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Width Parameters of Matroids

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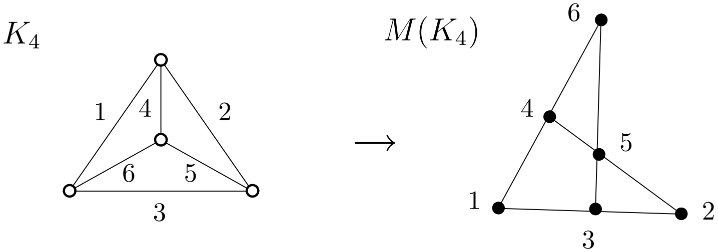
1 Introduction

Question: What really are **matroids**?

- A common combinatorial generalization of graphs and finite geometries.
- A new look at (some) structural graph properties.



Question: What can matroids bring into theoretical CS?

- So far not of deep interest among computer scientists. . .
- But, some interesting (and even surprising) applications and relations with important graph problems have been found recently!

Answers: We show two examples where matroids have given new view of algorithmically important graph width parameters. . .

Definition

A **matroid** M on E is a set system $\mathcal{B} \subseteq 2^E$ of *bases*, sat. the exchange axiom

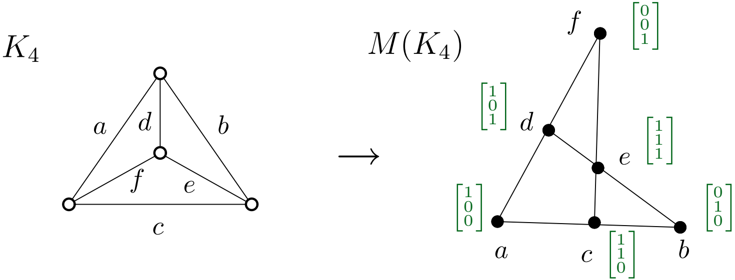
$$\forall B_1, B_2 \in \mathcal{B} \text{ a } \forall x \in B_1 - B_2,\ \exists y \in B_2 - B_1 :\ (B_1 - \{x\}) \cup \{y\} \in \mathcal{B}\,.$$

The subsets of bases are called *independent*.

Representations by graphs and vectors

Cycle matroid of a graph $M(G)$ – on the edges of G, where acyclic sets are independent.

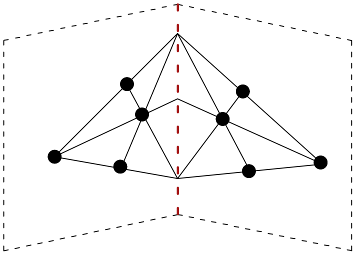
Vector matroid of a matrix $M(\boldsymbol{A})$ – on the (column) vectors of $\boldsymbol{A}$, with usual linear independence.



Matroid rank

The *rank function* of a matroid M is $\mathrm{r}_M : 2^{E(M)} \to \mathbb{N}$ where

$$\mathrm{r}_M(X) = \max\{|I| : \text{ independent } I \subseteq X\}.$$



Connectivity

The *connectivity function* of M is $\lambda_M : 2^{E(M)} \to \mathbb{N}$ where

$$\lambda_M(X) = \mathrm{r}_M(X) + \mathrm{r}_M(E - X) - \mathrm{r}(M) + 1\,.$$

Geometrically, $\lambda_M(X)$ is the rank of the intersection of the spans of X and $E - X$ plus 1.

(In graphs, $\lambda_G(X)$ equals the number of vertices shared between X and $E-X$.)

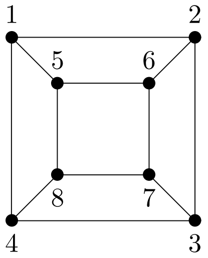
2 The First Example: TREE-WIDTH

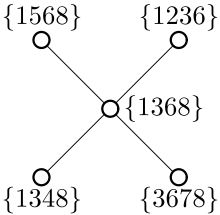
– Introduced [Robertson + Seymour, 80's] – the "Graph minor" project.

Definition: Tree *decomposition* of a graph G

- "bags" (subsets) of vertices at the tree nodes,
- each edge of G belongs to some bag, and
- the bags containing some vertex form a subtree (interpolation).

Tree-width $= \min_{\text{decomposition } G} \max\{|B|-1 : B \text{ bag in a decomp.}\}$.





– Alternative tree-width definitions; some even before R+S. . .
(For ex. by linear ordering of vertices, cf. simplicial decomposition.)

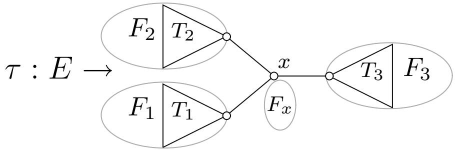
– The notion appears in many areas and relations, mainly algorithmic.
In parametrized computation – linear-time FPT [Bodlaender, 96].

A "vertex-free" definition

– Proposed by [PH + Whittle, 03].

A tree *edge decomposition* of a graph G

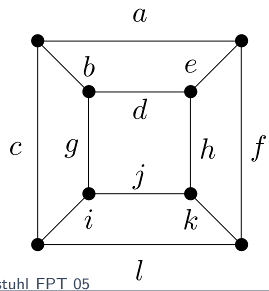
– arbitrary $\tau : E(G) \to V(T)$, without further restrictions.

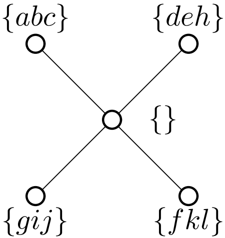


Node with of x $= |V(G)| + (d-1) \cdot c(G) - \sum_{i=1}^{d} c(G - F_i)$,

where F_i mapped to the comps. of $T - x$, and $c()$ the number of comps.

VF Tree-width $= \min_{\text{decomposition } G} \max(\,\text{node width in a decomp.}\,)$.



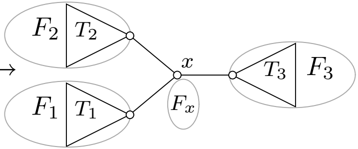


2.1 Matroid Tree-Width

– Introduced [PH + Whittle, 03] (following [Geelen, unpublished]).

Tree *decomposition* of a matroid M

– arbitrary $\tau : E(M) \to V(T)$, without further restrictions.

$\tau : E \longrightarrow$ 

Node width of $x = \sum_{i=1}^{d} \mathrm{r}_M\left(E(M) - F_i\right) - (d-1)\cdot \mathrm{r}(M).$

(M) Tree-width $= \min_{\text{decomposition } M} \max(\text{node width in a decomp.})$.

Theorem [PH + Whittle, 03]. If a matroid M has tree-width k and branch-width b, then $b - 1 \leq k \leq \max(2b-1, 1)$.

Theorem [PH + Whittle, 03]. Let a graph G has an edge, and $M = M(G)$ be the cycle matroid. Then the tree-width of G equals the tree-width of M.

So our VF tree-width definition seems OK. . .

Computing matroid tree-width

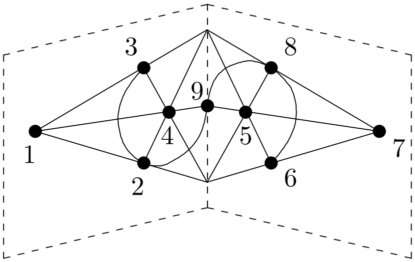
Is tree-width a "*good*" structural parameter?
(Can we compute the tree-width and a corresponding decomposition if bounded?)

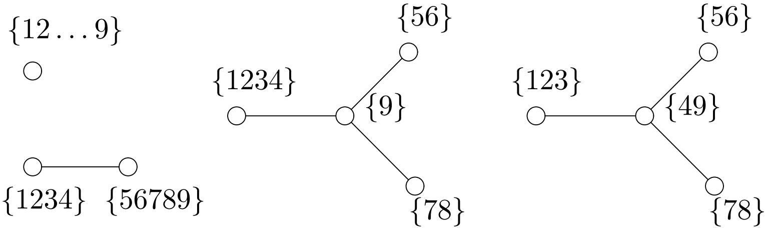
Theorem [PH, 03]. Computing the tree-width of a matroid represented over a finite field is FPT in $O(n^3)$.

A sketch:
tree-width $\rightarrow$ branch-width of the matroid $\rightarrow$ solved in FPT $O(n^3)$ by [PH, 02] $\rightarrow$ test the excluded minors for small tree-width.

(A decomposition is computed only approximately. . .)

Examples of matroid tree decompositions





widths: 4,3 3 4

3 The Second Example: CLIQUE-WIDTH

– Introduced [Courcelle + Olariu, 00] (implicitly [Courcelle et al, 93]).

Definition: A construction of a vertex-labeled graph G using

- creating a new (labeled) vertex,
- a disjoint union of graphs,
- relabeling of all vertices labeled i to labels j,
- adding all edges between vertices of labels i and labels j.

Clique-width $= \min$ number of labels needed to construct G (arb. labeling).

– Extends older *cographs* (graphs without induced P_4), which are constructed using disjoint and "complete" unions of smaller cographs (ie. clique-width $= 2$).

– Bounded clique-width allows efficient parametrized solutions of problems described in the MSO logic of adjacency graphs (called MS_1) – quantif. only over vertices and their sets. [Courcelle, Makowsky, Rotics, 00]

(Bounded tree-width allows efficient parametrized solutions of problems in MS_2 of incidence graphs.)

Comparing those widths

– Mostly recently found relations. . .

Tree-width	*Clique-width*
$\mathrm{tw}(K_n) = n-1$ $>>$	$\mathrm{clw}(K_n) = 2$
NPC, but linear FPT – a good parameter	NPC, but parametrized **???** a recent FPT approximation – possibly bad, but not "ugly"!
decidability of MS_2 theory	decidability of $\sim\mathrm{MS}_1$ theory
minors	vertex minors
excluding a large grid Q_n	excl. large "bipartized" grid S_n (only over bipartite graphs)
GM theory (Robertson+Seymour)	**???** so far WQO for bd. clique-width

A common denominator – branch-width of binary matroids.

3.1 Computing Clique-Width (via Rank-Width)

The notion of **rank-width**:

- Introduced by [Oum + Seymour, 03], related to prev. [Bouchet 93].
- Definition: see [Oum - Dagstuhl 05].
  Similar to branch-width, but decomposing vertices, and using a *"cut-rank"* function (the binary rank of the matrix ind. by the edges across vertex separation).

Theorem [Oum + Seymour, 03].
If a graph G has rank-width r and clique-width c, then $r \le c \le 2^{r+1} - 1$.

Computing rank-width

- The only known way to efficiently *approximate bounded clique-width*.
- The first algorithm $O(n^9)$ [Oum + Seymour, 03], now improved $O(n^4)$.
  (The decomposition is only approximate – factor 3.)
- A faster indirect algorithm via matroids:
  r-w of graphs $\rightarrow$ bipartite graphs $\rightarrow$ branch-width of binary matroids $\rightarrow$ solved in FPT $O(n^3)$ by [PH, 02] $\rightarrow$ back to graph rank-decompositions $\rightarrow$ testing excluded vertex minors for small rank-width [Oum, 04].

Theorem [Oum, 05]. Computing rank-width of a graph is FPT in $O(n^3)$.

3.2 Decidability of MSO theories

(Decidability $\sim$ model-checking...)

– A *theory* $\mathrm{Th}_L(\mathcal{K})$ — a class of structures $\mathcal{K}$, plus a language L (logic).

– *Decidability* of a theory — can algorithmically decide whether $\varphi \in L$ is true on all $K \in \mathcal{K}$, ie. whether $\mathcal{K} \models \varphi$.

MSO theory – L *monadic second-order* logic, which means quantification over the (structure) elements and their sets, but not over larger predicates.

** MSO decidability ?⇔? efficient decision of MSO-definable properties: **

No direct formal relation known so far, but the relevant results usually come hand in hand in considered structures...

– *Graphs:* CMS_2 theory (incidence graphs plus modulo-counting) decidable on the graphs of bounded tree-width, [Courcelle 88].
Decidability of MS_2 theory implies bounded tree-width, [Seese 91].

– Analogously for *matroids:*

Theorem [PH 02]. CMSO theory of matroids of bounded tree-width over a finite field is decidable.

Thm. [PH + Seese, 04]. Decidability of matroid MSO theory implies bounded tree-width (using matr. "excl. grid" [Geelen, Gerards, Whittle, 03]).

Similar for clique-width

Theorem [Courcelle, Makowsky, Rotics, 00]. CMS_1 theory (adjacency graphs plus mod-counting) is decidable on graphs of bounded clique-width.

Theorem [Courcelle + Oum, 04] Decidability of $\mathrm{C}_2\mathrm{MS}_1$ theory (with parity-counting) implies bounded clique-width.

A proof sketch:

unbounded clique-width on graphs $\rightarrow$ unbounded rank-width $\rightarrow$
$\rightarrow$ interpretation in bipartite graphs $\rightarrow$ interpretation in binary matroids $\rightarrow$
$\rightarrow$ undecidability of matroidal MSO for unbounded branch-width.

Seese's conjecture

– Formulated [Seese 91]:

Let $\mathcal{K}$ be a class of countable structures with decidable MSO theory. Then there is a class $\mathcal{T}$ of trees such that $\mathrm{Th}_{MSO}(\mathcal{K})$ is interpretable in $\mathrm{Th}_{MSO}(\mathcal{T})$.

($\rightarrow$ Decidability of the (tree) MSO theory S2S by [Rabin 69].)

– A new evidence for the conjecture in matr. MSO and (almost) graph MS_1.

4 Conclusions

- We have show that a matroidal (geometric) view can bring new and interesting notions and properties of ordinary graphs.
- Matroids provide a new nontrivial evidence for Seese's conjecture.
- Our research tries to contribute to (nowadays popular) extensions of the Graph-Minor project from graphs to matroids. . .

- And finally – **MACEK** [PH 01–05],
  a software tool for practical structural computations with matroids:

  `http://www.cs.vsb.cz/hlineny/MACEK`

  (Now with a new online interface - TRY IT yourself!)