complexity of Finding Embeddings in a k-tree*, SIAM J. Alg. Disc. Methods 8 (1987), 277–284.
2. S. Arnborg, J. Lagergren, D. Seese, *Problems easy for tree-decomposable graphs (extended abstract)*, Proc. 15th ICALP, Springer, LNCS 317, (1988) pp. 38-51.
3. S. Arnborg, J. Lagergren, D. Seese, *Easy Problems for Tree-Decomposable Graphs*, Journal of Algorithms 12 (1991), 308–340.
4. K. Barthelmann, *On equational graphs*, Tech. Rep. 9, Universitt Mainz, Institut fr Informatik, 1997.
5. A. Baudisch, *Zur Methatheorie abelscher Gruppen (translation: to the meta-theory of abelian groups*, Dissertation, Humboldt-Universitaet zu Berlin,1974.
6. A. Baudisch, D.G. Seese, P. Tuschik, M. Weese, *Decidability and Generalized Quantifiers*, Akademie-Verlag, Berlin, (1980), xii+235 pp.
7. A. Baudisch, D.G. Seese, P. Tuschik M. Weese, *Decidability and quantifier-elimination*, Chapter VII in J. Barwise, S. Feferman (Eds.): Model-Theoretic Logics; Springer-Verlag New York 1985., 1985, pp. 235-268.
8. A. Blumensath, *Axiomatising Tree-interpretable Structures*, Tech. Rep. AIB-2001-10, RWTH Aachen, 2001, pp. 1–23, `http://www-mgi.informatik-.rwth-aachen.de/~blume/Publications/2001-10.ps.gz`.
9. A. Blumensath, *A model theoretic characterization of clique width*, preprint RWTH Aachen, 2002 (see `http://www-mgi.informatik.rwth-aachen.de-/~blume/Publications.html`).
10. A. Blumensath, *Axiomatising tree-interpretable structures*, In Proc. 19th Int. Symp. on Theoretical Aspects of Computer Science, LNCS 2285, 2002, pp. 596–607, Springer-Verlag.
11. H.L. Bodlaender, *A Linear Time Algorithm for Finding Tree-Decompositions of Small Treewidth*, SIAM J. Computing 25 (1996), 1305–1317.
12. E. Börger, E. Grädel, Y. Gurevich, *The Classical Decision Problem*, Springer-Verlag, Berlin 1997.
13. J.R. Büchi, *Weak second-order arithmetic and finite automata*, ZML, 6, 1960, pp. 66–92.
14. J.R. Büchi, *The monadic second order theory of ω_1*, In: Decidable Theories, II: Monadic Second Order Theory of All Countable Ordinals, edited by G.H. Müller and D. Siefkes. Springer-Verlag Lecture Notes in Mathematics, vol. 328 (1973), 1–127.

15. J.R. Büchi, D. Siefkes, *Axiomatization of the monadic second order theory of* ω_1, In: Decidable Theories, II: Monadic Second Order Theory of All Countable Ordinals, edited by G.H. Müller and D. Siefkes. Springer-Verlag Lecture Notes in Mathematics, vol. 328 (1973), 129–217.
16. J.R. Büchi, D. Siefkes, *Axiomatization of the monadic second order theory of* ω_1, In: Decidable Theories, II: Monadic Second Order Theory of All Countable Ordinals, edited by G.H. M"'uller and D. Siefkes. Springer-Verlag Lecture Notes in Mathematics, vol. 328 (1973), pp. 129 – 217.
17. B. Courcelle, *The decidability of the monadic second order theory of certain sets of finite and infinite graphs*, LICS'88, Logic in Computer Science, Edinburg, 1988.
18. B. Courcelle, *The monadic second order theory of graphs III: Tree decompositions, minors, and complexity issues*, Information and Computation 85 (1990), 12–75. In: Theoret. Inform. Applic., Vol. 26 (2) (1992) pp. 257-286.
19. B. Courcelle, *The monadic second order logic of graphs VI: on several representations of graphs by relational structures*, Discrete Appl. Math. 54 (1994) 117–149. Erratum 63 (1995) 199–200.
20. B. Courcelle, *Monadic second order definable graph transductions: a survey*, Theoret. Comput. Sci. 126 (1994), no. 1, 53–75, Seventh Colloquium on Trees in Algebra and Programming (CAAP'92) and European Symposium on Programming (ESOP) (Rrennes, 1992).
21. B. Courcelle, *Structural properties of context-free sets of graphs generated by vertex replacement*, In: Information and Computation, 116, 1995, pp. 275–293.
22. B. Courcelle, *The monadic second order theory of graphs X: Linear orderings*, In: Theoret. Computer Sci., Vol. 160 (1996) pp. 87–144.
23. B. Courcelle, *The expression of graph properties and graph transformations in monadic second-order logic*, Handbook of graph grammars and computing by graph transformation, Vol. 1, World Sci. Publishing, River Edge, NJ, 1997, pp. 313–400.
24. B. Courcelle, *The monadic second-order logic of graphs XII: Planar graphs and planar maps*, In: Theoret. Comput. Sci. 237, pp. 1 - 32, 2000.
25. B. Courcelle, *Clique-width of countable graphs: a compactness property*, International Conference of Graph Theory, Marseille, 2000. Available from `http://dept-info.labri.u-bordeaux.fr/~courcell/Cwd-Inf.ps`.
26. B. Courcelle, *The monadic second-order logic of graphs XIV: Uniformly sparse graphs and edge set quantification*, Theoretical Computer Science 299 (2003) 1–36.
27. B. Courcelle, *The Monadic Second-Order Logic of Graphs XV: On a Conjecture by D. Seese*, manuscript, LaBri, Bordeaux 1 University, March 8, 2004, 1–40.
28. B. Courcelle, J. Engelfriet, G. Rozenberg, *Handle-rewriting hypergraph grammars*, Journal of Computer ans System Sciences, 46, 1993, pp. 218–270.
29. B. Courcelle, J. Engelfriet, *A logical characterization of the sets of hypergraphs defined by hyperedge replacement grammars*, Math. Systems Theory 28 (1995), no. 6, pp. 515–552.
30. B. Courcelle, J.A. Makowsky, U. Rotics, *Linear Time Solvable Optimization Problems on Graphs of Bounded Clique-Width*, Theory Comput. Systems 33 (2000) 125–150.
31. B. Courcelle, M. Mosbah, *Monadic second-order evaluations on tree-decomposable graphs*, Theoret. Comput. Sci. 109 (1993), 49–82.
32. B. Courcelle, M. Olariu, *Upper bounds to the clique-width of graphs*, Discrete Applied Mathematics 101, 2000, pp. 77 - 114.
33. B. Courcelle, Sang-Il Oum, *Vertex-minors, monadic second-order logic, and a conjecture by Seese*, preprint July 23, 2004, pp. 37.

34. H.-D. Ebbinghaus, J. Flum, W. Thomas, Mathematical Logic, Springer Verlag, 2nd edition, 1994.
35. H.-D. Ebbinghaus, J. Flum, Finite Model Theory, Springer Verlag, 1995.
36. S. Feferman, R.L. Vaught, *The first-order properties of algebraic systems*, Fundamenta Mathematicae, vol. 47, 1959, pp. 57–103.
37. M.R. Garey, D.S. Johnson, *Computers and Intractability*, Bell Telephone Laboratories 1979.
38. S. Garfunkel, J.H Schmerl, *The undecidability of theories of groupoids with an extra predicate*, Proc. Amer. Math. Soc. 42 (1974), pp. 286–289.
39. J.F. Geelen, A.H.M. Gerards, G.P. Whittle, *Branch-Width and Well-Quasi-Ordering in Matroids and Graphs*, J. Combin. Theory Ser. B 84 (2002), 270–290.
40. J.F. Geelen, A.H.M. Gerards, N. Robertson, G.P. Whittle, *On the Excluded Minors for the Matroids of Branch-Width k*, J. Combin. Theory Ser. B 88 (2003), 261–265.
41. J.F. Geelen, A.H.M. Gerards, G.P. Whittle, *Excluding a Planar Graph from a $GF(q)$-Representable Matroid*, manuscript, 2003.
42. Y. Gurevich, *Monadic second-order theories*, In: Chapter XIII from J. Barwise, S. Feferman: Model-Theoretic Logics; Springer-Verlag New York 1985, 479–506.
43. Y. Gurevich, M. Magidor, S. Shelah, *The monadic theory of ω_2*, In: Journal of Symbolic Logic, vol. 48 (1983), pp. 387 - 398.
44. H. Hauschild, H. Herre, W. Rautenberg, *Entscheidbarkeit der monadischen Theorie 2. Stufe der n-separierten Graphen (translation: Decidability of the monadic second order theory of n-separated graphs)*, In: Wissensch. Zeitschr. der Humboldt-Universitaet, Math.-Nat. R. XXI (1972) 5, pp. 507–511.
45. H. Hauschild, W. Rautenberg, *Interpretierbarkeit in der Gruppentheorie (translation: interpretability in the theory of groups)*, In: algebra universalis, Birkhaeuser Verlag Basel, vol. 1, fasc. 2, 1971, pp. 136–151.
46. H. Herre, *Unentscheidbarkeit in der Graphentheorie (translated: undecidability in graph theory*, In: Bulletin de L'Academie Polonaise des Sciences, Serie des Sciences math., astr. et phys. - Vol. XXI, No. 3, 1973, pp. 201–208.
47. P. Hliněný, *Branch-Width, Parse Trees, and Monadic Second-Order Logic for Matroids*, submitted.
Extended abstract in: STACS 2003, Lecture Notes in Computer Science 2607, Springer Verlag (2003), 319–330.
48. P. Hliněný, *On Matroid Properties Definable in the MSO Logic*, In: Math Foundations of Computer Science MFCS 2003, Lecture Notes in Computer Science 2747, Springer Verlag Berlin (2003), 470–479.
49. P. Hliněný, *It is Hard to Recognize Free Spikes*, submitted, 2002.
50. P. Hliněný, *A Parametrized Algorithm for Matroid Branch-Width*, submitted, 2002.
51. P. Hliněný, *On the Computational Complexity of Matroid Minors*, manuscript 2004, `http://www.cs.vsb.cz/hlineny/papers`.
52. P. Hliněný, D. Seese, *Decidability of MSO Theories of Representable Matroids*, In: Parameterized and Exact Computation, Proceedings of the First International Workshop on Parameterized and Exact Computation, IWPEC 2004, Bergen, Norway, September 14 - 17, 2004, Rod Downey, Michael Fellows, Frank Dehne (Eds.), Lecture Notes in Computer Science 3162, Springer Verlag Berlin (2004), 96–107.
53. P. Hliněný, D. Seese, *Interpretability and decidability for monadic theories of graphs and matroids*, In preparation 2005.
54. P. Hliněný, G.P. Whittle, *Matroid Tree-Width*, submitted, 2003.
Extended abstract in: Eurocomb'03, ITI Series 2003–145, Charles University, Prague, Czech Republic, 202–205.

55. P. Lapoire, *Recognizability equals Monadic Second-Order definability for sets of graphs of bounded tree-width*, Proceedings 15th Annual Symposium on Theoretical Aspects of Computer Science Paris, France, February 1998, STACS, 1998, M. Morvan, Ch. Meinel, Da. Krob (Eds.), Springer Lecture Notes in Computer Science 1373, 618–628.
56. J.A. Makowsky, *Algorithmic uses of the Feferman-Vaught Theorem*, Annals of Pure and Applied Logic 126 (2004), pp. 159–213.
57. J.G. Oxley, Matroid Theory, Oxford University Press, 1992.
58. Sang-Il Oum, P. Seymour, *Approximation Algorithm to the Clique-Width of a Graph*, manuscript, 2003.
59. Sang-Il Oum, *Rank-Width and Well-Quasi-Ordering*, manuscript, 2004.
60. M.O. Rabin, *A simple method for undecidability proofs and some applications*, Logic, Methodology and Philosophy of Sciences 1 (ed. Bar-Hillel), North-Holland, Amsterdam 1964, 58–68.
61. M.O. Rabin, *Decidability of second order theories and automata on infinite trees*, In: Trans. Am. Math. Soc., 141, 1969, 1–35.
62. M.O. Rabin, *Decidable theories*, In: Chapter C.3 of Handbook of Mathematical Logic, J. Barwise, North-Holland Publ.Co. 1977, 595–629.
63. W. Rautenberg, *Lectures on decidability theory*, Banach Center Warsaw, Logical Semester Warsaw 1973, Lecture I - VII, Warsaw 1973, pp. 67.
64. N. Robertson, P.D. Seymour, *Graph Minors – A Survey*, Surveys in Combinatorics, Cambridge Univ. Press 1985, 153–171.
65. N. Robertson, P.D. Seymour, *Graph Minors X. Obstructions to Tree-Decomposition*, J. Combin. Theory Ser. B 52 (1991), 153–190.
66. N. Robertson, P.D. Seymour, *Graph Minors XX. Wagner's Conjecture*, submitted.
67. S. Shelah, *The monadic theory of order*, Annals of Mathematics, 102 (1975), 375–419.
68. D. Seese, *Entscheidbarkeitsfragen der Theorien 2. Stufe einiger "netzartiger" Graphen (translation: Questions of decidability of second order theories of some "netlike" graphs)*, Diploma thesis, Humboldt-University, 1972, pp. 45.
69. D. Seese, *Entscheidbarkeits- und Definierbarkeitsfragen der Theorie "netzartiger" Graphen - I(translation: Questions of decidability and definability of the theory of "'netlike"' graphs - I)*, Wissenschaftliche Zeitschrift der Humboldt-Universitaet, Math.-Nat. R. XXI, 1972, pp. 513–517.
70. D. Seese, *Zur Entscheidbarkeit der monadischen Theorie 2. Stufe baumartiger Graphen (translation: To the decidability of the monadic second order theory of graphs with a similarity to trees)*, Wissenschaftliche Zeitschrift der Humboldt-Universitaet zu Berlin, Math.-Nat. R. XXIV, 1975, 6, pp. 768–772.
71. D. Seese, *Ein Unentscheidbarkeitskriterium (translation: A criterion of undecidability)*, Wissenschaftliche Zeitschrift der Humboldt-Universitaet zu Berlin, Math.-Nat. R. XXIV, 1975, 6, pp. 772–780.
72. D. Seese, *Entscheidbarkeits- und Interpretierbarkeitsfragen monadischer Theorien zweiter Stufe gewisser Klassen von Graphen (translation to English: Questions of decidability and interpretability of monadic second order theories of certain classes of graphs)*, Dissertation, Humboldt-Universitt zu Berlin, 1976, pp. iv+172.
73. D. Seese, *Decidability of ω-trees with bounded sets - a survey*, In: Fundamentals of Computation Theory, edited by M. Karpinski. Springer-Verlag Lecture Notes in Computer Science, vol. 56 (1977), pp. 511–515.
74. D. Seese, *Decidability of ω-trees with bounded sets)*, Akademie der Wissenschaften der DDR, Zentralinstitut für Mathematik und Mechanik, (1978); Report R-01/78, pp. 1–52.

75. D. Seese, *Ordered tree representations of infinite graphs*, Akademie der Wissenschaften der DDR, Karl-Weierstrass-Institut fuer Mathematik, Preprint P-MATH-12/88, Berlin 1988, pp. 1–50.
76. D. Seese, *The structure of the models of decidable monadic theories of graphs*, Annals of Pure and Applied Logic 53 (1991), pp. 169–195.
77. D. Seese, *Interpretability and Tree Automata: a simple way to solve Algorithmic Problems on Graphs closely related to Trees*, In: Tree Automata and Languages, M. Nivat and A. Podelski (editors), 1992, Elsevier Sceince Publishers, pp. 83–114.
78. S. Shelah, *The monadic theory of order*, In: Annals of Mathematics, vol. 102 (1975), pp. 379–419.
79. P.D. Seymour, *Recognizing Graphic Matroids*, Combinatorica 1 (1981), 75–78.
80. H. Wang, *Proving theorems by pattern recognition II*, In: Bell System Technical Journal, 40, 1961, pp. 1–41.