Planar Covers of Graphs: Negami's Conjecture

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PLANAR COVERS OF GRAPHS: NEGAMI'S CONJECTURE

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CONTENTS

GLOSSARY OF SYMBOLS

$\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}$	graphs
$V(\boldsymbol{G})$	vertex set of a graph $\boldsymbol{G}$
$E(\boldsymbol{G})$	edge set of a graph $\boldsymbol{G}$
$F(\boldsymbol{G})$	set of faces of an embedded graph $\boldsymbol{G}$
uv	a shortcut for an edge with ends u and v
$d_G(v)$	degree of a vertex v in $\boldsymbol{G}$
$N_G(v)$	neighbors of a vertex v in $\boldsymbol{G}$
$\boldsymbol{K}_n$	complete graph on n vertices
$\boldsymbol{K}_{n_1,\ldots,n_p}$	complete p-partite graph with parts of sizes $n_1, \ldots, n_p$
$\boldsymbol{C}_n$	cycle on n vertices
$\boldsymbol{P}_n$	path on $n+1$ vertices
$\boldsymbol{F} \simeq \boldsymbol{G}$	$\boldsymbol{F}$ and $\boldsymbol{G}$ are isomorphic
$\boldsymbol{F} \subseteq \boldsymbol{G}$	$\boldsymbol{F}$ is a subgraph of $\boldsymbol{G}$
$\boldsymbol{G} \restriction V$	the subgraph of $\boldsymbol{G}$ induced on the vertex set V

$\boldsymbol{G} - X$	deletion of a set of vertices X from $\boldsymbol{G}$
$\boldsymbol{G} - F$	deletion of a set of edges F from $\boldsymbol{G}$
$\boldsymbol{G} - v$	deletion of a vertex v from $\boldsymbol{G}$
$\boldsymbol{G} - e$	deletion of an edge e from $\boldsymbol{G}$
$\boldsymbol{G} + e$	addition of an edge e to $\boldsymbol{G}$
$\boldsymbol{G}/e$	contraction of an edge e in $\boldsymbol{G}$
$\boldsymbol{G} \,{\scriptstyle Y\Delta}\, \{w\}$	$Y\Delta$-transformation of a vertex w in $\boldsymbol{G}$
$\boldsymbol{G} \angle\, w \left\{ {N_1 \atop N_2} \right\}$	splitting of a vertex w in $\boldsymbol{G}$
$\boldsymbol{G} \dashv v \{u_1, u_2\}$	triad addition for an edge $u_1 u_2$ in $\boldsymbol{G}$
$\boldsymbol{G} \triangleleft\, w \left\{ {N_1 \atop N_2} \right\}$	triangle explosion of a vertex w in $\boldsymbol{G}$
Λ	the family of 32 connected minor-minimal nonprojective graphs (Appendix A)
$\boldsymbol{\mathcal{B}}_3, \boldsymbol{\mathcal{C}}_2, \boldsymbol{\mathcal{D}}_4$, etc.	graphs from the family Λ (Appendix A)
Λ_0	the family $\Lambda - \{\boldsymbol{K}_{1,2,2,2}, \boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{D}}_2\}$ (Appendix A)
Π	the family of 16 graphs from Appendix B
$\boldsymbol{\mathcal{B}}'_7, \boldsymbol{\mathcal{C}}^\circ_3, \boldsymbol{\mathcal{D}}^\bullet_2$, etc.	graphs from the family Π (Appendix B)

SUMMARY

Planar covers of graphs, with an extension to covers on other surfaces, were studied. A simple graph $\boldsymbol{H}$ is a cover of a simple graph $\boldsymbol{G}$ if there exists a mapping φ from $V(\boldsymbol{H})$ onto $V(\boldsymbol{G})$ such that for every vertex v of $\boldsymbol{G}$, φ maps the neighbors of v in $\boldsymbol{H}$ bijectively onto the neighbors of $\varphi(v)$ in $\boldsymbol{G}$. In 1986, S. Negami conjectured that a connected graph has a finite planar cover if and only if it embeds in the projective plane.

The "Kuratowski theorem for the projective plane" by D. Archdeacon implies that Negami's conjecture holds as long as none of the 32 connected minor-minimal nonprojective graphs has a planar cover. Results by D. Archdeacon, M. Fellows, and S. Negami from 1987–1988 stated that 25 of these graphs had no planar covers. In this thesis, the conjecture was verified for three other graphs ($\boldsymbol{K}_{4,4}-e$, $\boldsymbol{\mathcal{C}}_4$, and $\boldsymbol{\mathcal{D}}_2$) of the 32. Using those results, it was proved that, up to obvious constructions, there were at most 16 possible counterexamples to Negami's conjecture. (This was joint work with R. Thomas.) A consequence of this work is that in order to prove Negami's conjecture it suffices to prove that $\boldsymbol{K}_{1,2,2,2}$ has no planar cover. However, the conjecture is still open.

A reformulation of Negami's conjecture, which had a straightforward generalization to nonorientable surfaces, was proposed. Some support for the generalized conjecture was given in the case of the Klein bottle.

CHAPTER I

INTRODUCTION

1.1 Graphs

Graphs and Subgraphs; Graph Isomorphism

There are many books on graph theory available, among them [7],[8],[11],[40]. Our definitions are based on [8].

A *graph* $\boldsymbol{G}$ is an ordered triple $\boldsymbol{G} = (V, E, \varepsilon)$ consisting of a nonempty set of *vertices* V, a set of *edges* E (disjoint from V), and an *incidence function* ε that associates with each edge of $\boldsymbol{G}$ an unordered pair of (not necessarily distinct) vertices of $\boldsymbol{G}$. The vertex set, the edge set, and the incidence function of a graph $\boldsymbol{G}$ are also denoted by $V(\boldsymbol{G})$, $E(\boldsymbol{G})$, and ε_G respectively. If e is an edge and u, v are vertices such that $\varepsilon(e) = uv$, then e is said to *join* u and v; the vertices u, v are called the *ends* of e. A graph is *finite* if both its vertex set and edge set are finite. All graphs in this work are considered to be finite.

Graphs are commonly depicted such that the vertices are dots and edges are curves joining them. (See examples in appendices.) If e is an edge, then the ends of e are said to be *incident* with e, and vice versa. Two vertices which are incident with a common edge are *adjacent*. An edge is called a *loop* if its ends are identical. Two edges are called *parallel* if they have the same ends. If there is no danger of misunderstanding between parallel edges, $e = \{u, v\}$, or shortly $e = uv$, is used to denote an edge e with ends u, v.

The number of vertices of a graph $\boldsymbol{G}$ is the cardinality of the set $V(\boldsymbol{G})$, and the number of edges is the cardinality of $E(\boldsymbol{G})$. The *degree* $d_G(v)$ of a vertex v in $\boldsymbol{G}$ is the number of edges of $\boldsymbol{G}$ incident with v, each loop counting as two edges. A graph $\boldsymbol{G}$ is *k-regular* if $d_G(v) = k$ for all vertices $v \in V(\boldsymbol{G})$. It is a well known fact that $\sum_{v \in V(G)} d_G(v) = 2|E(\boldsymbol{G})|$.

A graph is called *simple* if it has no loops or parallel edges. Our definition of a graph allows loops and parallel edges, because some of the graph operations used here may create them. However, loops or parallel edges are allowed just for convenience; our results are not more general than if they had been formulated using simple graphs.

Two graphs $\boldsymbol{G} = (V, E, \varepsilon)$ and $\boldsymbol{H} = (V', E', \varepsilon')$ are *identical* (written $\boldsymbol{G} = \boldsymbol{H}$) if $V = V'$, $E = E'$, and $\varepsilon = \varepsilon'$. The graphs $\boldsymbol{G} = (V, E, \varepsilon)$ and $\boldsymbol{H} = (V', E', \varepsilon')$ are *isomorphic* (written $\boldsymbol{G} \simeq \boldsymbol{H}$) if there exists a pair of bijections $\theta : V \to V'$ and $\varphi : E \to E'$, called an *isomorphism*, such that $\varepsilon(e) = uv$ if and only if $\varepsilon'(\varphi(e)) = \theta(u)\theta(v)$. For simple graphs, it is enough to describe an isomorphism by the vertex bijection θ. An *automorphism* is an isomorphism of a graph onto itself.

A graph $\boldsymbol{F}$ is a *subgraph* of a graph $\boldsymbol{G}$ (written $\boldsymbol{F} \subseteq \boldsymbol{G}$) if $V(\boldsymbol{F}) \subseteq V(\boldsymbol{G})$, $E(\boldsymbol{F}) \subseteq E(\boldsymbol{G})$, and ε_F is the restriction of ε_G to $E(\boldsymbol{F})$. A subgraph $\boldsymbol{F} \subseteq \boldsymbol{G}$ is *proper* if $\boldsymbol{F} \neq \boldsymbol{G}$. A subgraph $\boldsymbol{F} \subseteq \boldsymbol{G}$ is a *spanning subgraph* if $V(\boldsymbol{F}) = V(\boldsymbol{G})$. Suppose that $\boldsymbol{G}$ is a graph and V' is a nonempty subset of $V(\boldsymbol{G})$. The subgraph $\boldsymbol{F}$ of $\boldsymbol{G}$ whose vertex set is V' and whose edge set is the set of those edges of $\boldsymbol{G}$ that have both ends in V' is called the subgraph of $\boldsymbol{G}$ *induced* by V' (written $\boldsymbol{F} = \boldsymbol{G} \upharpoonright V'$). It is also said that $\boldsymbol{F}$ is an induced subgraph of $\boldsymbol{G}$.

Some Classes of Graphs

Some useful common graph and subgraph classes are introduced now. A *complete graph* on n vertices, denoted by $\boldsymbol{K}_n$, is a simple graph with n vertices and $\binom{n}{2}$ edges joining each pair of vertices. A *path* of length k, denoted by $\boldsymbol{P}_k$, is the graph with a vertex set $\{v_1, v_2, \ldots, v_{k+1}\}$ and an edge set $\{v_1v_2, v_2v_3, \ldots, v_kv_{k+1}\}$. (Notice that the path of length k has $k+1$ vertices.) A cycle of length k, denoted by $\boldsymbol{C}_k$, is the graph with a vertex set $\{v_1, v_2, \ldots, v_k\}$ and an edge set $\{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k, v_kv_1\}$. A cycle of length k is called a *k-cycle*, and a 3-cycle is called a *triangle.* (In particular, a 1-cycle is a loop, and a 2-cycle is a pair of parallel edges.)

A graph $\boldsymbol{G}$ is called *bipartite* if the vertex set can be partitioned into two sets $V(\boldsymbol{G}) = A \cup B$ such that both $\boldsymbol{G} \restriction A$, $\boldsymbol{G} \restriction B$ have no edges. The sets A, B are called the *parts* of $\boldsymbol{G}$. A *complete bipartite (p-partite) graph* is a simple graph with a vertex set $V = V_1 \cup V_2$, $p = 2$ $(V = V_1 \cup \ldots \cup V_p)$ where the sets $V_1, \ldots, V_p$ are pairwise disjoint, and with the edge set consisting of all edges with ends in distinct sets V_i and V_j. The complete p-partite graph is denoted by $\boldsymbol{K}_{n_1,\ldots,n_p}$, where $n_i = |V_i|$, $i = 1, \ldots, p$.

An *independent set* in a graph $\boldsymbol{G}$ is a subset $I \subseteq V(\boldsymbol{G})$ of vertices that induces a graph with no edges. A *clique* in a graph $\boldsymbol{G}$ is a subgraph of $\boldsymbol{G}$ that is isomorphic to a complete graph, and that is inclusion-wise maximal with this property. A *k-clique* is a clique on k vertices.

A *walk* in a graph $\boldsymbol{G}$ is a finite sequence $v_0e_1v_1e_2v_2\ldots e_kv_k$ such that e_i is an edge with ends v_{i-1}, v_i for $i = 1, 2, \ldots, k$. The vertices v_0, v_k are called the *ends* of the walk, and all other vertices are *internal vertices*; the walk is said to *connect* v_0 and v_k. The *length* of the walk is the number k. If $\boldsymbol{G}$ is a simple graph, a walk can be denoted simply by the sequence of its vertices $v_0v_1\ldots v_k$. Clearly, if there exists a walk connecting vertices u, v in a graph $\boldsymbol{G}$, then there also exists a path connecting u, v in $\boldsymbol{G}$. A walk $v_0 e_1 v_1 e_2 v_2 \ldots e_k v_k$ is said to be *closed* if $v_0 = v_k$. A *cycle* is a closed walk such that all vertices $v_1, v_2, \ldots, v_k$ are pairwise distinct.

Graph Operations

If $\boldsymbol{G}$ is a graph and X is a subset of its vertices, then $\boldsymbol{G} - X$ denotes the subgraph of $\boldsymbol{G}$ induced on the vertex set $V(\boldsymbol{G}) - X$. For a graph $\boldsymbol{G}$, and a set of edges $E \subseteq E(\boldsymbol{G})$, $\boldsymbol{G} - E$ denotes the graph obtained by deleting all edges of E from $\boldsymbol{G}$. If e is an edge of $\boldsymbol{G}$, $\boldsymbol{G} - e$ is used as a shortcut for $\boldsymbol{G} - \{e\}$.

If u, v are two vertices of a graph $\boldsymbol{G}$, then $\boldsymbol{G} + uv$ denotes the graph obtained by adding a new edge e with ends u, v to $\boldsymbol{G}$. The graph obtained from $\boldsymbol{G}$ by adding a set of edges E' is denoted by $\boldsymbol{G} + E'$. The notation $\boldsymbol{G} + e$ is used as a shortcut for $\boldsymbol{G} + \{e\}$. If $\boldsymbol{G}_1, \boldsymbol{G}_2$ are subgraphs of a common graph $\boldsymbol{G}$, then the *union* $\boldsymbol{G}_1 \cup \boldsymbol{G}_2$ is the graph with vertex set $V(\boldsymbol{G}_1) \cup V(\boldsymbol{G}_2)$ and edge set $E(\boldsymbol{G}_1) \cup E(\boldsymbol{G}_2)$.

Let $\boldsymbol{G}$ be a graph, and let $\boldsymbol{F}$ be a spanning subgraph of $\boldsymbol{G}$ obtained by deleting all loops, and, for every pair of adjacent vertices, all but one of the edges joining them. Then $\boldsymbol{F}$ is called the *underlying simple graph* of $\boldsymbol{G}$.

A graph $\boldsymbol{H}$ results from a graph $\boldsymbol{G}$ by *subdividing* an edge $e = uv$ with vertices $v_1, \ldots, v_k$ that are not in $V(\boldsymbol{G})$ if $\boldsymbol{H} = (\boldsymbol{G} - e) \cup \boldsymbol{P}$, where $\boldsymbol{P} = uv_1 v_2 \ldots v_k v$ is a path disjoint from $\boldsymbol{G}$ except for the ends u, v. A graph $\boldsymbol{H}$ is a *subdivision* of a graph $\boldsymbol{G}$ if $\boldsymbol{H}$ is obtained from $\boldsymbol{G}$ by subdividing some edges. A graph $\boldsymbol{G}$ results from a graph $\boldsymbol{H}$ by *suppressing* a vertex w if $\boldsymbol{H}$ is obtained from $\boldsymbol{G}$ by subdividing an edge with a vertex $w \notin V(\boldsymbol{G})$. Two graphs $\boldsymbol{G}, \boldsymbol{H}$ are *homeomorphic* if they are isomorphic to subdivisions of the same graph.

A graph $\boldsymbol{F}$ results from a graph $\boldsymbol{G}$ by *contracting* an edge $e = uv$ (written $\boldsymbol{F} = \boldsymbol{G}/e$) if $V(\boldsymbol{F}) = V(\boldsymbol{G}) - \{u, v\} \cup \{w\}$, $E(\boldsymbol{F}) = E(\boldsymbol{G}) - \{e\}$, and the incidence function ε_F is obtained from the incidence function ε_G by replacing each occurrence of u or v in the image of ε_G with w. If e is a loop, then $\boldsymbol{G}/e = \boldsymbol{G} - e$. Informally, the edge $e = uv$ is deleted from the graph, and the vertices u, v are identified.

A vertex of degree 3 with three distinct neighbors in a graph is called *cubic*. If w is a cubic vertex in a graph $\boldsymbol{G}$ with neighbors v_1, v_2, v_3, then the operation of deleting the vertex w and adding three new edges forming a triangle on the vertices v_1, v_2, v_3 is called a *$Y\Delta$-transformation* (of w), written $\boldsymbol{G} \,{\scriptstyle Y\Delta}\, \{w\}$.

Connectivity and Distance in Graphs

A graph $\boldsymbol{G}$ is *connected* if every pair of its vertices is connected by a path. A *component* of $\boldsymbol{G}$ is an inclusion-wise maximal induced subgraph of $\boldsymbol{G}$ that is connected. A *(vertex) cut* in a graph $\boldsymbol{G}$ is a set $X \subseteq V(\boldsymbol{G})$ such that $\boldsymbol{G} - X$ has more components than $\boldsymbol{G}$. A *k-cut* is a cut of size k. A graph is *k-connected* if it is connected, has at least $k+1$ vertices, and has no cut of size less than k.

A *separation* in a graph $\boldsymbol{G}$ is a pair of sets (A, B) such that $A \cup B = V(\boldsymbol{G})$ and there is no edge in $\boldsymbol{G}$ between the sets $A - B$ and $B - A$. A separation (A, B) is nontrivial if both $A - B$ and $B - A$ are nonempty. The *order of a separation* (A, B) equals $|A \cap B|$, and a *k-separation* is a separation of order k. Clearly, a connected graph has a nontrivial k-separation if and only if it has a k-cut.

Let $\boldsymbol{G}$ be a graph, and let $C, D \subseteq V(\boldsymbol{G})$. A C–D path is a path P in $\boldsymbol{G}$ that has one end in C and the other end in D, and that the internal vertices of P are disjoint from $C \cup D$. The following important result is known as Menger's theorem.

Theorem 1.1.1. *Let $\boldsymbol{G}$ be a graph, and let $C, D \subseteq V(\boldsymbol{G})$. Then the maximum number of vertex-disjoint C–D paths equals to the minimum order of a separation (A, B) in $\boldsymbol{G}$ such that $A \supseteq C$ and $B \supseteq D$.*

Corollary 1.1.2. *A graph* $\boldsymbol{G}$ *on at least* $k+1$ *vertices is* k*-connected if and only if every two distinct vertices* u, v *in* $\boldsymbol{G}$ *can be connected by* k *paths that are vertex-disjoint except for their ends* u, v.

An *edge-cut* in a graph $\boldsymbol{G}$ is a set $Y \subseteq E(\boldsymbol{G})$ such that $\boldsymbol{G} - Y$ has more components than $\boldsymbol{G}$. A k*-edge-cut* is an edge-cut of size k. A graph is k*-edge-connected* if it is connected and has no edge-cut of size less than k. The following is an edge-variant of Menger's theorem.

Theorem 1.1.3. *Let* $\boldsymbol{G}$ *be a graph, and let* $C, D \subseteq V(\boldsymbol{G})$. *Then the maximum number of edge-disjoint* C–D *paths equals to the minimum number of edges separating* C *from* D.

Corollary 1.1.4. *A graph* $\boldsymbol{G}$ *is* k*-edge-connected if and only if every two distinct vertices in* $\boldsymbol{G}$ *can be connected by* k *edge-disjoint paths.*

Corollary 1.1.5. *A graph* $\boldsymbol{G}$ *is* 2*-edge-connected if and only if every edge in* $\boldsymbol{G}$ *is contained in some cycle of* $\boldsymbol{G}$.

The *distance* between two vertices of a graph $\boldsymbol{G}$ is defined as the length of the shortest path connecting them in $\boldsymbol{G}$. In particular, a vertex is at distance 0 from itself, and two distinct adjacent vertices are at distance 1. It is easy to see that the distance in graphs satisfies the triangle inequality.

Graph Minors

A graph $\boldsymbol{F}$ is a *minor* of a graph $\boldsymbol{G}$ if $\boldsymbol{F}$ is obtained from a subgraph of $\boldsymbol{G}$ by a sequence of edge contractions. (The order in which contractions are applied does not matter.) A graph $\boldsymbol{G}$ is said to have an $\boldsymbol{F}$ minor if there exists a minor $\boldsymbol{F}'$ of $\boldsymbol{G}$ such that $\boldsymbol{F}'$ is isomorphic to $\boldsymbol{F}$. Let Γ be a finite family of graphs. A graph $\boldsymbol{G}$ is said to have a Γ minor if there exists a minor $\boldsymbol{F}'$ of $\boldsymbol{G}$ such that $\boldsymbol{F}'$ is isomorphic to some member of Γ. It can be easily checked that the minor relation is transitive, i.e. if $\boldsymbol{F}$ is a minor of $\boldsymbol{G}$, and $\boldsymbol{G}$ is a minor of $\boldsymbol{H}$, then $\boldsymbol{F}$ is a minor of $\boldsymbol{H}$, too.

The following deep result is known as the Graph Minor Theorem [34], formerly Wagner's Conjecture.

Theorem 1.1.6. (N. Robertson, P.D. Seymour) *If Φ is an infinite family of (finite) graphs, then there exist two distinct graphs $\boldsymbol{F}, \boldsymbol{G} \in \Phi$ such that $\boldsymbol{F}$ is isomorphic to a minor of $\boldsymbol{G}$.*

An important corollary of this result is that if some graph property $\mathcal{P}$ is preserved under taking minors, then there exists a finite family Γ of graphs such that a graph $\boldsymbol{G}$ has the property $\mathcal{P}$ if and only if it has no Γ minor. (In other words, the property $\mathcal{P}$ can be characterized by a finite list of "forbidden minors".) Unfortunately, this argument is non-constructive, and gives no clue as how to construct the family Γ.

1.2 Graphs on Surfaces

This section presents basic terms for graphs embedded on surfaces. Our definitions follow the monograph [28], another book about topological graph theory is [15]. For basic topological definitions refer, for example, to [27] or [38].

Surfaces

A *surface* is a connected compact Hausdorff topological space S such that every point of S has an open neighborhood homeomorphic to $\mathbb{R}^2$. It can be shown that every surface is homeomorphic to a triangulated surface, defined as follows. Suppose that $\tau_1, \ldots, \tau_{2k}$ is a collection of disjoint unit triangles in the plane, and that the $6k$ sides of these triangles are partitioned into pairs. Let X be the topological space obtained from the collection of triangles $\tau_1, \ldots, \tau_{2k}$ by identifying the given pairs of sides (in a chosen orientation). If X is connected, and the graph defined by the vertices and sides of $\tau_1, \ldots, \tau_{2k}$ in X is simple, then X is called a *triangulated surface.*

Let S be a surface, and let $\mathcal{R}$ be an open neighborhood of some point of S homeomorphic to $\mathbb{R}^2$. Suppose that D_1 is a closed disc contained in $\mathcal{R}$. If a surface S' results from S by deleting the interior of D_1, and by identifying the opposite points on the boundary of D_1, then S' is said to be obtained from S by *adding a crosscap.* Suppose that D_1, D_2 are two disjoint closed discs contained in $\mathcal{R}$. Let us choose an orientation of $\mathcal{R}$, and let us give the boundaries of D_1 and D_2 opposite orientations. If a surface S' results from S by deleting the interiors of D_1 and D_2, and by identifying the boundary of D_1 with the boundary of D_2 with respect to the chosen orientations, then S' is said to be obtained from S by *adding a handle.*

The 2-sphere is denoted by S_0. The surface obtained from S_0 by adding $h > 0$ handles is called the *orientable surface of genus* h, and is denoted by S_h. A surface obtained from S_0 by adding $k > 0$ crosscaps is called the *nonorientable surface of genus* k, and is denoted by N_k. The surfaces S_1, S_2, S_3, N_1, N_2 are also called the *torus*, the *double-torus*, the *triple-torus*, the *projective plane*, and the *Klein bottle*, respectively. The Classification Theorem for Surfaces states the following:

Theorem 1.2.1. *Every surface is homeomorphic to precisely one of the surfaces* S_h *(*$h \geq 0$*), or* N_k *(*$k \geq 1$*).*

A *covering space* of S is a pair (X, p) where $p : X \to S$ is a continuous onto mapping satisfying the following condition: For each $x \in X$, there exists an open neighborhood U of x such that the restriction of p onto U is a homeomorphism. For example, the 2-sphere is a covering space of the projective plane via the "vertical projection". (In fact, the 2-sphere is the so called "universal covering space" of the projective plane.)

Graph Embeddings

A *curve* in a topological space S is the image of a continuous function $f : [0,1] \to S$. The curve is *simple* if f is injective. The curve is said to *connect* its endpoints $f(0)$ and $f(1)$. A *simple closed curve* is defined analogously except that f is injective on $[0,1)$ and $f(1) = f(0)$. A graph $\boldsymbol{G}$ is *embedded* in S if the vertices of $\boldsymbol{G}$ are distinct points of S, and every edge of $\boldsymbol{G}$ is a simple curve (a simple closed curve for a loop) connecting its ends. Moreover, it is required that no two edges intersect except at a common endvertex, and that no edge contains an isolated vertex. If $\boldsymbol{G}$ is embedded in S, then $\boldsymbol{G}$ is used to denote the point set $V(\boldsymbol{G}) \cup \left(\bigcup_{e \in E(G)} e\right)$. An *embedding* of a graph $\boldsymbol{G}$ in S is an isomorphism of $\boldsymbol{G}$ with a graph $\boldsymbol{G}'$ embedded in S. Notice that if a graph $\boldsymbol{G}$ has an embedding in S, then so does every minor of $\boldsymbol{G}$.

Let $\boldsymbol{G}$ be a graph embedded in a surface S. A *face* of $\boldsymbol{G}$ is an arcwise connected component of $S - \boldsymbol{G}$. The set of faces of $\boldsymbol{G}$ is denoted by $F(\boldsymbol{G})$. Each face of $\boldsymbol{G}$ is bounded by a closed walk, called a *facial walk*. A face f of $\boldsymbol{G}$ is said to be *incident* with those vertices and edges of $\boldsymbol{G}$ contained in the boundary of f. The graph $\boldsymbol{G}$ is said to be 2-cell embedded if each face of $\boldsymbol{G}$ is homeomorphic to $\mathbb{R}^2$. A simple 2-cell embedded graph $\boldsymbol{G}$ is called a *triangulation* if each face of $\boldsymbol{G}$ is incident with exactly three edges. The *Euler characteristic* $\chi(S)$ of a surface S is defined as $\chi(S_h) = 2 - 2h$, and $\chi(N_k) = 2 - k$. The next result is well-known as Euler's formula.

Theorem 1.2.2. *If $\boldsymbol{G}$ is a graph that is 2-cell embedded in a surface S, then* $|V(\boldsymbol{G})| + |F(\boldsymbol{G})| - |E(\boldsymbol{G})| = \chi(S)$.

Planar Graphs

Planar graphs are the most natural examples of graph embeddings. According to the above definitions of surfaces and embeddings, a graph $\boldsymbol{G}$ is *planar* if it has an embedding in the sphere S_0. However, it is more common to treat planar graphs as embeddings in the plane $\mathbb{R}^2$, and that is also our approach. This clearly does not matter for an embeddability of a graph since the plane is homeomorphic to any open disc in the sphere.

A *plane graph* is a graph embedded in the plane. Since the point set of a plane graph $\boldsymbol{G}$ is compact, exactly one face of $\boldsymbol{G}$ (i.e. an arcwise connected component of $\mathbb{R}^2 - \boldsymbol{G}$) is unbounded, and it is called the *outer face* of $\boldsymbol{G}$. It is a useful fact that every planar graph has a planar embedding in which each edge is a piecewise linear curve.

The Kuratowski Theorem [26] characterizes planar graphs.

Theorem 1.2.3. (K. Kuratowski, 1930) *A graph is planar if and only if it does not contain a subgraph isomorphic to a subdivision of* $\boldsymbol{K}_5$ *or* $\boldsymbol{K}_{3,3}$.

Corollary 1.2.4. *A graph is planar if and only if it does not contain a minor isomorphic to* $\boldsymbol{K}_5$ *or* $\boldsymbol{K}_{3,3}$.

The following statement is a reasonably easy but important result.

Proposition 1.2.5. *If* $\boldsymbol{G}$ *is a 2-connected plane graph, then each facial walk in* $\boldsymbol{G}$ *is a cycle.*

Two embeddings of a planar graph are *equivalent* if they have the same collection of facial walks. If $\boldsymbol{G}$ is a 3-connected planar graph, then $\boldsymbol{G}$ has a unique planar embedding, i.e. every two planar embeddings of $\boldsymbol{G}$ are equivalent. A graph

$\boldsymbol{G}$ is called *outerplanar* if $\boldsymbol{G}$ has a planar embedding in which all of its vertices are incident with the outer face. It can be proved that a graph is outerplanar if and only if it does not contain a subgraph isomorphic to a subdivision of $\boldsymbol{K}_4$ or $\boldsymbol{K}_{2,3}$ [9].

Projective Graphs

A graph $\boldsymbol{G}$ is called *projective-planar* (or shortly *projective*) if $\boldsymbol{G}$ has an embedding in the projective plane. Projective-planar graphs can be characterized in a similar way as planar graphs. Glover, Huneke and Wang [14] found a family Λ' of 35 graphs such that each member of Λ' had no embedding in the projective plane, and was minor-minimal with that property. (See Appendix A for a complete list of Λ'.) Archdeacon [2],[3] then proved that those were the only such graphs.

Theorem 1.2.6. (D. Archdeacon, 1981) *A graph $\boldsymbol{G}$ has an embedding in the projective plane if and only if $\boldsymbol{G}$ has no minor isomorphic to a member of Λ'.*

Robertson and Seymour proved [32] that a similar characterization holds for graphs embeddable in any fixed surface. Unfortunately, their proof gives no way how to find the collections of "forbidden minors" for other surfaces than the plane or the projective plane. Seymour found [36] a double-exponential upper bound on the number of vertices of a minor-minimal nonembeddable graph for any fixed surface. So, theoretically, the "forbidden minors" can be found by examining the finite collection of all nonisomorphic graphs satisfying this bound. However, this is not a practical approach.

CHAPTER II

NEGAMI'S PLANAR COVER CONJECTURE

2.1 Covers of Graphs

Definition and Basic Properties

A graph $\boldsymbol{H}$ is a *cover* of a graph $\boldsymbol{G}$ if there exist a pair of onto mappings (φ, ψ), $\varphi : V(\boldsymbol{H}) \to V(\boldsymbol{G})$, $\psi : E(\boldsymbol{H}) \to E(\boldsymbol{G})$, called a (cover) *projection*, such that ψ maps the edges incident with each vertex v in $\boldsymbol{H}$ bijectively onto the edges incident with $\varphi(v)$ in $\boldsymbol{G}$. In particular, this definition clearly implies the following:

Observation. For every edge e in $\boldsymbol{H}$ with endvertices u, v, the edge $\psi(e)$ in $\boldsymbol{G}$ has endvertices $\varphi(u), \varphi(v)$.

Therefore for simple graphs, it is enough to specify the vertex projection φ that maps the neighbors of each vertex v in $\boldsymbol{H}$ bijectively onto the neighbors of $\varphi(v)$ in $\boldsymbol{G}$ (a traditional approach).

If $\boldsymbol{G}'$ is a subgraph of $\boldsymbol{G}$, then the graph $\boldsymbol{H}'$ with the vertex set $\varphi^{-1}(V(\boldsymbol{G}'))$ and the edge set $\psi^{-1}(E(\boldsymbol{G}'))$ is called a *lifting of* $\boldsymbol{G}'$ *into* $\boldsymbol{H}$. A cover $\boldsymbol{H}$ of a connected graph $\boldsymbol{G}$ is called an *n-fold cover* if $|\varphi^{-1}(v)| = n$ for some vertex $v \in V(\boldsymbol{G})$. (It follows from Corollary 2.1.2 below that this a is well-defined term.) A 2-fold cover is also called a *double cover*.

Of course, every graph covers itself by the identity mapping, or by any other automorphism. A nontrivial example is a 9-cycle covering a triangle (the projection "winds" the 9-cycle three times around the triangle). Another nontrivial example is presented further in Fig. 2.1. A useful way to represent a cover projection $\boldsymbol{H} \to \boldsymbol{G}$ is to label the vertices or (possibly) edges of $\boldsymbol{H}$ by the names of the vertices or edges of $\boldsymbol{G}$ they are projected to. To illustrate, several basic properties of covers are presented now.

Observation. Suppose that $\boldsymbol{H}$ is a cover of $\boldsymbol{G}$ via projection (φ, ψ). Then $d_H(v) = d_G(\varphi(v))$ for each vertex $v \in V(\boldsymbol{H})$.

Observation. If $\boldsymbol{H}$ is a cover of $\boldsymbol{G}$, and $\boldsymbol{H}_1$ is a component of $\boldsymbol{H}$, then $\boldsymbol{H}_1$ is also a cover of $\boldsymbol{G}$.

Lemma 2.1.1. *Suppose that $\boldsymbol{H}$ is a cover of $\boldsymbol{G}$ via the projection (φ, ψ), and P is a path in $\boldsymbol{G}$. Then the lifting of P into $\boldsymbol{H}$ is a collection of disjoint paths such that the restrictions of the projection (φ, ψ) to each of these paths are isomorphisms with P.*

Proof. The statement is proved by induction on the length k of the path $P = v_0e_1v_1 \dots e_kv_k \subseteq \boldsymbol{G}$. The case of $k = 1$ follows from the definition of a cover. For $k > 1$, let $P' = v_0e_1v_1 \dots e_{k-1}v_{k-1}$, and let $P^1, \dots, P^n$ be paths (components) isomorphic to P' that form the lifting of P' into $\boldsymbol{H}$, by the induction assumption. Then for $i = 1, \dots, n$, the vertex $w^i \in V(P^i)$, with $\varphi(w^i) = v_{k-1}$ is incident with exactly one edge $f^i = w^it^i$ such that $\psi(f^i) = e_k$, and $\varphi(t^i) = v_k$. Since the vertex t^i is disjoint from all $V(P^j)$, $j = 1, \dots, n$, the path Q^i obtained by adding t^i and f^i to P^i is a component in the lifting of P into $\boldsymbol{H}$, and Q^i is isomorphic to P via a restriction of the projection (φ, ψ). ∎

Corollary 2.1.2. *If $\boldsymbol{H}$ is a cover of a connected graph $\boldsymbol{G}$ via the projection (φ, ψ), then $|\varphi^{-1}(v)|$ is the same number for all vertices $v \in V(\boldsymbol{G})$.*

Lemma 2.1.3. *Suppose that $\boldsymbol{H}$ is a cover of $\boldsymbol{G}$, and C is a cycle of length n in $\boldsymbol{G}$. Then the lifting of C into $\boldsymbol{H}$ is a collection of disjoint cycles, each cycle of length divisible by n.*

Proof. Let (φ, ψ) be the projection of $\boldsymbol{H}$ onto $\boldsymbol{G}$, and let $\boldsymbol{F} \subseteq \boldsymbol{H}$ be the lifting of C into $\boldsymbol{H}$. By the above observations, all vertices of $\boldsymbol{F}$ have degree 2, so $\boldsymbol{F}$ is a collection of disjoint cycles. Let $\boldsymbol{C'}$ be one cycle (component) in $\boldsymbol{F}$, and let $e \in E(C)$. Then $\boldsymbol{C'}$ itself covers C, and the lifting of the path $C - e$ into $\boldsymbol{C'}$ is the graph $\boldsymbol{C'} - (\psi^{-1}(e) \cap E(\boldsymbol{C'}))$, which is a collection of $p = |\psi^{-1}(e) \cap E(\boldsymbol{C'})|$ disjoint paths of length $n-1$ by Lemma 2.1.1. Thus $\boldsymbol{C'}$ has $(n-1)p + p = np$ edges. ∎

Planar Covers

A cover is called *planar* if it is a planar graph. (Notice that every graph can be covered by an infinite tree, but that is not our objective. Recall that all graphs are supposed to be *finite.*)

Every planar graph has a planar cover by the identity projection, but there are also nonplanar graphs having planar covers. If a graph $\boldsymbol{G}$ has an embedding in the projective plane, then the lifting of the embedding of $\boldsymbol{G}$ into the sphere (which is a covering surface of the projective plane) is a double planar cover of $\boldsymbol{G}$. See an example in Fig. 2.1. So it is concluded:

Proposition 2.1.4. *Every projective-planar graph has a double planar cover.*

The following properties of planar covers are important for further arguments.

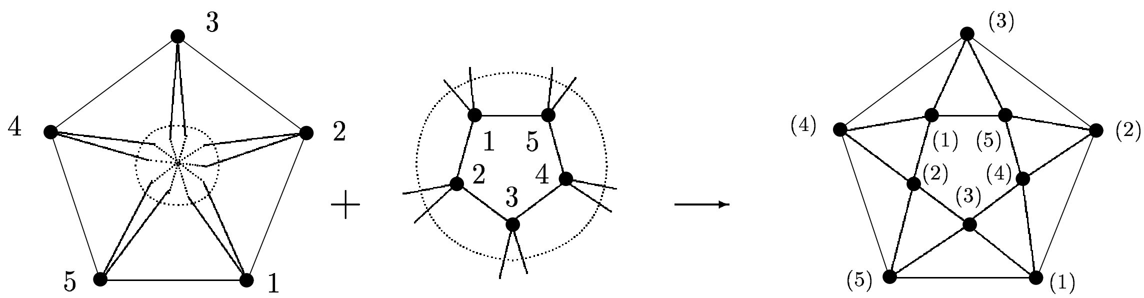


Figure 2.1: A double planar cover of $\boldsymbol{K}_5$, constructed by lifting its projective drawing into the sphere. The cover projection is determined by labels of vertices.

Lemma 2.1.5. *If a graph $\boldsymbol{G}$ has a planar cover, then so does every minor of $\boldsymbol{G}$.*

Proof. Let $\boldsymbol{H}$ be a cover of $\boldsymbol{G}$, and (φ, ψ) be the projection. If $\boldsymbol{G}'$ is a subgraph of $\boldsymbol{G}$, then the lifting of $\boldsymbol{G}'$ into $\boldsymbol{H}$ is a cover of $\boldsymbol{G}'$ by definition. If $\boldsymbol{G}'$ is obtained from $\boldsymbol{G}$ by contracting an edge e, then it is easy to see that the graph $\boldsymbol{H}'$ obtained from $\boldsymbol{H}$ by contracting all edges of $\psi^{-1}(e)$ is a cover of $\boldsymbol{G}'$. So the statement follows by an induction. ∎

The next statement is reasonably easy to see, and, in fact, it is a special case of Lemma 6.1.2(b).

Proposition 2.1.6. (D. Archdeacon, 1987) *Let $\boldsymbol{G}$ be a graph, and let e be an edge of $\boldsymbol{G}$ such that some cubic vertex of $\boldsymbol{G}$ is adjacent to both endvertices of e. If $\boldsymbol{G} - e$ has a planar cover, then so does $\boldsymbol{G}$.*

Corollary 2.1.7. *Let a graph $\boldsymbol{G}$ be obtained from a graph $\boldsymbol{H}$ by a sequence of $Y\Delta$-transformations. If $\boldsymbol{H}$ has a planar cover, then so does $\boldsymbol{G}$.*

Related Research

The definition of a graph cover is related to topology. Suppose that a graph $\boldsymbol{H}$ is a cover of a graph $\boldsymbol{G}$. If $\boldsymbol{G}$ ($\boldsymbol{H}$) is embedded in some surface, then this embedding is a topological subspace of the surface in the usual sense. If $\boldsymbol{G}$ has no vertices of degree 2, then the concept of the cover defined earlier coincides with the notion of a covering space in general topology. However, as it was presented above, graph covers admit a purely combinatorial definition, and our research is concerned with combinatorial aspects of graph covers. The concept of the graph cover can be also found in [15].

Negami [29] showed how to use double planar covers to enumerate the equivalence classes of embeddings of graphs in the projective plane. In particular, he proved that a graph has a double planar cover if and only if it embeds in the projective plane. In [30], Negami defined regular covers for simple connected graphs as follows: A cover $\varphi : V(\boldsymbol{H}) \to V(\boldsymbol{G})$ is said to be *regular* if there is a subgroup A of the automorphism group of $\boldsymbol{H}$ such that for $u, v \in V(\boldsymbol{H})$, $\varphi(u) = \varphi(v)$ if and only if $\tau(u) = v$ for some automorphism $\tau \in A$. He proved:

Theorem 2.1.8. (S. Negami, 1986) *A connected graph has a regular planar cover if and only if it has an embedding in the projective plane.*

Fellows studied planar emulators in his thesis [12]. The definition of an *emulator* is a relaxation of the definition of a cover in which the edge projection is only required to be surjective on the neighborhoods of vertices. (See Section 7.4 for a more precise definition.) Fellows was inspired by computational applications, such as the following example. Suppose that some circuit consists of five units, and each of the units needs to communicate with all others. In such situation, it is not possible to realize that circuit in the plane without crossings. But if two identical copies of each unit can be used, then the inter-unit communication is realizable in the plane – the network is arranged as a planar cover of the graph $\boldsymbol{K}_5$, as the example in Fig. 2.1 shows. In [13], Fellows considered the relation between planar emulators and planar covers. Kitakubo [22] later studied emulators under the name of *branched planar covers*. Some additional results by Abello and Chen related to Fellows' research are contained in [1].

Another interesting question is what fold numbers planar covers admit. Of course, a planar graph has an n-fold planar cover for every natural n. Similarly, a projective-planar graph has an n-fold planar cover for every even n, as follows from Proposition 2.1.4. On the other hand, Archdeacon and Richter proved [4]:

Theorem 2.1.9. (D. Archdeacon, R.B. Richter, 1990) *If $\boldsymbol{H}$ is an n-fold planar cover of a nonplanar graph $\boldsymbol{G}$, then n is even.*

Recently, Kratochvíl, Proskurowski and Telle [23],[24],[25] considered the computational complexity of graph covers. The cover problem is to decide whether an input graph $\boldsymbol{H}$ is a cover of a fixed graph $\boldsymbol{F}$. They studied the question of when it is polynomialy solvable, or when it is an NP-complete problem.

2.2 Negami's Conjecture

As it was concluded in Proposition 2.1.4 above, every projective-planar graph has a double planar cover. The converse is false in general, because for instance the graph consisting of two disjoint copies of $\boldsymbol{K}_5$ has a planar cover, and yet it has no embedding in the projective plane. On the other hand, Negami made the following interesting conjecture [30], a generalization of Theorem 2.1.8.

Conjecture 2.2.1. (S. Negami, 1986) *A connected graph has a finite planar cover if and only if it has an embedding in the projective plane.*

Curiously, in order to prove the conjecture it suffices to prove that certain finite number of graphs have no planar covers. Let us explain that now. By Lemma 2.1.5, the property of having a planar cover is closed on taking minors. Let Λ denote the family of 32 connected minor-minimal nonprojective graphs. (See Appendix A.) The next statement follows easily from Archdeacon's Theorem 1.2.6.

Theorem 2.2.2. (D. Archdeacon) *A connected graph has an embedding in the projective plane if and only if it has no minor isomorphic to a member of* Λ.

If a graph is projective-planar, then it has a double planar cover by Proposition 2.1.4. Otherwise, it has a minor isomorphic to a member of Λ by Theorem 2.2.2. Thus in order to prove Conjecture 2.2.1 it suffices to show that no member of Λ has a planar cover. The number of graphs to check can be further reduced using $Y\Delta$-transformations, as observed by Archdeacon (cf. Corollary 2.1.7). Known results are listed in the next sections.

2.3 Previously Known Cases

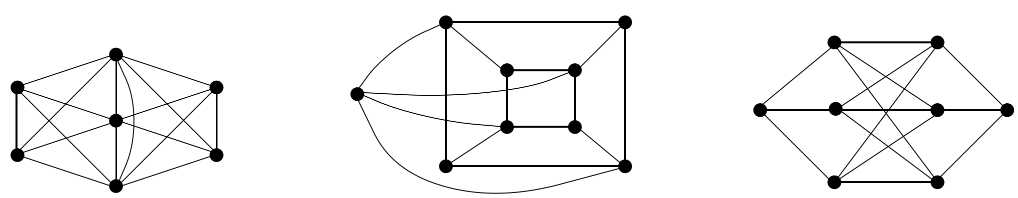
A short counting argument shows that the graph $\boldsymbol{K}_{3,5}$ has no planar cover. This was independently discovered by several people, among them Fellows and Archdeacon. A variant of this argument was published by Huneke [20] in a note written on Negami's conjecture.

A large subfamily of Λ can be handled by the following argument. A subgraph $\boldsymbol{F}$ of a graph $\boldsymbol{G}$ is called a *k-graph* if there exists a graph $\boldsymbol{F}' \subseteq \boldsymbol{G}$ such that the following holds: $\boldsymbol{F}$ is an induced subgraph of $\boldsymbol{F}'$ isomorphic to a subdivision of $\boldsymbol{K}_{2,3}$

or $\boldsymbol{K}_4$, the graph $\boldsymbol{F}' - V(\boldsymbol{F})$ is connected, and contracting the vertices of $\boldsymbol{F}' - V(\boldsymbol{F})$ into one vertex creates a subgraph isomorphic to a subdivision of $\boldsymbol{K}_{3,3}$ or $\boldsymbol{K}_5$. (The definition of k-graphs is based on [14].) The following theorem was independently discovered by Negami [31] and Archdeacon [5].

Theorem 2.3.1. (S. Negami, 1987, independently D. Archdeacon) *Let $\boldsymbol{G}$ be a graph containing two disjoint k-graphs $\boldsymbol{F}_1, \boldsymbol{F}_2$ such that both $\boldsymbol{G} - V(\boldsymbol{F}_1)$ and $\boldsymbol{G} - V(\boldsymbol{F}_2)$ are connected. Then $\boldsymbol{G}$ has no planar cover.*

Corollary 2.3.2. *The graphs $\boldsymbol{K}_{3,3} \cdot \boldsymbol{K}_{3,3}$, $\boldsymbol{K}_5 \cdot \boldsymbol{K}_{3,3}$, $\boldsymbol{K}_5 \cdot \boldsymbol{K}_5$, $\boldsymbol{\mathcal{B}}_3$, $\boldsymbol{\mathcal{C}}_2$, $\boldsymbol{\mathcal{C}}_7$, $\boldsymbol{\mathcal{D}}_1$, $\boldsymbol{\mathcal{D}}_4$, $\boldsymbol{\mathcal{D}}_9$, $\boldsymbol{\mathcal{D}}_{12}$, $\boldsymbol{\mathcal{D}}_{17}$, $\boldsymbol{\mathcal{E}}_6$, $\boldsymbol{\mathcal{E}}_{11}$, $\boldsymbol{\mathcal{E}}_{19}$, $\boldsymbol{\mathcal{E}}_{20}$, $\boldsymbol{\mathcal{E}}_{27}$, $\boldsymbol{\mathcal{F}}_4$, $\boldsymbol{\mathcal{F}}_6$, $\boldsymbol{\mathcal{G}}_1$ have no planar covers.*



$\boldsymbol{K}_7 - \boldsymbol{C}_4$ $\boldsymbol{K}_{4,5} - \boldsymbol{M}_4$ $\boldsymbol{K}_{4,4} - e$

Figure 2.2:

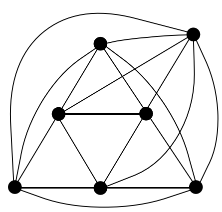
Still in 1987, Archdeacon proved [5] that two other graphs $\boldsymbol{K}_7 - \boldsymbol{C}_4$ and $\boldsymbol{K}_{4,5} - \boldsymbol{M}_4$ of the family Λ (see Fig. 2.2) have no planar covers.

Theorem 2.3.3. (D. Archdeacon, 1987) *The graphs $\boldsymbol{K}_7 - \boldsymbol{C}_4$ and $\boldsymbol{K}_{4,5} - \boldsymbol{M}_4$ have no planar covers.*

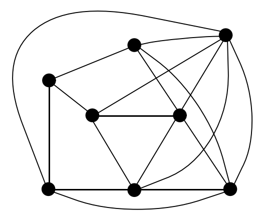
Using $Y\Delta$-transformations and Corollary 2.1.7, this result implies:

Corollary 2.3.4. *The graphs $\mathcal{D}_3$, $\mathcal{E}_5$, $\mathcal{F}_1$ have no planar covers.*

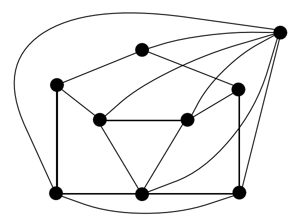
Since five graphs $\mathcal{B}_7, \mathcal{C}_3, \mathcal{C}_4, \mathcal{D}_2, \mathcal{E}_2$ from the remaining seven graphs of Λ (Appendix A) can be $Y\Delta$-transformed to $\boldsymbol{K}_{1,2,2,2}$ (Fig. 2.3), the above stated results and Corollary 2.1.7 imply that Conjecture 2.2.1 is equivalent to the statement that two graphs $\boldsymbol{K}_{4,4}-e$ (Fig. 2.2) and $\boldsymbol{K}_{1,2,2,2}$ have no planar covers. However, no further progress had been made during next several years.



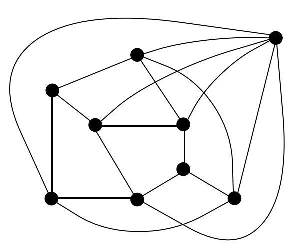
$\boldsymbol{K}_{1,2,2,2}$



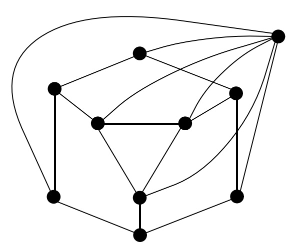
$\mathcal{B}_7$



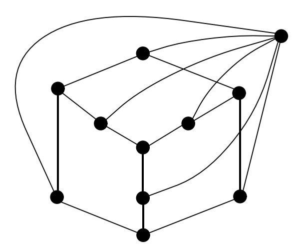
$\mathcal{C}_3$



$\mathcal{C}_4$



$\mathcal{D}_2$



$\mathcal{E}_2$

Figure 2.3:

2.4 New Contributions

In 1995, the author proved [16] that there is no planar cover of $\boldsymbol{K}_{4,4}-e$. This proof is included in Chapter III.

Theorem 2.4.1. *The graph* $\boldsymbol{K}_{4,4}-e$ *has no planar cover.*

It remains to prove that $\boldsymbol{K}_{1,2,2,2}$ has no planar cover in order to finish the proof of Negami's conjecture. Thus it is tempting to say that Negami's conjecture is almost proven, but that is not quite accurate. Testing whether $\boldsymbol{K}_{1,2,2,2}$ (or any given graph) has a planar cover does not seem to be a finite problem, because no apriori bound on the size of a possible planar cover is known. Moreover, the arguments outlined above seem to imply little about possible counterexamples.

The author proved [18] in 1998 that there are no planar covers of the graphs $\boldsymbol{\mathcal{E}}_2$ or $\boldsymbol{\mathcal{C}}_4$ (Fig. 2.3). The proofs are presented in Chapters IV and V.

Theorem 2.4.2. *The graph* $\boldsymbol{\mathcal{E}}_2$ *has no planar cover.*

Theorem 2.4.3. *The graph* $\boldsymbol{\mathcal{C}}_4$ *has no planar cover.*

Using these particular results, R. Thomas and the author have recently proved [19] Theorem 6.1.3; stating that there are, up to obvious constructions, at most 16 possible counterexamples to Negami's conjecture. The proof of this theorem is included in Chapter VI. Notice that this result speaks just about *possible* counterexamples, since no counterexample to the conjecture has been found so far, and it is believed that none exists. Negami's conjecture is still open.

An equivalent formulation of Negami's conjecture [17], discovered by the author in 1998, is proposed in Section 7.2. The new formulation has a natural generalization to nonorientable surfaces other than the projective plane. Some support for the generalized conjecture in the case of the Klein bottle is provided.

Finally, a remark on Fellows' planar emulator conjecture [13] is added in Section 7.4. It is shown that Fellows' conjecture cannot be generalized to the triple-torus.

CHAPTER III

THE GRAPH $K_{4,4}-e$

3.1 Supposed Planar Cover of $K_{4,4}-e$

Let the vertices of the graph $\boldsymbol{K}_{4,4}-e$ be denoted by a, b, c, d, e, f, s, t as in Fig. 3.1. Suppose that a connected plane graph $\boldsymbol{H}$ is a cover of $\boldsymbol{K}_{4,4}-e$, determined by the projection $\varphi : V(\boldsymbol{H}) \to V(\boldsymbol{K}_{4,4}-e)$. Notice that $\boldsymbol{H}$ is a simple graph. The value $\varphi(v)$, for a vertex $v \in V(\boldsymbol{H})$, is called the *label* of v.

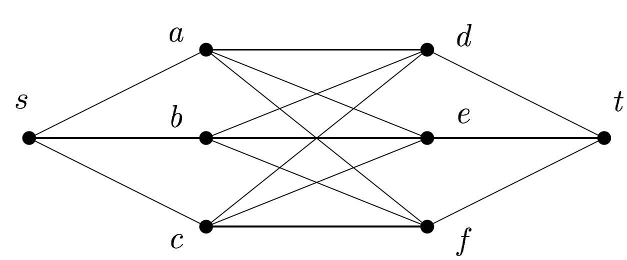


Figure 3.1: The graph $\boldsymbol{K}_{4,4}-e$.

Since $\boldsymbol{H}$ is a cover of $\boldsymbol{K}_{4,4}-e$, the following holds: If u is a vertex of $\boldsymbol{H}$ such that $\varphi(u) = s$ ($\varphi(u) = t$), then u has exactly three neighbors labeled a, b, c (d, e, f); and if u is a vertex of $\boldsymbol{H}$ such that $\varphi(u) \in \{a, b, c\}$ ($\varphi(u) \in \{d, e, f\}$), then u has exactly four neighbors labeled s, d, e, f (t, a, b, c). Let $V_s = \varphi^{-1}(s)$, $V_t = \varphi^{-1}(t)$. It follows that each vertex of $\boldsymbol{H}$ is at distance of at most 1 from some vertex of $V_s \cup V_t$. A plane graph $\boldsymbol{H}_1$ on the vertex set $V_s \cup V_t$ is constructed from $\boldsymbol{H}$ by contracting all edges incident with the vertices of $V_s \cup V_t$. (Recall that our definition of a graph allows multiple edges.) Then $\boldsymbol{H}_1$ is a 9-regular bipartite plane graph with bipartition V_s, V_t.

A sequence $\{e_1, e_2, \ldots, e_k\} \subseteq \boldsymbol{H}_1$ of edges of $\boldsymbol{H}_1$ is called a *bunch* of edges if it is an inclusion-wise maximal subset of $E(\boldsymbol{H}_1)$ satisfying

- all $e_1, e_2, \ldots, e_k$ have the same pair of endvertices (i.e. they are parallel edges),
- the pair e_i, e_{i+1} bounds a face of length 2 in $\boldsymbol{H}_1$, for all $i = 1, \ldots, k-1$.

Notice that each edge of $\boldsymbol{H}_1$ belongs to some bunch, possibly of size one, and that there may be more than one bunch of edges between the same pair of vertices. A spanning subgraph $\boldsymbol{H}_2 \subseteq \boldsymbol{H}_1$ is formed by deleting all but one edge from every bunch of edges in $\boldsymbol{H}_1$. If $e = uv$ is an edge of $\boldsymbol{H}_2$, and $E_e \ni e$ is the bunch of edges of $\boldsymbol{H}_1$ containing e, then the *thickness* of e is defined as the number of edges in E_e; and a subgraph $\boldsymbol{F}_e \subset \boldsymbol{H}$ is defined as follows: Let W_u be the set containing u and its three neighbors in $\boldsymbol{H}$, and let W_v be the set containing v and its three neighbors in $\boldsymbol{H}$. The vertex set of $\boldsymbol{F}_e$ is $V(\boldsymbol{F}_e) = W_u \cup W_v$, the edge set is $E(\boldsymbol{F}_e) = E(\boldsymbol{H} \restriction W_u) \cup E(\boldsymbol{H} \restriction W_v) \cup E_e$, and the incidence function ε_{F_e} is the restriction of ε_H to $E(\boldsymbol{F}_e)$. (Informally, $\boldsymbol{F}_e$ is the subgraph of $\boldsymbol{H}$ that corresponds to the bunch E_e "before the contraction".) Notice that $\boldsymbol{F}_e$ need not be an induced subgraph of $\boldsymbol{H}$ if there is another bunch of edges between u, v.

Lemma 3.1.1. *(a) The graph $\boldsymbol{H}_2$ is a bipartite plane graph without loops or faces of length 2 (but not necessarily simple).*

(b) For each $v \in V(\boldsymbol{H}_2)$, the sum of thicknesses of edges incident with v in $\boldsymbol{H}_2$ is 9.

(c) The graph $\boldsymbol{H}_2$ is 2-connected.

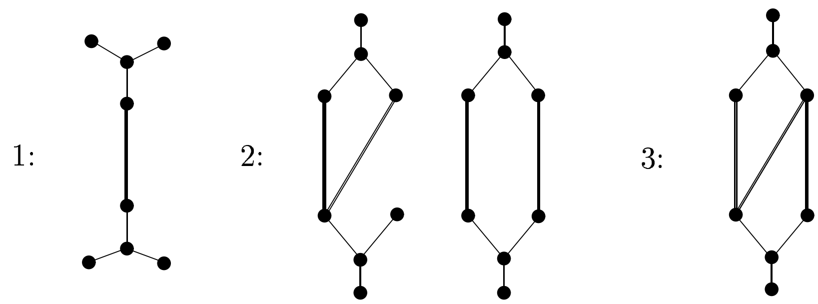


Figure 3.2:

(d) Each graph $\boldsymbol{F}_e$, $e \in E(\boldsymbol{H}_2)$ is isomorphic to one of the graphs from Fig. 3.2.

Proof. Parts (a),(b) follow easily from the fact that $\boldsymbol{H}_1$ is a 9-regular bipartite plane graph, and from the way in which $\boldsymbol{H}_2$ was constructed. To show (c) that $\boldsymbol{H}_2$ is 2-connected, suppose that (A, B) is a nontrivial separation of order 1 in $\boldsymbol{H}_2$, and that $\boldsymbol{H}'_2$ is the subgraph induced on A. Let $\{v\} = A \cap B$. Recall that $\boldsymbol{H}_2$ has the bipartition V_s,V_t, and assume that $v \in V_s$. Let p be the sum of thicknesses of edges between v and $\boldsymbol{H}_2 - V(\boldsymbol{H}'_2)$, then $1 \leq p \leq 8$. The total sum of thicknesses of edges of $\boldsymbol{H}'_2$ is equal to $9\,|V(\boldsymbol{H}'_2) \cap V_t|$ (counted according to the part induced by V_t), but, at the same time, it is equal to $9\,|V(\boldsymbol{H}'_2) \cap V_s| - p$ (counted according to the part induced by V_s), a contradiction.

Finally, to show that (d) holds, let $e = uv \in E(\boldsymbol{H}_2)$. The above definition of the graph $\boldsymbol{F}_e \subset \boldsymbol{H}$ implies that the vertices of $\boldsymbol{H} - V(\boldsymbol{F}_e)$ are all embedded in the same face of $\boldsymbol{F}_e$. Let u_1, u_2, u_3; v_1, v_2, v_3 be the neighbors of u, v, respectively, in $\boldsymbol{H}$. First, if each of the vertices u_1, u_2, u_3 is adjacent to one of v_1, v_2, v_3, then the graph $\boldsymbol{F}'_e$ obtained by contracting the edges vv_1, vv_2, vv_3 in $\boldsymbol{F}_e$ has a $\boldsymbol{K}_{2,3}$ spanning subgraph. Since $\boldsymbol{K}_{2,3}$ is not outerplanar, one of u_1, u_2, u_3, say u_1, is adjacent to no vertex of $\boldsymbol{H} - V(\boldsymbol{F}_e)$, and hence $u_1v_1, u_1v_2, u_1v_3 \in E(\boldsymbol{F}_e)$ by the definition of cover. Then $\boldsymbol{F}_e - \{u, u_2, u_3\}$ is isomorphic to $\boldsymbol{K}_{2,3}$, so, by the same argument, it can be assumed that one of v_1, v_2, v_3, say v_1, is adjacent to all three vertices u_1, u_2, u_3. One can check that $\boldsymbol{F}_e$ has no planar embedding in which the vertices u_2, u_3, v_2, v_3 are incident with the same face. So, by repeating the previous argument once more, it can be assumed, without loss of generality, that v_2 is also adjacent to all three vertices u_1, u_2, u_3. However, in such a case $\boldsymbol{F}_e$ has a $\boldsymbol{K}_{3,3}$ subgraph with the partition $\{u, v_1, v_2\}$, $\{u_1, u_2, u_3\}$, which is a contradiction to planarity of $\boldsymbol{H}$.

Therefore, without loss of generality, the vertices u_3, v_3 have degree 1 in the graph $\boldsymbol{F}_e$. If all four edges $u_1v_1, u_1v_2, u_2v_1, u_2v_2$ are present in $\boldsymbol{F}_e$, then $\boldsymbol{F}_e$ contains a non-outerplanar $\boldsymbol{K}_4$ minor on the vertices u_1, u_2, v_1, v_2, and hence, similarly as above, one of u_1, u_2, v_1, v_2, say u_1, has no neighbor in $\boldsymbol{H} - V(\boldsymbol{F}_e)$. But then u_1 is adjacent to no vertex other than u, v_1 or v_2, so it has degree at most 3 in $\boldsymbol{H}$, a contradiction. It follows that the graphs in Fig. 3.2 are the only possibilities, up to isomorphism, for $\boldsymbol{F}_e$. ∎

Corollary 3.1.2. *The graph $\boldsymbol{H}_2$ has no vertex of degree less than 3. If $e = uv, e' = uw$ are two edges incident with vertex u, consecutive in the embedding of $\boldsymbol{H}_2$ (i.e. the edges e, e' are incident with the same face), then $v \neq w$.*

Proof. The first statement follows immediately. Suppose, for a contradiction, that $v = w$ for the two consecutive edges $e = uv, e' = uw$. Since there are no 2-faces in $\boldsymbol{H}_2$, in such a case the open region bounded by the edges e, e' and not containing other edges incident with u, must contain some vertices of $\boldsymbol{H}_2$. Consequently, these vertices are not adjacent to u. Hence $v = w$ is a cut-vertex, which is a contradiction to Lemma 3.1.1(c). ∎

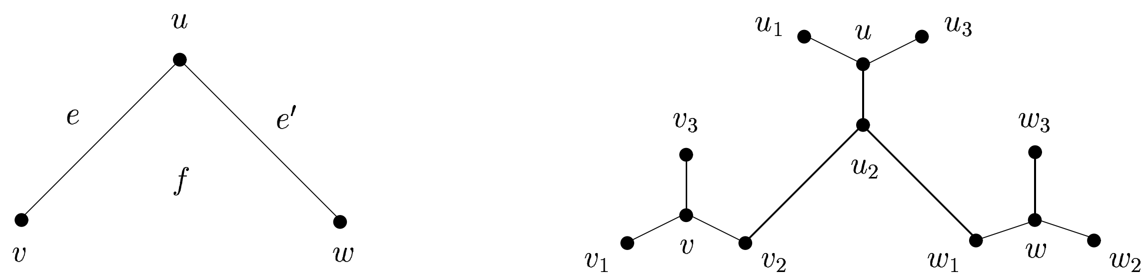


Figure 3.3: An illustration of a ◇-basis.

Further, a ◇-basis is defined. Recall that each vertex of $\varphi^{-1}(s) \cup \varphi^{-1}(t) = V(\boldsymbol{H}_2)$ has three distinct neighbors in $\boldsymbol{H}$. Two edges $e = uv$, $e' = uw$ of $\boldsymbol{H}_2$ sharing a common vertex u are said to form a ◇-*basis* if the following conditions are satisfied.

- The vertex v is distinct from w, and e, e' are two consecutive edges on a boundary of a face f of $\boldsymbol{H}_2$ such that w, u, v appear in counterclockwise orientation.
- The neighbors of u, v, w in $\boldsymbol{H}$ can be denoted in counterclockwise orientation by u_1, u_2, u_3; v_1, v_2, v_3; w_1, w_2, w_3 respectively, so that $\varphi(v_1) = \varphi(w_1)$, $\varphi(v_2) = \varphi(w_2)$, $\varphi(v_3) = \varphi(w_3)$, and that u_2v_2, u_2w_1 are edges of $\boldsymbol{F}_e$, $\boldsymbol{F}_{e'}$ respectively.

(See an illustration in Fig. 3.3; other possible edges between the vertices in the picture are not drawn.)

3.2 Discharging Rules

A discharging argument is used to show that the graph $\boldsymbol{H}_2$ cannot exist. Generally, a *discharging argument* first assigns certain *charge* to vertices, edges, and/or faces of a graph, then it redistributes the charge according to specified *discharging* *rules*, and finally it shows that the total sum of the charge has changed, which leads to a contradiction. In this particular case, the starting charges and the discharging rules are defined as follows.

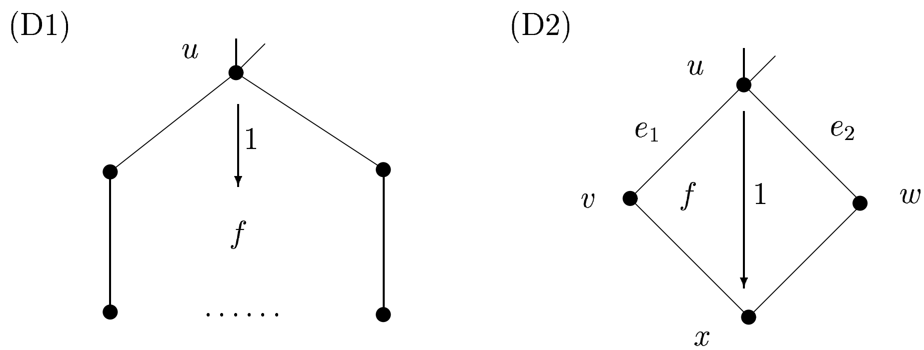


Figure 3.4: An illustration of the discharging rules.

Initial charges. If v is a vertex of $\boldsymbol{H}_2$ of degree d, then v starts with a charge of $3(4-d)$. If f is a face of $\boldsymbol{H}_2$ of length k, then f starts with a charge of $3(4-k)$. The edges of $\boldsymbol{H}_2$ have no charge.

Discharging rules.

(D1) Each vertex u of $\boldsymbol{H}_2$ sends a charge of 1 to each incident face of length greater than 4.

(D2) If f is a 4-face bounded by a cycle $wuvx$, and the edges $e_1 = uw$ and $e_2 = uw$ incident with f form a $\diamond$-basis, then the vertex u sends a charge of 1 to the vertex x.

(See Fig. 3.4.)

3.3 Finding a Contradiction

Notice that since $\boldsymbol{H}_2$ has no faces of length 2 by Lemma 3.1.1(a), all faces start with nonpositive charges. Vertices of degree 3 start with positive charges, all other vertices have nonpositive charges at the beginning. Our goal is to show that after applying the above discharging rules, all vertices and all faces end up with nonpositive charges.

Lemma 3.3.1. *Each face of $\boldsymbol{H}_2$ ends up with a nonpositive charge.*

Proof. As it was noted above, each face starts with a nonpositive charge. A face f of length k receives charge only through the rule (D1), provided that $k \geq 6$. (The graph $\boldsymbol{H}_2$ is bipartite, and hence k is even.) In such a case, f does not receive more than a charge of 1 from each of the incident vertices, thus it ends up with a charge of at most $3(4-k)+k = 12-2k \leq 0$. ∎

Lemma 3.3.2. *Suppose that $ue_1ve_3xe_4we_2$ is a 4-cycle in $\boldsymbol{H}_2$ bounding a face f, such that $e_1 = uv$ and $e_2 = uw$ form a $\diamond$-basis. (See Fig. 3.5.)*

(a) Each of the three neighbors of the vertex x in $\boldsymbol{H}$ has degree at most 3 in the subgraph $\boldsymbol{F}_{e_3} \cup \boldsymbol{F}_{e_4} \subseteq \boldsymbol{H}$.

(b) The vertex x has degree at least 4 in the graph $\boldsymbol{H}_2$.

(c) If x has degree 4 in $\boldsymbol{H}_2$, and e_3, e_5, e_6, e_4 are the four edges of $\boldsymbol{H}_2$ incident with x listed in the counterclockwise order, then e_5, e_6 form a $\diamond$-basis.

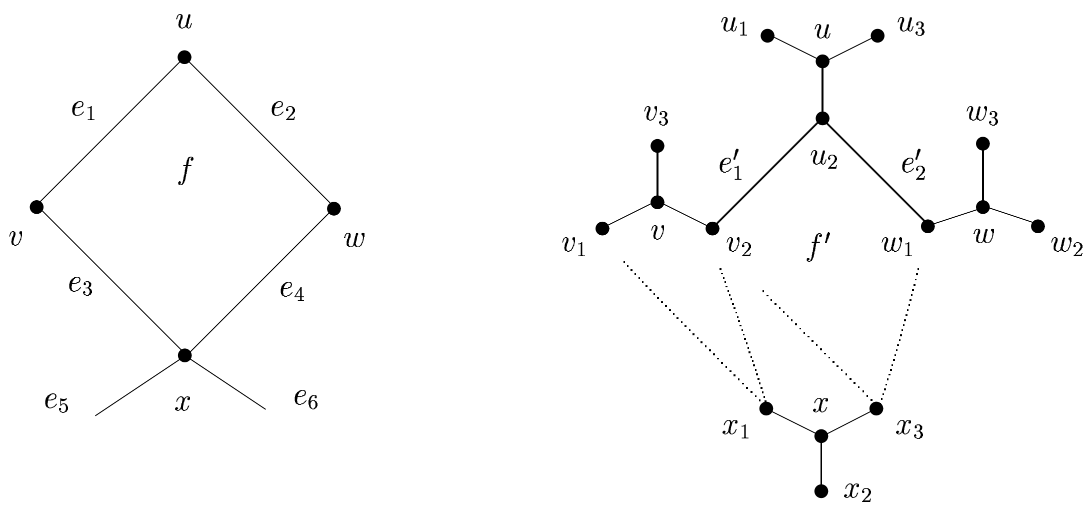


Figure 3.5:

Proof. Let u_1, u_2, u_3; v_1, v_2, v_3; w_1, w_2, w_3 denote the neighbors of u, v, w in $\boldsymbol{H}$, respectively, according to the definition of the $\diamond$-basis. Let x_1, x_2, x_3 be the neighbors of x in $\boldsymbol{H}$ in the counterclockwise orientation. Notice that since $\varphi(v_1) = \varphi(w_1)$ and $u_2w_1 \in E(\boldsymbol{F}_{e_2})$ by the definition of the $\diamond$-basis, there is no edge between u_2 and v_1 in $\boldsymbol{H}$. Similarly, there is no edge between u_2 and w_2 in $\boldsymbol{H}$. Let $e_1' = u_2v_2$ and $e_2' = u_2w_1$. If there was an edge e' incident with u_2 in $\boldsymbol{H}$ such that e_1',e',e_2' were consecutive edges in the counterclockwise order around u_2, then e' would belong, by the definition of the $\diamond$-basis, to one of the graphs $\boldsymbol{F}_{e_1}$ or $\boldsymbol{F}_{e_2}$, say $\boldsymbol{F}_{e_1}$, and hence $e' = u_2v_3$. But then, by planarity, the graph $\boldsymbol{F}_{e_1}$ would include all three edges incident with the vertex v_1 in $\boldsymbol{H} - v$, which contradicts Lemma 3.1.1(d). Hence the edges e_1' and e_2' are incident with the same face f' in $\boldsymbol{H}$. (The face f' corresponds to f.) Let e_3' be the edge of $\boldsymbol{F}_{e_3} - \{v, x\}$ incident with f' in $\boldsymbol{H}$. Then e_3', e_1' are two consecutive edges in the counterclockwise order around the vertex v

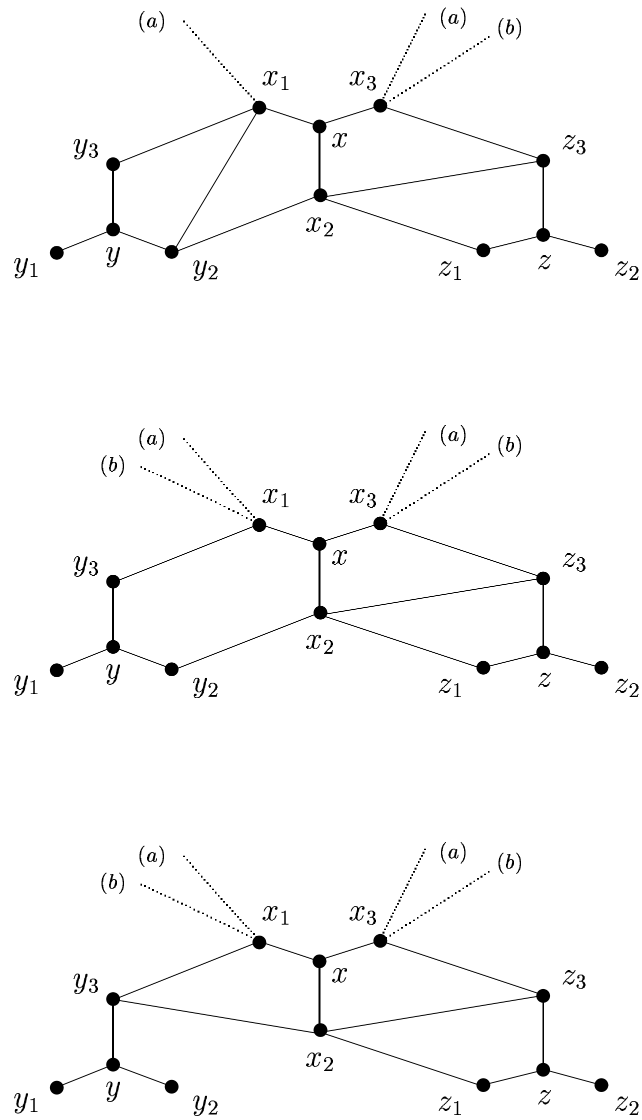
in the graph $\boldsymbol{H}_1$ (cf. Section 3.1). Suppose, for a contradiction, that there is an edge $e'' \in E(\boldsymbol{F}_{e_3} - v)$ incident with the vertex v_3. Then the edges e'' and e_3' belong to the same bunch of edges E_{e_3} in $\boldsymbol{H}_1$ by the definition of $\boldsymbol{F}_{e_3}$. Since the edge e_1' is incident with v_2 in $\boldsymbol{H}$, it follows, by planarity, that the bunch E_{e_3} contains all three edges incident with v_1 in $\boldsymbol{H} - v$, and hence v_1 has degree four in $\boldsymbol{F}_{e_3}$, which contradicts Lemma 3.1.1(d).

Therefore, there are no edges between v_3 and x_1, x_2, x_3 in $\boldsymbol{F}_{e_3}$. (However, this is *not* saying that there is no edge between v_3 and one of x_1, x_2, x_3 in $\boldsymbol{H}$.) Similarly, it is shown that there are no edges between w_3 and x_1, x_2, x_3 in $\boldsymbol{F}_{e_4}$. Since $\varphi(v_1) = \varphi(w_1)$ and $\varphi(v_2) = \varphi(w_2)$, each of the vertices x_1, x_2, x_3 is adjacent to at most two of the vertices v_1, v_2, w_1, w_2 in $\boldsymbol{F}_{e_3} \cup \boldsymbol{F}_{e_4}$, and the statement (a) follows.

In particular, the above conclusion implies that each of the vertices x_1, x_2, x_3 is adjacent to some vertex of the graph $\boldsymbol{H} - \{v_1, v_2, v_3, w_1, w_2, w_3\}$, and hence, by Lemma 3.1.1(d), there are at least two edges e_5, e_6 incident with x in $\boldsymbol{H}_2$ and distinct from e_3, e_4. Therefore, the statement (b) is proved.

So let x have degree 4 in $\boldsymbol{H}_2$, and let e_3, e_5, e_6, e_4 be the edges incident with x in $\boldsymbol{H}_2$, listed in the counterclockwise orientation. Let e_5 have ends x, y, and let e_6 have ends x, z. (It may happen that u, w, y, z are not all distinct, but $y \neq z$ by Corollary 3.1.2.) Let the neighbors of y, z in $\boldsymbol{H}$ be denoted by y_1, y_2, y_3; z_1, z_2, z_3 in the counterclockwise orientation. The set of edges of the graph $(\boldsymbol{F}_{e_3} \cup \boldsymbol{F}_{e_4}) - \{x, v, w\}$ is denoted by E_x, and the set of edges of $(\boldsymbol{F}_{e_5} \cup \boldsymbol{F}_{e_6}) - \{x, y, z\}$ is denoted by E_x'. Notice that $E_x \cap E_x' = \emptyset$ and $|E_x \cup E_x'| = 9$.

It has been shown above that no edge of E_x is incident with v_3 or w_3. Suppose, for a contradiction, that each of the vertices x_1, x_2, x_3 is incident with some edge of E_x. In such a situation, contracting the path $P = v_1 v v_2 u_2 w_1 w w_2$ creates a non-outerplanar minor $\boldsymbol{F}$ of $\boldsymbol{H}$ isomorphic to the graph $\boldsymbol{K}_{2,3}$, such that $\{x_1, x_2, x_3\}$ is one


(a)
(a)
(b)
x_1
x_3
x
y_3
z_3
x_2
y_1
y
y_2
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z_2
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Figure 3.6:

of the parts of $\boldsymbol{F}$, and that all vertices of $\boldsymbol{H} - V(P) - \{x, x_1, x_2, x_3\}$ are embedded in the same face of $\boldsymbol{F}$. Thus one of the vertices x_1, x_2, x_3, say x_1, is adjacent only to vertices from v_1, v_2, w_1, w_2, but none of them is labeled $\varphi(v_3)$, a contradiction. Hence assume, without loss of generality, that no edge of E_x is incident with the vertex x_2. This implies, in particular, that x_2 is incident with three edges of E'_x, and that $|E_x| \leq 4$, so $|E'_x| \geq 5$. On the other hand, $|E'_x| \leq 6$, so $|E_x| \geq 3$. Without loss of generality, it may be assumed that x_3 is incident with two edges from E_x, and that x_1 is incident with one or two edges from E_x. One may easily check that, by Lemma 3.1.1(d), the graph $\boldsymbol{F}_{e_5} \cup \boldsymbol{F}_{e_6}$ is isomorphic to one of the graphs in Fig. 3.6, up to symmetry between y, z in the latter two cases.

Assume, for simplicity, that the labels of the vertices adjacent to x_1, x_3 via edges from E_x are a, b (not necessarily in this order), as depicted in Fig. 3.6. Recall that each of the vertices x_1, x_2, x_3 should be adjacent to vertices of the three distinct labels a, b, c in $\boldsymbol{H}$. So the following can be deduced about the graphs in Fig. 3.6:

- For the first graph (top of the figure), it follows that $\varphi(z_3) = c$. One of the vertices y_2, z_1, z_3 must be labeled a, and it cannot be y_2 due to the vertex x_1; hence $\varphi(z_1) = a$. Consequently, $\varphi(y_2) = b = \varphi(z_2)$, $\varphi(y_3) = c = \varphi(z_3)$, and $\varphi(y_1) = a = \varphi(z_1)$.

- For the second graph (middle of the figure), it follows that $\varphi(y_3) = \varphi(z_3) = c$. Then $\{\varphi(y_1), \varphi(y_2)\} = \{\varphi(z_1), \varphi(z_2)\} = \{a, b\}$, and since $\varphi(y_2) \neq \varphi(z_1)$, it is concluded that $\varphi(y_1) = \varphi(z_1)$ and $\varphi(y_2) = \varphi(z_2)$.

- For the third graph (bottom of the figure), it again follows that $\varphi(y_3) = \varphi(z_3) = c$. However, x_2 is now adjacent to two vertices of the same label, a contradiction.

Consequently, the edges e_5, e_6 form a $\diamond$-basis in $\boldsymbol{H}_2$. ∎

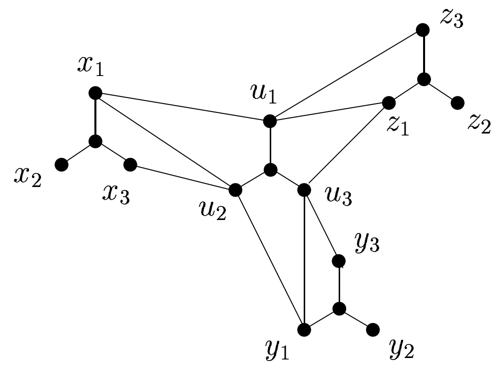


Figure 3.7:

Lemma 3.3.3. *Each vertex of $\boldsymbol{H}_2$ ends up with a nonpositive charge.*

Proof. By Corollary 3.1.2, the graph $\boldsymbol{H}_2$ has minimum degree at least 3. Suppose that u is a vertex of degree $d \geq 3$ in $\boldsymbol{H}_2$. Let e_i, $i = 1, \ldots d$ be the edges incident with u in $\boldsymbol{H}_2$ in the counterclockwise orientation. Let f_i, $i = 1, \ldots d$ be the faces incident with u in $\boldsymbol{H}_2$, listed in the order such that f_i is incident with e_i and e_{i+1}. (Indices are considered modulo d.) Since $\boldsymbol{H}_2$ is 2-connected, the faces $f_1, \ldots, f_d$ are pairwise distinct. The neighbors of u in $\boldsymbol{H}$ are denoted by u_1, u_2, u_3 in the counterclockwise orientation.

Suppose that $d = 3$, and that $e_1 = ux, e_2 = uy, e_3 = uz$. The vertices x, y, z are pairwise distinct by Corollary 3.1.2. In such a case, the vertex u starts with a charge of $3(4-3) = 3$. By Lemma 3.1.1(b,d), the subgraph $\boldsymbol{F}_{e_1} \cup \boldsymbol{F}_{e_2} \cup \boldsymbol{F}_{e_3}$ is isomorphic to the graph in Fig. 3.7. It follows from the cover properties that $\varphi(z_1) = \varphi(y_2)$, etc. Consequently, $\varphi(x_1) = \varphi(y_3) = \varphi(z_2)$, $\varphi(x_2) = \varphi(y_1) = \varphi(z_3)$, and $\varphi(x_3) = \varphi(y_2) = \varphi(z_1)$. If the face f_i, $i \in \{1, 2, 3\}$ has length greater than 4, then the rule (D1) applies, and hence u sends a charge of 1 to f_i. If the face f_i,

$i \in \{1, 2, 3\}$ has length 4, and t_i is the vertex opposite to u on the boundary of f_i, then it follows from Lemma 3.3.2(b) that u does not receive any charge from t_i by an application of the rule (D2). On the other hand, the pair of edges e_i, e_{i+1} incident with f_i forms a $\diamond$-basis by the above arguments, and hence u sends a charge of 1 to t_i by the rule (D2). Therefore in any case, u ends up with a charge of $3 - 3 = 0$.

Now, consider $d = 4$. If the face f_i, $i \in \{1, 2, 3, 4\}$ is of length 4, and the vertex u receives a charge of 1 from the vertex t_i that is opposite to u on the boundary of f_i, then the edges e_{i+2}, e_{i+3} incident with the opposite face f_{i+2} (indices modulo 4) form a $\diamond$-basis by Lemma 3.3.2(c). Hence, if the length of f_{i+2} is 4, then u sends a charge of 1 to the vertex t_{i+2} opposite to u on the boundary of f_{i+2} by the rule (D2); and, if the length of f_{i+2} is greater than 4, then u sends a charge of 1 to the face f_{i+2} by the rule (D1). Consequently, u ends up with a charge not larger than the charge of $3(4 - 4) = 0$ it started with.

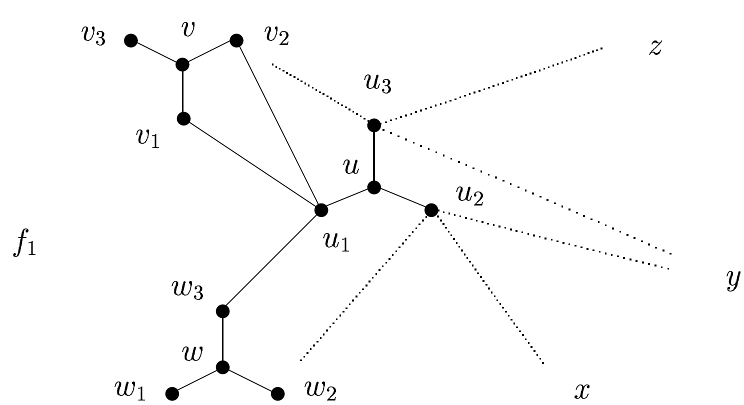


Figure 3.8:

Let $d = 5$, and let $e_1 = uv$, $e_2 = uw$, $e_3 = ux$, $e_4 = uy$, $e_5 = uz$ be the edges incident with u in $\boldsymbol{H}_2$. For $j = 1, 2, 3$, let L_j denote the set of all indices $l \in \{1, \ldots, 5\}$ such that the vertex u_j has degree at least two in the graph $\boldsymbol{F}_{e_l}$. Notice that, by planarity, each of L_1, L_2, L_3 consists of consecutive indices (where 5 and 1 are regarded as consecutive). It follows that $|L_j| \leq 3$, and that $|L_j| \geq 2$ by Lemma 3.1.1(d). If each of the sets L_1, L_2, L_3 has three elements, then u must be incident with at least 6 edges in $\boldsymbol{H}_2$ by the planarity of $\boldsymbol{H}$, a contradiction. So it may be assumed, without loss of generality, that $L_1 = \{1, 2\}$; which means that the vertex u_1 has degree 4 in the subgraph $\boldsymbol{F}_{e_1} \cup \boldsymbol{F}_{e_2}$. By symmetry, it may be assumed that the degree of u_1 in $\boldsymbol{F}_{e_1}$ is 3. Let the neighbors of v, w in $\boldsymbol{H}$ be denoted by v_1, v_2, v_3; w_1, w_2, w_3 in the counterclockwise orientation, so that u_1v_1, u_1v_2 are edges of $\boldsymbol{F}_{e_1}$, and that u_1w_3 is an edge of $\boldsymbol{F}_{e_2}$. Notice that $v \neq w$ by Corollary 3.1.2. (See Fig. 3.8.) If the face f_1 (incident with e_1, e_2 in $\boldsymbol{H}_2$) is of length 4, and t is the vertex opposite to u on the boundary of f_1, then it follows by Lemma 3.3.2(a) that the edges tv, tw incident with f may not form a $\diamond$-basis, and hence u receives no charge from t.

Consequently, u receives at most a charge of 4. If u receives a charge of 3 or less, then it ends up with at most a charge of $3(4-5)+3 = 0$, as desired. So suppose that u receives a total charge of 4. Observe that if any two edges $e = pr$, $e' = ps$ form a $\diamond$-basis, then the neighbors of r, s in $\boldsymbol{H}$ are labeled (by the cover projection) in the same cyclic order. Hence if, for each $i = 2, 3, 4, 5$, the face f_i has length 4, and the two edges not incident with u on the boundary of f_i form a $\diamond$-basis, then then the neighbors of w, x, y, z and v in $\boldsymbol{H}$ are labeled in the same cyclic order. Since $\varphi(w_3)$ is distinct from both $\varphi(v_1), \varphi(v_2)$, it follows that $\varphi(w_3) = \varphi(v_3)$, $\varphi(w_2) = \varphi(v_2)$, and $\varphi(w_1) = \varphi(v_1)$. Thus the edges e_1, e_2 form a $\diamond$-basis; so if the length of f_1 is 4, then u sends a charge of 1 to t; otherwise, u sends a charge of 1 to f_1. Again, u ends up with at most a charge of $3(4-5)+4-1=0$.

Finally, let $d \geq 6$. It is clear that the vertex u may receive at most a charge of 1 per each incident face in the discharging process, and hence it ends up with at most a charge of $3(4-d)+d = 12-2d \leq 0$, which is as desired. ∎

Proof of Theorem 2.4.3. Let $l_G(f)$ denote the length of a face f in a plane graph $\boldsymbol{G}$. According to the discharging rules and to the Euler's formula, the graph $\boldsymbol{H}_2$ starts with the total charge of

$$\sum_{v \in V(H_2)} 3(4 - d_{H_2}(v)) + \sum_{f \in F(H_2)} 3(4 - l_{H_2}(f)) =$$

$$12|V(\boldsymbol{H}_2)| - 6|E(\boldsymbol{H}_2) + 12|F(\boldsymbol{H}_2)| - 6|E(\boldsymbol{H}_2) = 24.$$

Since all charges of vertices and faces in $\boldsymbol{H}_2$ are nonpositive at the end of the discharging process by Lemmas 3.3.1 and 3.3.3, the graph $\boldsymbol{H}_2$ ends up with total charge of at most 0. However, the discharging process just redistributes existing charges, so no charge is lost during the process. This contradiction shows that the graph $\boldsymbol{H}_2$, and hence, also a planar cover of $\boldsymbol{K}_{4,4}-e$, cannot exist. ∎

CHAPTER IV

THE GRAPH $\mathcal{E}_2$

4.1 Handling a Supposed Planar Cover of $\mathcal{E}_2$

Let the vertices of the graph $\mathcal{E}_2$ be denoted by a_1, a_2, a_3, a_4, $b_1, b_2, b_3, b_4, b_5, b_6$, x, as in Fig. 4.1. Suppose, for a contradiction, that there exists a connected n-fold planar cover $\boldsymbol{H}$ of $\mathcal{E}_2$. Since $\mathcal{E}_2$ is simple, $\boldsymbol{H}$ is also simple, so the cover is described by the vertex projection $\varphi : V(\boldsymbol{H}) \to V(\mathcal{E}_2)$. The graph $\boldsymbol{H}$ is treated as a plane graph here.

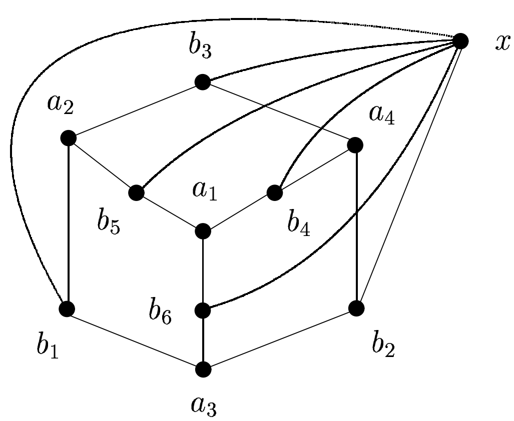


Figure 4.1: The graph $\mathcal{E}_2$.

A plane graph $\boldsymbol{H}_4$ on the vertex set $V(\boldsymbol{H}_4) = \varphi^{-1}(a_1) \cup \varphi^{-1}(a_2) \cup \varphi^{-1}(a_3) \cup \varphi^{-1}(a_4)$ is constructed from $\boldsymbol{H}$ by deleting all vertices u of $\boldsymbol{H}$ for which $\varphi(u) = x$, and by suppressing all vertices w of $\boldsymbol{H}$ such that $\varphi(w) \in \{b_1, b_2, b_3, b_4, b_5, b_6\}$. The planar embedding of $\boldsymbol{H}_4$ follows in a natural way from the planar embedding of $\boldsymbol{H}$.

Let the mappings $\psi : V(\boldsymbol{H}_4) \to \{1, 2, 3, 4\}$, $\eta : E(\boldsymbol{H}_4) \to \{1, 2, \ldots, n\}$, and $\lambda : \{1, 2, \ldots, n\} \to F(\boldsymbol{H}_4)$ be defined by the following rules: For a vertex v of $\boldsymbol{H}_4$, let $\psi(v) = i$ if $\varphi(v) = a_i$. Assuming $\varphi^{-1}(x) = \{x_1, x_2, \ldots, x_n\}$, define $\eta(e) = j$ if e is an edge of $\boldsymbol{H}_4$ that was formed by suppressing a vertex $w \in V(\boldsymbol{H})$ with $wx_j \in E(\boldsymbol{H})$. For $1 \leq j \leq n$, define $\lambda(j) = f$ if f is a face of $\boldsymbol{H}_4$ containing x_j. (Notice that x_j is a vertex of $\boldsymbol{H}$, but not of $\boldsymbol{H}_4$, so x_j lies inside some face of $\boldsymbol{H}_4$.) It follows easily from the fact that $\boldsymbol{H}$ covers $\boldsymbol{\mathcal{E}}_2$ that these mappings are well-defined.

Lemma 4.1.1. *The plane graph $\boldsymbol{H}_4$, and the mappings ψ, η, λ, satisfy the following properties.*

(a) $\boldsymbol{H}_4$ is a simple 2-connected 3-regular graph on $4n$ vertices, and ψ is a cover projection of $\boldsymbol{H}_4$ onto $\boldsymbol{K}_4$, the complete graph on the vertex set $\{1, 2, 3, 4\}$. In particular, any two vertices $v \neq w$ of $\boldsymbol{H}_4$ satisfying $\psi(v) = \psi(w)$ must be at distance of at least 3 apart.

(b) If e is an edge of $\boldsymbol{H}_4$, then $\lambda(\eta(e))$ is a face incident with e. Consequently, for a face f and $j \in \lambda^{-1}(f)$, the edges of $\eta^{-1}(j)$ lie on the boundary of f.

(c) For each $j \in \{1, 2, \ldots, n\}$, $\eta^{-1}(j)$ has six elements, and $\{\{\psi(u), \psi(v)\} : uv \in \eta^{-1}(j)\}$ is the collection of all six two-element subsets of $\{1, 2, 3, 4\}$.

(d) Let f be a face of $\boldsymbol{H}_4$, and $j_1, j_2 \in \lambda^{-1}(f)$. If e_1, e_2, e_3, e_4 are four edges of f in this cyclic order (not necessarily consecutive), and $\eta(e_1) = \eta(e_3) = j_1$, $\eta(e_2) = \eta(e_4) = j_2$, then $j_1 = j_2$.

Proof. (a) All these properties, except the first one, follow immediately from the definition of $\boldsymbol{H}_4$. Since $\boldsymbol{H}_4$ is connected and 3-regular, it is enough to show that it is edge 2-connected. Indeed, for each edge uv of $\boldsymbol{H}_4$ there is a triangle C in $\boldsymbol{K}_4$ containing the vertices $\psi(u), \psi(v)$. Hence uv belongs to a cycle that is a component of the lifting of C into $\boldsymbol{H}_4$. Thus $\boldsymbol{H}_4$ is edge 2-connected.

(b) Let $w \in V(\boldsymbol{H})$ be the vertex that was suppressed when forming the edge e, and let f, f' be the two faces of $\boldsymbol{H}_4$ incident with e. By definition, $\eta(e) = j$ if and only if wx_j is an edge in $\boldsymbol{H}$ ($\varphi(x_j) = x$). Since $\boldsymbol{H}$ is planar, the vertex x_j is embedded in one of the faces f, f' of $\boldsymbol{H}_4$, thus $\lambda(\eta(e)) \in \{f, f'\}$.

(c) For each $j \in \{1, 2, \ldots, n\}$, the vertex x_j has six neighbors $w_1, \ldots, w_6$, where $\varphi(w_i) = b_i$ in $\boldsymbol{H}$. Let $e_1, \ldots, e_6$ denote the edges of $\boldsymbol{H}_4$ formed by suppressing the vertices $w_1, \ldots, w_6$, respectively. Then $\eta^{-1}(j) = \{e_1, \ldots, e_6\}$. Moreover, each of the six vertices $b_1, \ldots, b_6$ of $\boldsymbol{\mathcal{E}}_2$ has a different pair of vertices a_1, a_2, a_3, a_4 as neighbors. Therefore $\{\psi(u_i), \psi(v_i)\}$ for $e_i = u_i v_i$, $i = 1, \ldots, 6$ are six different pairs of numbers from $1, 2, 3, 4$.

(d) Let w_1, w_2, w_3, w_4 be the vertices that were suppressed when forming the edges e_1, e_2, e_3, e_4, respectively, and let C be the cycle in $\boldsymbol{H}$ corresponding to the boundary of f. If $j_1 \neq j_2$, then $\{w_1, x_{j_1}, w_3\}$ and $\{w_2, x_{j_2}, w_4\}$ are the vertex sets of two disjoint paths embedded in the same face of C. However, this contradicts planarity of $\boldsymbol{H}$ since w_1, w_2, w_3, w_4 lie in this cyclic order on the boundary of C. ∎

4.2 Discharging Rules

A discharging argument is used to show that the graph $\boldsymbol{H}_4$ and the mappings ψ, η, λ with the properties described by Lemma 4.1.1 cannot exist. In this particular case, the starting charges and the discharging rules are defined as follows.

Initial charges. Each face f of $\boldsymbol{H}_4$ starts with a charge of $3k$, where k is the length of f. All edges of $\boldsymbol{H}_4$ start with no charge.

Discharging rules. For any face f of $\boldsymbol{H}_4$, and for any four consecutive vertices u_1, u_2, u_3, u_4 on the boundary of f such that $\psi(u_1) = \psi(u_4)$ (possibly $u_1 = u_4$), the following rule applies: If $\lambda(\eta(u_2u_3)) = f$, then the edge u_2u_3 receives a charge of 1 from f, otherwise u_2u_3 sends a charge of 1 to f. (See also Fig. 4.2.)

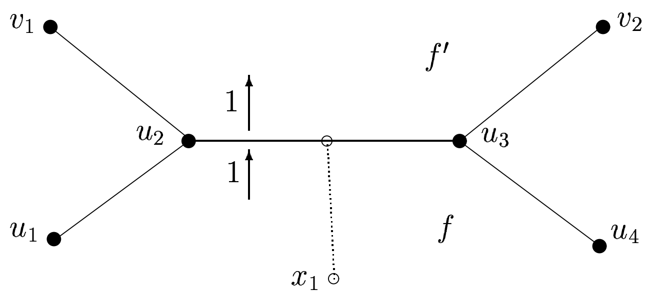


Figure 4.2: An illustration of the discharging rule, $\psi(u_1) = \psi(u_4)$, $\lambda(\eta(u_2u_3)) = f$.

Lemma 4.2.1. *Each edge of $\boldsymbol{H}_4$ ends up with a charge of* 0.

Proof. Let e be an edge of $\boldsymbol{H}_4$. By Lemma 4.1.1(b), $\lambda(\eta(e)) = f$ is a face incident with e. Let u_1, u_2, u_3, u_4 denote four consecutive vertices of f such that $e = u_2u_3$. Let f' denote the other face incident with e, and let v_1, u_2, u_3, v_2 be four consecutive vertices of f'. (See Fig. 4.2.) By Lemma 4.1.1(a), $\psi(u_1), \psi(u_2), \psi(u_3), \psi(v_1)$ form a permutation of $1, 2, 3, 4$. Similarly, $\psi(u_4), \psi(u_2), \psi(u_3), \psi(v_2)$ form a permutation of $1, 2, 3, 4$. Thus $\psi(u_1) = \psi(u_4)$ if and only if $\psi(v_1) = \psi(v_2)$. So if $\psi(u_1) \neq \psi(u_4)$, then no discharging rule applies to e. If $\psi(u_1) = \psi(u_4)$, then

$\psi(v_1) = \psi(v_2)$. Therefore the edge e receives a charge of 1 from the face f and sends a charge of 1 to f', and hence it ends up with no charge. ∎

4.3 Charge of a Face

Since $\boldsymbol{H}_4$ is a simple 2-connected graph, each face is bounded by a cycle of length of at least 3. In order to use induction in the proof of the next lemma, the assumptions about the graph $\boldsymbol{H}_4$ need to be restricted to each face of $\boldsymbol{H}_4$ alone. The following claim is an immediate corollary of Lemma 4.1.1.

Claim 1. Suppose that $\boldsymbol{C}$ is the cycle bounding a face f of $\boldsymbol{H}_4$. Let $Y \subseteq E(\boldsymbol{C})$ be the set defined by $Y = \eta^{-1}(\lambda^{-1}(f))$. Let ψ' be the restriction of ψ to $V(\boldsymbol{C})$, and let η' be the restriction of η to Y.

(a) If $v \neq w$ are two vertices of $\boldsymbol{C}$, and $\psi'(v) = \psi'(w)$, then the distance between v, w is at least 3.

(b) For $j \in \eta'[Y]$, $\eta'^{-1}(j)$ is a set of six edges of $\boldsymbol{C}$, and $\{\{\psi'(u), \psi'(v)\} : uv \in \eta'^{-1}(j)\}$ is the collection of all six two-element subsets of $\{1, 2, 3, 4\}$.
(The symbol $f[A]$ stands for the image of A under f.)

(c) If $e_1, e_2, e_3, e_4 \in Y$ are four edges of the cycle $\boldsymbol{C}$ in this cyclic order (not necessarily consecutive), and $\eta'(e_1) = \eta'(e_3) = j_1$, $\eta'(e_2) = \eta'(e_4) = j_2$, then $j_1 = j_2$.

□

The discharging rules are reformulated for the cycle $\boldsymbol{C}$ (which stands for the cycle bounding f now), the set Y, and the mapping ψ' as follows:

Claim 2. The cycle $\boldsymbol{C}$ starts with a charge of $3|V(\boldsymbol{C})|$. Whenever u_1, u_2, u_3, u_4 are four consecutive vertices of $\boldsymbol{C}$ (possibly $u_1 = u_4$) such that $\psi'(u_1) = \psi'(u_4)$, the edge u_2u_3 receives a charge of 1 from $\boldsymbol{C}$ if $u_2u_3 \in Y$, and u_2u_3 sends a charge of 1 to $\boldsymbol{C}$ else. □

Lemma 4.3.1. *Suppose that a cycle $\boldsymbol{C}$ of the length at least 3, a set $Y \subseteq E(\boldsymbol{C})$, and mappings $\psi' : V(\boldsymbol{C}) \to \{1, 2, 3, 4\}$, $\eta' : Y \to \{1, \ldots, n\}$ satisfy the conditions described by Claim 1. If the discharging rules from Claim 2 are applied to $\boldsymbol{C}$, then $\boldsymbol{C}$ ends up with a charge of at least $12 \cdot |\, \eta'[Y]\,| + 12$.*

Proof. Let $k = |V(\boldsymbol{C})|$, and $p = |\,\eta'[Y]\,|$. Notice that $|Y| = 6p$ by Claim 1(b). So the charge of $\boldsymbol{C}$ may decrease by at most $6p$ in the discharging process. If $k \geq 6p+4$, then $3k - 6p \geq 12p+12$, and hence the lemma holds. Thus it is necessary to consider only cycles with $k \leq 6p+3$. If $p = 0$, then $|Y| = 0$ and $k = 3$, so $\boldsymbol{C}$ is a triangle. In such a case, $u_1 = u_4$ holds for any four consecutive vertices u_1, u_2, u_3, u_4 of $\boldsymbol{C}$, so $\boldsymbol{C}$ receives a charge of 1 from each of its edges. Therefore it ends up with a charge of $9 + 3 = 12$, as desired.

The rest of the statement is proved by induction on $p \geq 1$. The base case $p = 1$ needs to be considered for cycles of length $k \leq 9$. Let the vertices of $\boldsymbol{C}$ be denoted $v_1, v_2, \ldots, v_k$ in order, see Fig. 4.3. By Claim 1(b), $\{\psi'(v_1), \psi'(v_2)\}$, $\{\psi'(v_2), \psi'(v_3)\}$, $\ldots$, $\{\psi'(v_k), \psi'(v_1)\}$ include all two-element subsets of $\{1, 2, 3, 4\}$. In other words, $\psi'(v_1)\psi'(v_2)\ldots\psi'(v_k)\psi'(v_1)$ is a closed walk in $\boldsymbol{K}_4$ visiting all edges. Hence, in particular, each of the four ψ'-values $1, 2, 3, 4$ occurs at least twice among the vertices of $\boldsymbol{C}$, so the length of $\boldsymbol{C}$ is 8 or 9.

Consider first $k = 8$ (Fig. 4.3 left). Assume, without loss of generality, that the edge v_2v_3 receives a charge of 1 from $\boldsymbol{C}$, so $v_2v_3 \in Y$, $\psi'(v_1) = \psi'(v_4) = 1$, and $\{\psi'(v_2), \psi'(v_3)\} = \{2, 3\}$. Since the ψ'-values of two of the vertices v_5, v_6, v_7, v_8

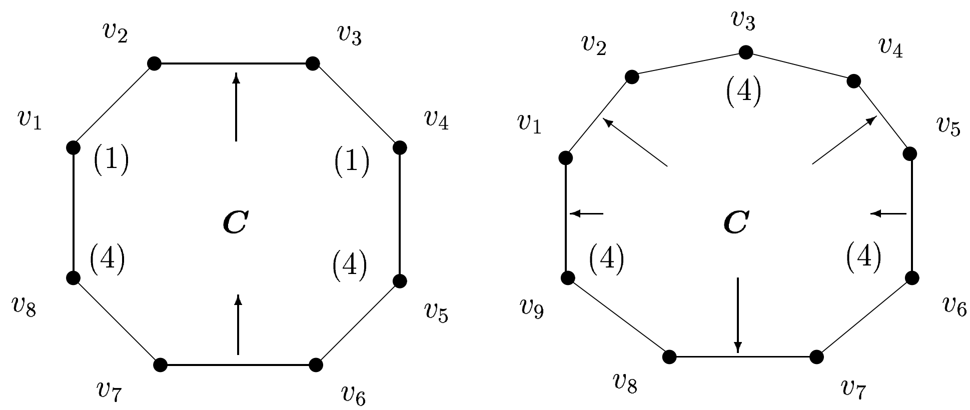


Figure 4.3:

should be 4, Claim 1(a) implies that necessarily $\psi'(v_5) = \psi'(v_8) = 4$, and $\{\psi'(v_6), \psi'(v_7)\} = \{2, 3\}$. Now, since $\{\psi'(v_2), \psi'(v_3)\} = \{\psi'(v_6), \psi'(v_7)\}$, at most one of the edges v_2v_3, v_6v_7 may be in Y by Claim 1(b), so $v_6v_7 \notin Y$, and hence v_6v_7 sends a charge of 1 to $\boldsymbol{C}$. That means, whenever an edge e of $\boldsymbol{C}$ receives a charge from $\boldsymbol{C}$, then the edge opposite to e sends a charge to $\boldsymbol{C}$. Therefore $\boldsymbol{C}$ always ends up with at least the initial charge of $3 \cdot 8 = 24 = 12p + 12$, as required.

Consider $k = 9$ now (Fig. 4.3 right). In this case, one of the ψ'-values $1, 2, 3, 4$ occurs three times at distance three on the boundary of $\boldsymbol{C}$, so let it be $\psi'(v_3) = \psi'(v_6) = \psi'(v_9) = 4$. Then, whatever the other values of ψ' are, the edges v_1v_2, v_4v_5, v_7v_8 receive a charge of 1 each from $\boldsymbol{C}$. If no other edge receives a charge from $\boldsymbol{C}$, then $\boldsymbol{C}$ ends up with a charge of $3 \cdot 9 - 3 = 24 = 12p + 12$, as desired. Otherwise, assume, without loss of generality, that the edge v_1v_9 receives a charge of 1 from $\boldsymbol{C}$, so $\psi'(v_2) = \psi'(v_8) = 1$. By Claim 1(a), there is only one possibility for the remaining values $\psi'(v_4) = \psi'(v_7) = 3$, $\psi'(v_1) = \psi'(v_5) = 2$, up to symmetry. Since $\{\psi'(v_1), \psi'(v_9)\} = \{\psi'(v_5), \psi'(v_6)\}$, the same argument as in the previous paragraph implies that the edge v_5v_6 sends a charge of 1 to $\boldsymbol{C}$. No discharging rule applies elsewhere, so $\boldsymbol{C}$ ends up with a charge of $3 \cdot 9 - 4 + 1 = 24 = 12p + 12$.

Assume it is proved that every cycle $\boldsymbol{C}$ satisfying the induction hypothesis for some $p \geq 1$ ends up with a charge of at least $12p+12$. Let $\boldsymbol{C}$ be a cycle for which $|\,\eta'[Y]\,| = p + 1 \geq 2$, and let $X = \eta'[Y]$. First, it is shown that there exist distinct $j_1, j_2 \in X$, and two disjoint paths $P_1 = s_1s_2 \ldots s_q \supseteq \eta'^{-1}(j_1)$, $P_2 = t_1t_2 \ldots t_{q'} \supseteq \eta'^{-1}(j_2)$ on the boundary of $\boldsymbol{C}$ such that, for $i = 1, 2$, $P_i \cap \eta'^{-1}(k) = \emptyset$ whenever $k \in X - \{j_1, j_2\}$. Let $j_1 \in X$ be chosen such that P_1 – the shortest path in $\boldsymbol{C}$ containing $\eta'^{-1}(j_1)$, has the smallest possible length. If there is some $j_1' \in X$ such that $\eta'^{-1}(j_1') \cap P_1 \neq \emptyset$, then $\eta'^{-1}(j_1')$ is strictly contained in P_1 by Claim 1(c), which is a contradiction to the choice of j_1. The other path P_2 is found in a similar way in $\boldsymbol{C} - V(P_1)$ (which is connected).

Similarly as in the base induction case, it follows from Claim 1(b) that $\psi'(s_1)\psi'(s_2) \ldots \psi'(s_q)$ and $\psi'(t_1)\psi'(t_2) \ldots \psi'(t_{q'})$ are walks (not necessarily closed) in $\boldsymbol{K}_4$, both visiting all of its edges. Hence, in particular, each of the ψ'-values $1, 2, 3, 4$ occurs at least twice among the vertices of P_1 and of P_2, so $q, q' \geq 8$. And since $|\eta'^{-1}(j)| = 6$ for each $j \in X - \{j_1, j_2\}$, the length of $\boldsymbol{C}$ is at least $k \geq 6(p + 1 - 2) + q - 1 + q' - 1 \geq 6(p + 1) + 2$. (Recall that $k \leq 6(p + 1) + 3$ can be assumed.)

If $k = 6(p + 1) + 3$, then, without loss of generality, $q = 8$ and $q' \leq 9$. In the case when the net charge edges of P_1 receive from $\boldsymbol{C}$ (considering also a charge that some edges of P_1 might send to $\boldsymbol{C}$) is at most 3, the cycle $\boldsymbol{C}$ ends up with a charge of at least $3k - 6p - 3 = 12p + 24 = 12(p + 1) + 12$, as desired. Similarly, if $k = 6(p + 1) + 2$, then $q = q' = 8$. In the case when the net charge edges of each of P_1, P_2 receive from $\boldsymbol{C}$ is at most 3, the cycle $\boldsymbol{C}$ ends up with a charge of at least $3k - 6(p - 1) - 3 - 3 = 12p + 24 = 12(p + 1) + 12$ again. Thus, up to symmetry, it remains to consider the case when the path P_1 of length 7 receives the net charge of at least 4 from the cycle $\boldsymbol{C}$ (regardless of k).

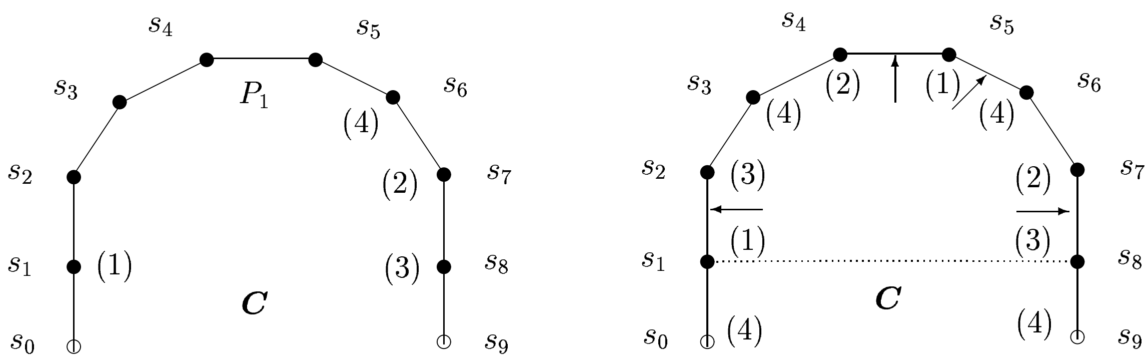


Figure 4.4:

Recall that each of the ψ'-values $1, 2, 3, 4$ occurs twice among the vertices of P_1. If $\psi'(s_1) = \psi'(s_8)$, then P_1 would correspond to a closed walk in $\boldsymbol{K}_4$ of length 7 visiting all edges, which is impossible. So assume, without loss of generality, that $\psi'(s_1) = 1$, $\psi'(s_8) = 3$, see Fig. 4.4. Since $\psi'(s_2) = 3$, $\psi'(s_7) = 1$ is not possible due to Claim 1(a), it can also be assume that $\psi'(s_7) = 2$. Now, if $\psi'(s_6) = 1$, then necessarily $\psi'(s_2) = \psi'(s_5) = 4$, and no edge of P_1 has values $\{1, 3\}$, which is a contradiction to Claim 1(b). So $\psi'(s_6) = 4$. Further, the three possible values of $\psi'(s_2)$ are considered.

If $\psi'(s_2) = 4$, then necessarily $\psi'(s_4) = 1$, hence $\psi'(s_3) = 2$ and $\psi'(s_5) = 3$ by Claim 1(a). In such situation, only the edges s_1s_2, s_7s_8, and one of s_2s_3, s_6s_7 may receive charges from $\boldsymbol{C}$, so P_1 receives the net charge of at most 3 from $\boldsymbol{C}$, which is an already covered case. If $\psi'(s_2) = 2$, then $\psi'(s_3) = 4$, and P_1 again receives the net charge of at most 3 from $\boldsymbol{C}$ for both choices $\{\psi'(s_4), \psi'(s_5)\} = \{1, 3\}$. So the remaining possibility is $\psi'(s_2) = 3$, hence $\psi'(s_3) = 4$, $\psi'(s_5) = 1$, and $\psi'(s_4) = 2$. In such a situation, $\boldsymbol{C}$ may a send charge of up to 4 to the edges s_1s_2, s_4s_5, s_5s_6, s_7s_8 of P_1, provided that $\psi'(s_0) = 4$ and $\psi'(s_9) = 4$, see Fig. 4.4 right.

If the latter case happens, a new cycle $\boldsymbol{C}'$ is formed by replacing the path P_1 with the edge s_1s_8, a new set $Y' = Y - E(P_1)$ is defined, and the mappings ψ', η' are restricted to $E(\boldsymbol{C}')$, Y', respectively. It is easy to check that the conditions in Claim 1(b,c) are still satisfied for $\boldsymbol{C}'$ by the choice of P_1. Also, validity of Claim 1(a) is preserved in this special case. (See the picture.) Since $|\,\eta'[Y']\,| = |\,\eta'[Y]\,| - 1 = p$, the new cycle $\boldsymbol{C}'$ ends up with a charge of at least $12p + 12$ by the induction assumption. Now, the cycle $\boldsymbol{C}$ is longer by 6 than $\boldsymbol{C}'$, hence $\boldsymbol{C}$ starts with a charge larger by 18 than $\boldsymbol{C}'$ does. The same discharging rules (cf. Claim 2) apply in $\boldsymbol{C}$ as in $\boldsymbol{C}'$ to all edges of $\boldsymbol{C}'$ except for two, namely s_1s_8 and s_8s_9. (The edge s_1s_8 does not exist in $\boldsymbol{C}$, and s_8s_9 has a neighbor of a different ψ'-value in $\boldsymbol{C}$ than in $\boldsymbol{C}'$.) Additionally, exactly four edges of P_1 receive a charge of 1 from $\boldsymbol{C}$. Therefore the cycle $\boldsymbol{C}$ ends up with a charge of at least $(12p+12)+18-2-4 = 12(p+1)+12$, as desired. ∎

Corollary 4.3.2. *Each face f of $\boldsymbol{H}_4$ ends up with a charge of at least* $12|\lambda^{-1}(f)|+12$.

Proof. Lemma 4.3.1 is applied to the cycle $\boldsymbol{C}$ bounding f, and to the set Y and the mappings ψ', η' defined as in Claim 1. Notice that $\lambda^{-1}(f) = \eta'[Y]$ by definition. ∎

4.4 Discharging Contradiction

Finally, a contradiction is derived from the above facts about the graph $\boldsymbol{H}_4$, and from Euler's formula, which shows that a planar cover of $\boldsymbol{\mathcal{E}}_2$ cannot exist.

Proof of Theorem 2.4.2. Since $\boldsymbol{H}_4$ is a 3-regular graph on $4n$ vertices, the total charge at the beginning is

$$3 \sum_{f \in F(H_4)} |f| = 3 \cdot 12n = 36n.$$

The number of faces of $\boldsymbol{H}_4$ is $2n+2$ by Euler's formula. By Lemma 4.2.1 and Corollary 4.3.2, the total sum of charges at the end of the discharging process is at least

$$0 + \sum_{f \in F(H_4)} \Big(12|\lambda^{-1}(f)| + 12\Big) = 12 \cdot |F(\boldsymbol{H}_4)| + 12 \cdot \sum_{f \in F(H_4)} |\lambda^{-1}(f)| =$$

$$= 12(2n+2) + 12n = 36n + 24 > 36n.$$

However, the discharging process just redistributes existing charges, no new charge is introduced during the process. This contradiction shows that the graph $\boldsymbol{H}_4$, and hence also a planar cover of $\boldsymbol{\mathcal{E}}_2$, cannot exist. ∎

CHAPTER V

THE GRAPH $\mathcal{C}_4$

5.1 Semi-Covers and the Necklace Property

Let $\boldsymbol{H}$ be a plane graph, and let f be the outer face of $\boldsymbol{H}$. The graph $\boldsymbol{H}$ is called a *semi-cover* of a graph $\boldsymbol{G}$ if there exists a pair of onto mappings $\varphi : V(\boldsymbol{H}) \to V(\boldsymbol{G})$, $\psi : E(\boldsymbol{H}) \to E(\boldsymbol{G})$, called a *semi-projection*, satisfying the following: For each vertex v of $\boldsymbol{H}$ not incident with f, ψ maps the edges incident with v bijectively onto the edges incident with $\varphi(v)$ in $\boldsymbol{G}$; and for each vertex w of $\boldsymbol{H}$ incident with f, ψ maps the edges incident with w bijectively onto a subset of the edges incident with $\varphi(w)$ in $\boldsymbol{G}$. (Informally, the vertices of the outer face are "allowed to miss some neighbors" in a semi-cover.) Clearly, each cover is a semi-cover, but the converse is false.

Assume that a connected plane graph $\boldsymbol{H}$ is a semi-cover of a simple connected graph $\boldsymbol{G}$, and $\varphi : V(\boldsymbol{H}) \to V(\boldsymbol{G})$, $\psi : E(\boldsymbol{H}) \to E(\boldsymbol{G})$ is a semi-projection. If $\boldsymbol{G}'$ is a subgraph of $\boldsymbol{G}$, then the graph $\boldsymbol{H}'$ with vertex set $\varphi^{-1}(V(\boldsymbol{G}'))$ and edge set $\psi^{-1}(E(\boldsymbol{G}'))$ is called a *lifting* of $\boldsymbol{G}'$ into $\boldsymbol{H}$. Assuming C is a cycle in $\boldsymbol{G}$, the semi-cover $\boldsymbol{H}$ is said to be *C-fixed* if the lifting of C into $\boldsymbol{H}$ consists of finite facial cycles of the same length as C.

The idea of a necklace was introduced by Archdeacon in [5]. For our purpose it is formally defined as follows. Suppose that C is an induced 4-cycle in $\boldsymbol{G}$, w is a

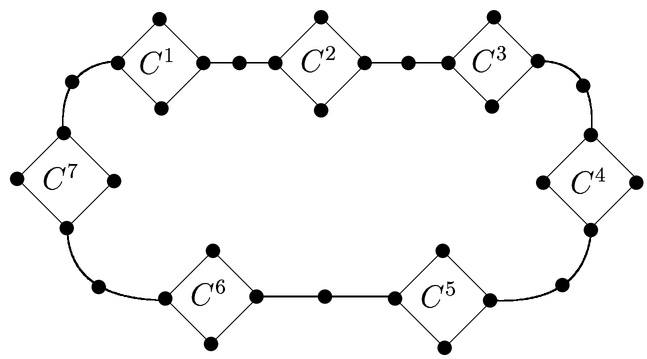


Figure 5.1: An illustration of a necklace.

vertex of $\boldsymbol{G} - V(C)$, and $\boldsymbol{F}$ is a subgraph isomorphic to $\boldsymbol{K}_{2,3}$ such that $C \subset \boldsymbol{F} \subset \boldsymbol{G}$, $V(\boldsymbol{F}) = V(C) \cup \{w\}$. Furthermore, suppose that a plane graph $\boldsymbol{H}$ is a semi-cover of $\boldsymbol{G}$, and that φ is the corresponding semi-projection. A connected component $\boldsymbol{N}$ of the lifting of $\boldsymbol{F}$ into $\boldsymbol{H}$ is called a (C, w)-*necklace*, if $\boldsymbol{N}$ is C-fixed, and the restriction $(\,\varphi \restriction V(\boldsymbol{N})\,,\, \psi \restriction E(\boldsymbol{N})\,)$ is a projection onto $\boldsymbol{F}$ (and hence $\boldsymbol{N}$ is actually a cover of $\boldsymbol{F}$). Let the lifting of C into $\boldsymbol{N}$ consist of l facial 4-cycles $C^1, C^2, \ldots, C^l$; then these cycles are called the *beads* of the necklace, and l is the *length* of the necklace. (See Fig. 5.1 for an illustration.) The finite face of $\boldsymbol{N}$ not bounded by any of the beads is called the *interior* of $\boldsymbol{N}$.

Let C_1, C_2 be two induced 4-cycles in a graph $\boldsymbol{G}$. The graph $\boldsymbol{G}$ is said to have the (C_1, C_2, w)-*necklace property* if

- the sets $V(C_1), V(C_2), \{w\}$ are pairwise disjoint, and $V(C_1) \cup V(C_2) \cup \{w\} = V(\boldsymbol{G})$, i.e. $\boldsymbol{G}$ has 9 vertices;

- for $i = 1, 2$, the vertices of C_i can be denoted by a, b, c, d in this cyclic order so that aw, cw are edges of $\boldsymbol{G}$, and that each of b, d is adjacent by an edge to exactly one vertex of the other cycle C_{3-i} in $\boldsymbol{G}$.

Examples of two graphs having the necklace property are shown in Fig. 5.2. (At the first look, the necklace property may seem to be similar to the property of "having two disjoint k-graphs", as defined in Chapter II. However, unlike the latter one, the necklace property may hold also for some projective-planar graphs, such as for the right-hand side graph in Fig. 5.2.)

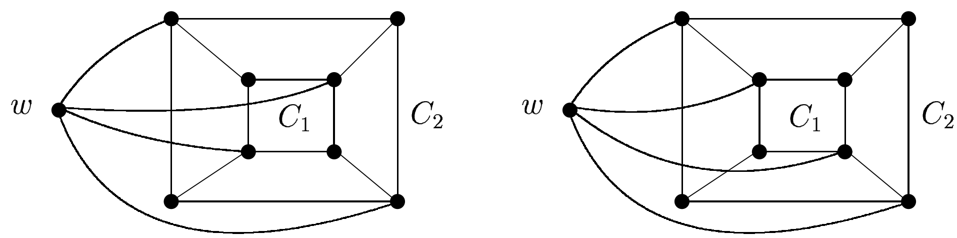


Figure 5.2: Examples of two graphs having the necklace property.

5.2 Finding a Short Necklace

The arguments given in this section generalize the proofs used by Archdeacon [5] to show that the graphs $\boldsymbol{K}_7 - \boldsymbol{C}_4$ and $\boldsymbol{K}_{4,5} - \boldsymbol{M}_4$ (Fig. 5.2 left) have no planar covers. From now on, it is assumed that $\boldsymbol{G}$ is a graph having a (C_1, C_2, w)-necklace property. Let a connected plane graph $\boldsymbol{H}$ be a semi-cover of $\boldsymbol{G}$. Suppose that $V(\boldsymbol{H})$ can be partitioned into the vertex set of a (C_1, w)-necklace $\boldsymbol{N}$, and the vertex set of a C_2-fixed lifting of C_2 into $\boldsymbol{H}$. Furthermore, suppose that the only vertices incident with the outer face of $\boldsymbol{H}$ are those of $\boldsymbol{N}$, i.e. $\boldsymbol{H} - V(\boldsymbol{N})$ is embedded in the interior of the necklace $\boldsymbol{N}$. Then $\boldsymbol{H}$ is called a *reduced semi-cover* of $\boldsymbol{G}$ bounded by the necklace $\boldsymbol{N}$.

Lemma 5.2.1. *Suppose that a graph $\boldsymbol{G}$ has a (C_1, C_2, w)-necklace property. If $\boldsymbol{G}$ has a planar cover, then, for some $i \in \{1, 2\}$, $\boldsymbol{G}$ has a reduced semi-cover bounded by a (C_i, w)-necklace.*

Proof. Let a connected plane graph $\boldsymbol{H}$ be a cover of $\boldsymbol{G}$, and let $\boldsymbol{H}^o$ denote the lifting of the graph $C_1 \cup C_2$ into $\boldsymbol{H}$. Clearly, $\boldsymbol{H}^o$ is a collection of disjoint cycles of $\boldsymbol{H}$. Notice that if C' is a cycle in the lifting of C_i, and C' is longer than C_i, then the cover projection "winds" C' several times around C_i. So if C' is a face of $\boldsymbol{H}$, it can be easily broken down into facial 4-cycles covering C_i. Hence it may be assumed that all cycles of $\boldsymbol{H}^o$ that are faces in $\boldsymbol{H}$ have size 4, and that they bound finite faces.

Let C^o denote some cycle of $\boldsymbol{H}^o$ that bounds an inclusion-wise minimal open disc containing at least one vertex of $\boldsymbol{H}$. By the previous assumption, the subgraph embedded inside C^o is C_1- and C_2-fixed. Without loss of generality, assume that C^o belongs to the lifting of C_2. Let $C_1 = abcd$, and let $\boldsymbol{G}'$ be the subgraph of $\boldsymbol{G}$ with vertex-set $V(C_1) \cup \{w\}$ and edge-set $E(C_1) \cup \{aw, cw\}$. (See the definition of the (C_1, C_2, w)-necklace property.) Now, since $\boldsymbol{G} - V(C_2) \supseteq \boldsymbol{G}'$ is connected, and since C^o is not a face of $\boldsymbol{H}$, some component $\boldsymbol{N}$ of the lifting of $\boldsymbol{G}'$ into $\boldsymbol{H}$ must be embedded inside C^o. Hence $\boldsymbol{N}$ is a (C_1, w)-necklace.

Let $\boldsymbol{N}^o$ be a (C_1, w)-necklace inside C^o with minimal interior with respect to inclusion. Then all vertices in the interior of $\boldsymbol{N}^o$ belong to the lifting of C_2 (which is C_2-fixed); since otherwise there would be a (C_1, w)-necklace with its interior properly contained in the interior of $\boldsymbol{N}^o$, by repeating the previous argument. Thus $\boldsymbol{N}^o$ bounds a reduced semi-cover of $\boldsymbol{G}$. ∎

Lemma 5.2.2. *Suppose that a graph $\boldsymbol{G}$ has a (C_1, C_2, w)-necklace property. If there exists, for some $i \in \{1, 2\}$, a reduced semi-cover of $\boldsymbol{G}$ bounded by a (C_i, w)-necklace* *of length $l > 2$, then there exists a reduced semi-cover of $\boldsymbol{G}$ bounded by a (C_i, w)-necklace of length smaller than l.*

Proof. The proof of this lemma is the heart of our argument. Without loss of generality, assume that $\boldsymbol{H}$ is a reduced semi-cover of $\boldsymbol{G}$ bounded by a (C_1, w)-necklace $\boldsymbol{N}$, and $\varphi : V(\boldsymbol{H}) \to V(\boldsymbol{G})$ is the semi-projection. Let the vertices of C_1 be a, b, c, d in this cyclic order so that $\boldsymbol{N}$ is the lifting of $(V(C_1) \cup \{w\}, E(C_1) \cup \{aw, cw\})$. Notice that if C^1, C^2 are two beads of the necklace $\boldsymbol{N}$, and v_1, v_2 are the vertices of C^1, C^2, respectively, encountered first when traversing the necklace in the clockwise orientation, then $\varphi(v_1) = \varphi(v_2) \in \{a, c\}$.

Since $\boldsymbol{N}$ is not outerplanar, and $\boldsymbol{G} - \{a, c, w\} \supset C_2$ is connected, there exists some cycle C' in $\boldsymbol{H}$ with $\varphi(C') = C_2$. From the necklace property of $\boldsymbol{G}$ it follows that there exists exactly one vertex $b_1 \in V(\boldsymbol{N})$ such that $\varphi(b_1) = b$, and that b_1 is adjacent to some vertex of C'. Similarly, there exists exactly one vertex $d_2 \in V(\boldsymbol{N})$, $\varphi(d_2) = d$ adjacent to some vertex of C'. Let C^1 and C^2 be the beads of $\boldsymbol{N}$ such that $b_1 \in V(C^1)$ and $d_2 \in V(C^2)$. Clearly, $C^1 \neq C^2$ since $\boldsymbol{H}$ is a plane graph. The subgraph of $\boldsymbol{H}$ induced on $V(C') \cup \{b_1, d_2\}$ is denoted by $\boldsymbol{B}$ (for "bridge").

Since the length of the necklace $\boldsymbol{N}$ is greater than two, one of the two regions that $\boldsymbol{B}$ separates the interior of $\boldsymbol{N}$ into, say $\mathcal{R}$, has at least one bead other than C^1, C^2 on its boundary. The left-hand side of Fig. 5.3 illustrates the situation. Notice, however, that the other ends of edges joining b_1 and d_2 with the cycle C' (according to the necklace property of $\boldsymbol{G}$) need not be diagonally opposite on C'. Let the vertices of C^1 be a_1, d_1, c_1, b_1, the vertices of C^2 be a_2, b_2, c_2, d_2 (both in clockwise orientation), and $\varphi(c_1) = c$. Then it follows that $\varphi(a_2) = a$.

Let $e_1, e_2, \ldots, e_k$ be the edges that have exactly one endpoint in $V(\boldsymbol{B})$ and that belong to the interior of $\mathcal{R}$, ordered by their appearance on the boundary of $\mathcal{R}$

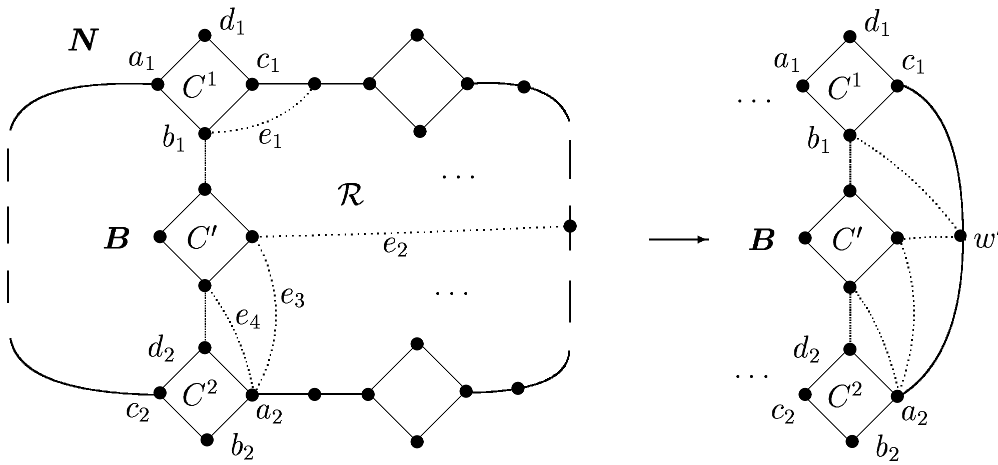


Figure 5.3:

from b_1 to d_2. Let $u_1, \ldots, u_k$ be the ends of $e_1, \ldots, e_k$, respectively, not in $V(\boldsymbol{B})$. Since the subgraph $\boldsymbol{B}$ is actually isomorphic to $\boldsymbol{G} - \{a, c, w\}$, it follows that $\varphi(u_i) \in \{a, c, w\}$ for $i = 1, \ldots, k$, and hence $u_1, \ldots, u_k$ are incident with the outer face of $\boldsymbol{H}$. Now, suppose that there exist $0 \leq i \leq j \leq k$ such that $\varphi(u_1) = \ldots = \varphi(u_i) = c$, $\varphi(u_{i+1}) = \ldots = \varphi(u_j) = w$, and $\varphi(u_{j+1}) = \ldots = \varphi(u_k) = a$. In such a case, the part of $\boldsymbol{H}$ embedded in $\mathcal{R}$ is deleted, and the section of the bounding necklace between c_1 and a_2 is replaced by a new path $c_1 w' a_2$. Instead of the edges $e_1, \ldots, e_k$, corresponding new edges $e'_1, \ldots, e'_k$ between vertices of $\boldsymbol{B}$ and $\{c_1, w', a_2\}$ are drawn, as needed. Clearly, the new graph $\boldsymbol{H'}$ is a reduced semi-cover of $\boldsymbol{G}$ bounded by a necklace of shorter length.

Otherwise, if the above case does not happen, then there exists $1 \leq i < k$ such that $\varphi(u_i) \in \{w, a\}$, $\varphi(u_{i+1}) \in \{c, w\}$, and $\varphi(u_i) \neq \varphi(u_{i+1})$. (See Fig. 5.4, where $\varphi(u_2) = w$, $\varphi(u_3) = c$.) Each of the edges e_i, e_{i+1} separates $\mathcal{R}$ into two regions, and e_i, e_{i+1} are disjoint up to possible common endvertex in $\boldsymbol{B}$. Let $\mathcal{R}_1$ be

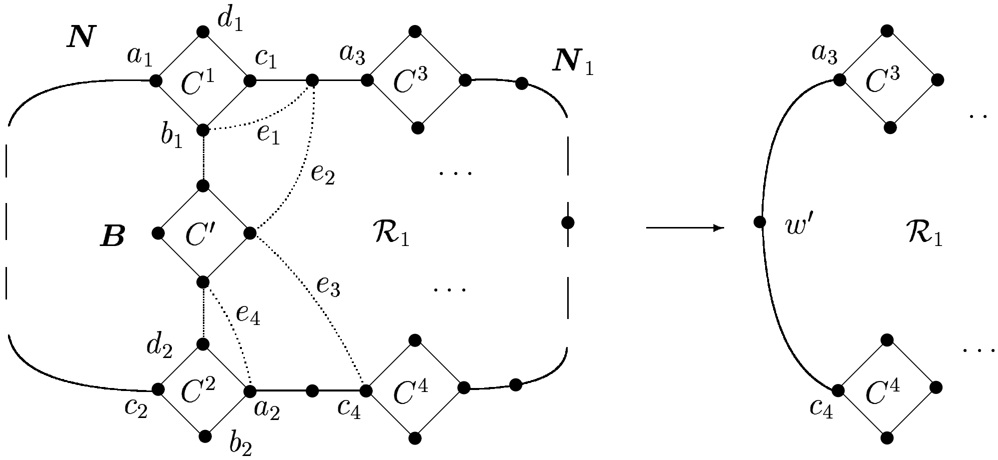


Figure 5.4:

the connected component of the set $\mathcal{R} - e_i - e_{i+1}$ incident with both e_i, e_{i+1}, and let $\boldsymbol{N}_1$ be the section of $\boldsymbol{N}$ incident with the boundary of $\mathcal{R}_1$. In this case, by the choice of u_i, u_{i+1}, $\boldsymbol{N}_1$ must contain at least one bead. So let C^3, C^4 be the ending beads of $\boldsymbol{N}_1$, and let a_3, c_4 be the vertices of C^3, C^4 closest to u_i, u_{i+1}, respectively. (Notice that C^3, C^4 are not necessarily next to C^1, C^2, and they may be equal.) Then at most one of the vertices u_i, u_{i+1}, say u_i, is distinct from both a_3, c_4, and if this happens, then $\varphi(u_i) = w$. A new graph $\boldsymbol{H}'$ is formed as the part of $\boldsymbol{H}$ embedded in the region $\mathcal{R}_1$, bounded by the section $\boldsymbol{N}_1$ and a new path $c_4 w' a_3$. Possible edges between some vertices of $\boldsymbol{H}'$ and u_i if $u_i \neq a_3, c_4$ are rerouted to the endvertex w'. Again, $\boldsymbol{H}'$ is clearly a reduced semi-cover of $\boldsymbol{G}$ bounded by a necklace of shorter length than the length of $\boldsymbol{N}$. ∎

5.3 Conclusion of the Proof

Lemma 5.3.1. *Suppose that a graph $\boldsymbol{G}$ has a (C_1, C_2, w)-necklace property. If $\boldsymbol{G}$ has a planar cover, then $\boldsymbol{G}$ has an embedding in the projective plane.*

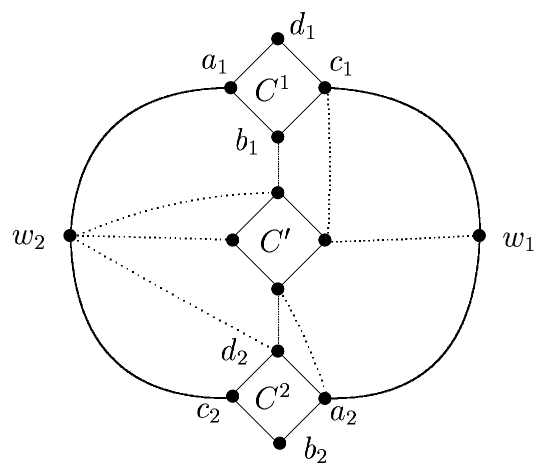


Figure 5.5:

Proof. By Lemma 5.2.1, there exists a reduced semi-cover of $\boldsymbol{G}$ bounded by a (C_i, w)-necklace for some $i \in \{1, 2\}$, so assume that $i = 1$. By repeatedly applying Lemma 5.2.2, it can be deduced that there exists a reduced semi-cover $\boldsymbol{H}_2$ of $\boldsymbol{G}$, $\varphi : V(\boldsymbol{H}_2) \to V(\boldsymbol{G})$, bounded by a (C_1, w)-necklace of length at most two. As it was implicitly found in the proof of Lemma 5.2.2, the lifting of C_1 into $\boldsymbol{H}_2$ consists of two cycles C^1, C^2, and the lifting of C_2 is a single cycle C' in the interior of the necklace. Let the vertices of C^1, C^2 be $C^1 = a_1b_1c_1d_1$ and $C^2 = a_2d_2c_2b_2$ in the counterclockwise orientation so that b_1, d_2, $\varphi(b_1) = b, \varphi(d_2) = d$ are the vertices not incident with the outer face of $\boldsymbol{H}_2$, and hence adjacent to the cycle C'. Let w_1

be the common neighbor of c_1, a_2, and w_2 be the common neighbor of c_2, a_1, in the bounding necklace. (See Fig. 5.5 for an example.)

An embedded projective-planar graph $\boldsymbol{H}_p$ is formed from $\boldsymbol{H}_2$ by deleting the vertices d_1, b_2, and identifying the opposite pairs $w_1 = w_2$, $a_1 = a_2$, $c_1 = c_2$. A mapping φ' is the restriction of φ onto $V(C') \cup \{a_1, b_1, c_1, w_1, d_2\}$. It is claimed that $\varphi' : V(\boldsymbol{H}_p) \to V(\boldsymbol{G})$ is an isomorphism. Indeed, the vertices of $V(C') \cup \{b_1, d_2\}$ are not incident with the outer face of $\boldsymbol{H}_2$, and hence they are incident with all the required edges in the isomorphism relation by the definition of φ and φ'. In particular, all the required edges between the sets $V(C') \cup \{b_1, d_2\}$ and $\{a_1, c_1, w_1\}$ are present also in $\boldsymbol{H}_p$, and the edges a_1w_1, c_2w_1 are in $\boldsymbol{H}_p$ as well. (Recall that there is no edge between a, c in $\boldsymbol{G}$.) ∎

It is shown that Lemma 5.3.1 can be applied, in particular, to the graph $\boldsymbol{\mathcal{C}}_4$.

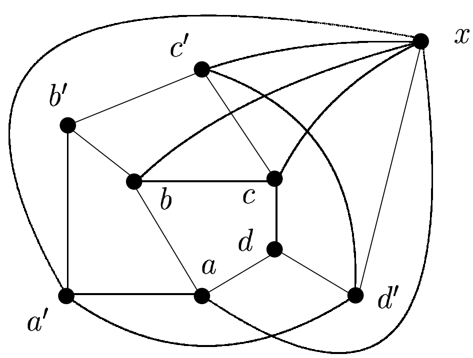


Figure 5.6: The graph $\boldsymbol{\mathcal{C}}_4$.

Proof of Theorem 2.4.3. Let the vertices of the graph $\boldsymbol{\mathcal{C}}_4$ be denoted by $a, b, c, d, a', b', c', d', x$ as depicted in Fig. 5.6. It is easy to verify that $\boldsymbol{\mathcal{C}}_4$ has the

(C_1, C_2, x)-necklace property for $C_1 = abcd$ and $C_2 = a'b'c'd'$. By [14], $\mathcal{C}_4$ has no projective embedding, and hence it has no planar cover by Lemma 5.3.1. ∎

Remark. It is possible to generalize the definition of a necklace, allowing it to be a lifting of a subgraph isomorphic to $\boldsymbol{K}_4$, with pairs of edge-sharing facial triangles as the beads. (A variant that was used by Archdeacon in [5].) Then the above arguments work as well, and they also include the graph $\boldsymbol{K}_7 - \boldsymbol{C}_4$.

Remark. It is likely that Lemma 5.3.1 can be proved for much wider definition of a necklace, assuming more cycles of different sizes to be potential beads of a necklace, and allowing more additional vertices and interconnecting edges. Unfortunately, this does not seem to be useful for any one of the 16 graphs that might be (up to obvious constructions) possible counterexamples to Negami's conjecture, cf. Theorem 6.1.3.

CHAPTER VI

SEARCH FOR POSSIBLE COUNTEREXAMPLES

6.1 Separations and Expansions

Results of Archdeacon, Fellows, and Negami, and Theorems 2.4.1, 2.4.2, and 2.4.3 proved in the previous Chapters are summarized in the next statement.

Corollary 6.1.1. (D. Archdeacon, M. Fellows, S. Negami, P. Hliněný) *No member of the family* $\Lambda - \{\boldsymbol{K}_{1,2,2,2}, \boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{D}}_2\}$ *has a planar cover.*

The aim of this chapter is to show that Negami's conjecture cannot be "far from the truth", i.e. that there are only several possible internally 4-connected counterexamples. However, notice that no real counterexample to the conjecture was found so far, and the conjecture is still believed to be true.

Recall the notion of a separation in a graph from Chapter I. A separation (A, B) in $\boldsymbol{G}$ is called *flat* if the graph $\boldsymbol{G} \upharpoonright B$ has a planar embedding with all the vertices of $A \cap B$ incident with the outer face.

Let $\boldsymbol{G}$ be a graph. Let $\boldsymbol{F}$ be a connected planar graph on the vertex set $V(\boldsymbol{F})$ disjoint from $V(\boldsymbol{G})$, and let $x_1 \in V(\boldsymbol{F})$. If y_1 is a vertex of $\boldsymbol{G}$, and the graph $\boldsymbol{H}_1$ is obtained from $\boldsymbol{G} \cup \boldsymbol{F}$ by identifying the vertices x_1 and y_1, then $\boldsymbol{H}_1$ is called a 1-*expansion* of $\boldsymbol{G}$. Let $x_1, x_2 \in V(\boldsymbol{F})$ be two distinct vertices that are incident

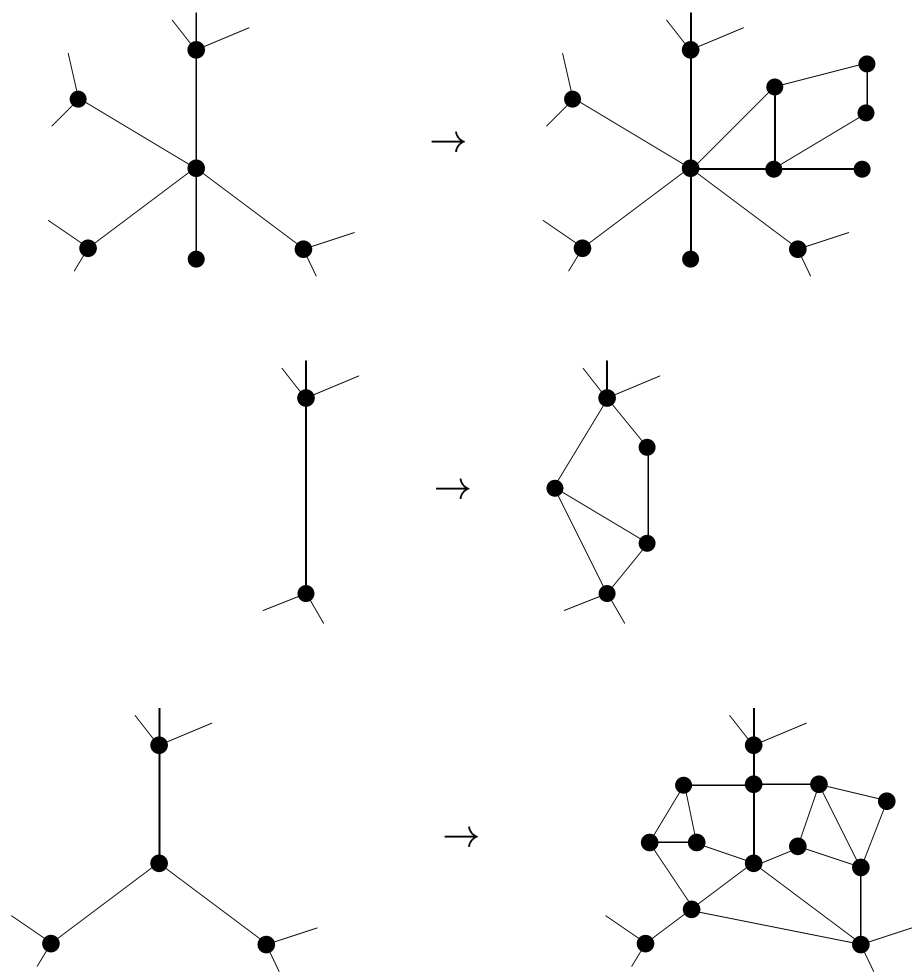


Figure 6.1: Illustrations of 1-, 2- and 3-expansion.

with the same face in a planar embedding of $\boldsymbol{F}$. If $e = y_1y_2$ is an edge of $\boldsymbol{G}$, and the graph $\boldsymbol{H}_2$ is obtained from $(\boldsymbol{G} - e) \cup \boldsymbol{F}$ by identifying the vertex pairs (x_1, y_1) and (x_2, y_2), then $\boldsymbol{H}_2$ is called a 2-*expansion* of $\boldsymbol{G}$. Let $x_1, x_2, x_3 \in V(\boldsymbol{F})$ be three distinct vertices such that $\boldsymbol{F} - \{x_1, x_2, x_3\}$ is connected. Moreover, let each of the vertices x_1, x_2, x_3 be adjacent to some vertex of $V(\boldsymbol{F} - \{x_1, x_2, x_3\})$, and let all three vertices x_1, x_2, x_3 be incident with the same face in a planar embedding of $\boldsymbol{F}$. If w is a cubic vertex of $\boldsymbol{G}$ with the neighbors y_1, y_2, y_3, and the graph $\boldsymbol{H}_3$ is obtained from $(\boldsymbol{G} - w) \cup \boldsymbol{F}$ by identifying the vertex pairs (x_1, y_1), (x_2, y_2) and (x_3, y_3), then $\boldsymbol{H}_3$ is called a 3-*expansion* of $\boldsymbol{G}$.

A graph $\boldsymbol{H}$ is an *expansion* of a graph $\boldsymbol{G}$ if there is a sequence of graphs $\boldsymbol{G}_0 = \boldsymbol{G}, \boldsymbol{G}_1, \ldots, \boldsymbol{G}_l = \boldsymbol{H}$ such that $\boldsymbol{G}_i$ is a 1-,2-, or 3-expansion of $\boldsymbol{G}_{i-1}$ for all $i = 1, \ldots, l$. (See examples in Fig. 6.1.)

Lemma 6.1.2. *Let $\boldsymbol{H}$ be an expansion of a graph $\boldsymbol{G}$.*

(a) $\boldsymbol{G}$ has an embedding in the projective plane if and only if so does $\boldsymbol{H}$.

(b) $\boldsymbol{G}$ has a planar cover if and only if so does $\boldsymbol{H}$.

(c) $\boldsymbol{G}$ is a minor of $\boldsymbol{H}$.

Proof. Using induction, it is enough to prove the statements (a),(b) when $\boldsymbol{H} = \boldsymbol{H}_k$ is a k-expansion of $\boldsymbol{G}$ for $k = 1, 2, 3$. Let, for simplicity, the same notation as in the definition of the k-expansion be used now. Suppose that the graph $\boldsymbol{G}$ is embedded in the projective plane. If $k = 1$, then $\boldsymbol{F}$ can be embedded in any face incident with y_1 in $\boldsymbol{G}$. If $k = 2$, and $e = y_1y_2$ is an edge of $\boldsymbol{G}$, then there is a face f of $\boldsymbol{G} - e$ incident with both y_1, y_2. Similarly, if $k = 3$, and w is a cubic vertex of $\boldsymbol{G}$, then there is a face f of $\boldsymbol{G} - w$ incident with all three neighbors of w. In both cases, the graph $\boldsymbol{F}$ can be embedded in the face f.

If a plane graph $\boldsymbol{G}'$ is a cover of $\boldsymbol{G}$, then the same construction as in the previous paragraph is applied to every vertex of $\boldsymbol{G}'$ that is projected to y_1 ($k = 1$), or to every edge of $\boldsymbol{G}'$ that is projected to $e = y_1y_2$ ($k = 2$), or to the neighbors of every cubic vertex that is projected to w ($k = 3$). The resulting graph $\boldsymbol{H}'$ is then a planar cover of $\boldsymbol{H}_k$.

Finally, $\boldsymbol{G}$ is a subgraph of $\boldsymbol{H}_1$ if $k = 1$, and $\boldsymbol{G}$ is a minor of $\boldsymbol{H}_k$ if $k = 2, 3$ by the connectivity assumptions. Since the minor relation is transitive, $\boldsymbol{G}$ is a minor of $\boldsymbol{H}$. This fact in particular implies that if $\boldsymbol{H}$ has a projective embedding (planar cover), then so does $\boldsymbol{G}$. ∎

A graph $\boldsymbol{G}$ would be a counterexample to Conjecture 2.2.1 if $\boldsymbol{G}$ had a planar cover but no projective embedding. Thus if $\boldsymbol{K}_{1,2,2,2}$ had a planar cover, then Lemma 6.1.2 would enable us to generate infinitely many counterexamples to Conjecture 2.2.1. However, our result is:

Theorem 6.1.3. *Let* Π *be the family of 16 graphs listed in Appendix B. If a connected graph* $\boldsymbol{G}$ *has a planar cover but no embedding in the projective plane, then* $\boldsymbol{G}$ *is an expansion of some graph from* Π.

Before proving Theorem 6.1.3 in Section 6.4, a lot of preparatory work needs to be done. Let $\Lambda_0 = \Lambda - \{\boldsymbol{K}_{1,2,2,2}, \boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{D}}_2\}$ denote the family of all connected minor-minimal nonprojective graphs which are known to have no planar cover. It immediately follows from Theorem 2.2.2 and Corollary 6.1.1:

Corollary 6.1.4. *If* $\boldsymbol{G}$ *is a connected graph that has a planar cover but no embedding in the projective plane, then* $\boldsymbol{G}$ *has a minor isomorphic to one of* $\boldsymbol{K}_{1,2,2,2}$, $\boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{D}}_2$, *but* $\boldsymbol{G}$ *has no minor isomorphic to a member of* Λ_0.

A graph $\boldsymbol{G}$ is *internally* 4*-connected* if it is simple and 3-connected, has at least five vertices, and for every separation (A, B) of order 3, either $\boldsymbol{G} \restriction A$ or $\boldsymbol{G} \restriction B$ has at most three edges. The following lemma and its corollary show that the search for a possible counterexample to Negami's conjecture may be restricted to internally 4-connected graphs. However, proving this lemma is quite a technical task, which needs several additional results, and so the proof is postponed until Section 6.3.

Lemma 6.1.5. *Suppose $\boldsymbol{G}$ is a connected graph that has no embedding in the projective plane, and that has no minor isomorphic to a member of Λ_0. If $k \in \{1, 2, 3\}$ is the least integer such that there is a nontrivial separation (A, B) of order k in $\boldsymbol{G}$, then either (A, B) or (B, A) is flat.*

Corollary 6.1.6. *If $\boldsymbol{G}$ is a connected graph that has no embedding in the projective plane, and that has no minor isomorphic to a member of Λ_0, then there exists an internally 4-connected graph $\boldsymbol{F}$ with the same properties such that $\boldsymbol{G}$ is an expansion of $\boldsymbol{F}$.*

Proof. The proof proceeds by induction on the number of edges in $\boldsymbol{G}$. Since the graph $\boldsymbol{K}_6$ has a projective embedding, $\boldsymbol{G}$ has at least 7 vertices. If $\boldsymbol{G}$ is an internally 4-connected graph, then there is nothing to prove. If $\boldsymbol{G}$ is not simple, then it is an expansion of its underlying simple graph. So suppose, for some $k \in \{1, 2, 3\}$, that $\boldsymbol{G}$ is a simple k-connected graph, and that there exists a nontrivial separation (A, B) in $\boldsymbol{G}$ of order k. If $k = 3$, then also suppose that both $\boldsymbol{G} \restriction A$ and $\boldsymbol{G} \restriction B$ have more than three edges. By Lemma 6.1.5, it can be assumed that (A, B) is flat, and hence the graph $\boldsymbol{G} \restriction B$ has a planar embedding in which all vertices of $A \cap B$ are incident with the same face.

Observe that $\boldsymbol{G} \restriction B$ is connected. If $k = 1$, then $\boldsymbol{G}$ is a 1-expansion of $\boldsymbol{G}' = \boldsymbol{G} \restriction A$. If $k = 2$, and say $A \cap B = \{u_1, u_2\}$, then $\boldsymbol{G}$ is a 2-expansion of

$\boldsymbol{G}' = (\boldsymbol{G} \restriction A) + u_1u_2$. So suppose that $k = 3$, and $A \cap B = \{u_1, u_2, u_3\}$. Since $\boldsymbol{G}$ is 3-connected, every vertex $v \in B - A$ is connected by three disjoint (except for v) paths to each of the vertices u_1, u_2, u_3. In particular, each of u_1, u_2, u_3 is adjacent to some vertex of $B - A$. If the graph $\boldsymbol{G} \restriction (B - A)$ is not connected, let us say that v, w belong to distinct components of $\boldsymbol{G} \restriction (B - A)$, then the six paths connecting each of v, w with u_1, u_2, u_3 are pairwise disjoint except for their ends, and hence they form a subdivision of $\boldsymbol{K}_{2,3}$. But then there is no planar embedding of $\boldsymbol{G} \restriction B$ in which u_1, u_2, u_3 are incident with the same face, a contradiction. Thus the planar graph $\boldsymbol{F} = \boldsymbol{G} \restriction B$ fulfills all conditions in the definition of the 3-expansion, and hence $\boldsymbol{G}$ is a 3-expansion of the graph $\boldsymbol{G}'$, obtained from $\boldsymbol{G}$ by contracting the set $B - A$ into one vertex and deleting all edges with both ends in $A \cap B$.

In all three cases outlined above, the graph $\boldsymbol{G}'$ has no embedding in the projective plane by Lemma 6.1.2, and it is a proper minor of $\boldsymbol{G}$. So the statement follows by induction. ∎

6.2 A Splitter Theorem

This section presents a basic tool that is used in the search for possible counterexamples to Negami's conjecture. Understanding the following convention will be important in the next definitions and theorem: Formally, a graph is a triple consisting of a vertex set, an edge set, and an incidence relation between vertices and edges. Contracting an edge e in a graph $\boldsymbol{G}$ means deleting the edge and identifying its ends. Thus if $\boldsymbol{H}$ denotes the resulting graph, then $E(\boldsymbol{H}) = E(\boldsymbol{G}) - \{e\}$.

Suppose that a simple graph $\boldsymbol{G}$ is obtained from a simple graph $\boldsymbol{G}_s$ by contracting an edge $e \in E(\boldsymbol{G}_s)$ to a vertex $w \in V(\boldsymbol{G})$. If the degrees of the endvertices of e in $\boldsymbol{G}_s$ are at least 3, then $\boldsymbol{G}_s$ is said to be obtained from $\boldsymbol{G}$ by *splitting* the vertex w. Notice that this definition implies that the edge e belongs to no triangle of $\boldsymbol{G}_s$. The graph $\boldsymbol{G}_s$ is formally denoted by $\boldsymbol{G} \angle w \left\{ {N_1 \atop N_2} \right\}$ where N_1, N_2 are the neighborhoods of endvertices u, v of e, respectively, excluding u, v themselves.

A graph is *almost* 4-*connected* if it is simple and 3-connected, has at least five vertices, and for every separation (A, B) of order 3, either $\boldsymbol{G} \restriction A$ or $\boldsymbol{G} \restriction B$ has at most four edges. A pair (w, e) is called a *violating pair* in $\boldsymbol{G}$ if w is a cubic vertex of $\boldsymbol{G}$, and e is an edge of $\boldsymbol{G}$ having both endvertices adjacent to w. (Such pair is called violating because it violates the condition of being internally 4-connected. However, this is the only possible violation for almost 4-connected graphs.) An edge is *violating* if it is in some violating pair.

Given a violating edge $e = \{u_1, u_2\}$ in a simple graph $\boldsymbol{G}$, the operation of a *triad addition* is defined as follows. If v is a vertex of $\boldsymbol{G}$ such that v is not equal or adjacent to any of u_1 or u_2, and there is no violating pair (w, e) in $\boldsymbol{G}$ for which v, w are adjacent, then the triad addition produces a graph $\boldsymbol{G}_t$ from $\boldsymbol{G}$ by subdividing the edge e with a new vertex v', and by connecting v' to v by an edge. The graph $\boldsymbol{G}_t$ is formally denoted by $\boldsymbol{G} \dashv v \{u_1, u_2\}$.

Given a violating pair (w, e) in a simple graph $\boldsymbol{G}$, the operation of a *triangle explosion* (of w) is defined as follows. Let u be the neighbor of w which is not incident with e, and assume that u has degree at least 5 in $\boldsymbol{G}$. Then the triangle explosion produces a graph $\boldsymbol{G}_x$ from $\boldsymbol{G}$ by splitting the vertex u into vertices u_1, u_2, and by adding the missing one of edges $\{w, u_1\}, \{w, u_2\}$, so that the degrees of both u_1, u_2 in the resulting graph are at least 4. The graph $\boldsymbol{G}_x$ is formally denoted by $\boldsymbol{G} \triangleleft w \left\{ {N_1 \atop N_2} \right\}$ where N_1, N_2 are the neighborhoods of u_1, u_2 in $\boldsymbol{G}_x$, respectively, excluding u_1, u_2, w themselves.

See Fig. 6.2 for an illustration of a vertex splitting, of a triad addition, and of a triangle explosion.

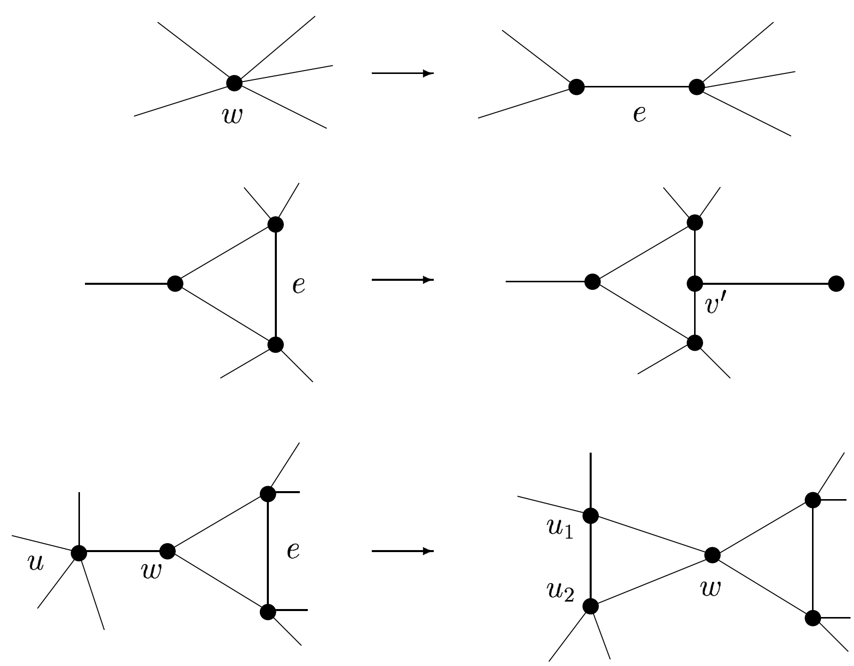


Figure 6.2: An illustration of a vertex splitting, of a triad addition, and of a triangle explosion.

The following theorem is a simplified version of a result proved in [21].

Theorem 6.2.1. (T. Johnson, R. Thomas, 1997) *Suppose* $\boldsymbol{G}$ *is an internally* 4-*connected minor of an internally* 4-*connected graph* $\boldsymbol{H}$ *such that* $\boldsymbol{G}$ *has no embedding in the projective plane. Then there exist a sequence* $\boldsymbol{J}_0 = \boldsymbol{G}, \boldsymbol{J}_1, \ldots, \boldsymbol{J}_k \simeq \boldsymbol{H}$ *of almost* 4-*connected graphs such that for* $i = 1, 2, \ldots, k$, *the graph* $\boldsymbol{J}_i$ *is obtained from* $\boldsymbol{J}_{i-1}$ *by adding an edge, or by splitting a vertex, or by a triad addition, or by a triangle explosion. Moreover, each* $\boldsymbol{J}_i$ *has at most one violating edge, and if an edge* e *is contained in both* $\boldsymbol{J}_{i-1}, \boldsymbol{J}_i$, *then it is not violating in at least one of them.*

A practical consequence of the theorem is that an internally 4-connected graph can be "built" from its internally 4-connected minor by using the above described operations. Notice that a triad addition or a triangle explosion can be used only in graphs that have a violating edge. Each of the four operations might create a violating edge, but only those sequences need to be considered that have at most one violating edge at each step, and such edge must be "repaired" by the next step.

A violating pair (w, e) can be "repaired" using one of the following methods. Either an edge incident with w is added in the next step, or one of the endvertices of e is split so that w is no longer adjacent to both endvertices of e, or a triad addition is applied to e, or a triangle explosion is applied to w. Furthermore, only "repairs" that leave no other violating pair (w', e), and that create at most one new violating edge, need to be considered.

6.3 Assorted Lemmas

Recall that $\Lambda_0 = \Lambda - \{\boldsymbol{K}_{1,2,2,2}, \boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{D}}_2\}$ is the family of those minor-minimal nonprojective graphs that are known to have no planar cover. Let

$\boldsymbol{K}_{3,5}, \boldsymbol{K}_7-\boldsymbol{C}_4, \boldsymbol{\mathcal{D}}_3,\ \boldsymbol{K}_{4,4}-e, \boldsymbol{K}_{4,5}-\boldsymbol{M}_4, \boldsymbol{\mathcal{D}}_{17} \in \Lambda_0$ denote the graphs depicted in Fig. 6.3. Recall also the graphs $\boldsymbol{\mathcal{C}}_4, \boldsymbol{\mathcal{E}}_2 \in \Lambda_0$ from Fig. 2.3. Let Φ' be the family of all simple graphs $\boldsymbol{G}$ such that one of the graphs $\boldsymbol{K}_7-\boldsymbol{C}_4$, $\boldsymbol{\mathcal{D}}_3$, or $\boldsymbol{\mathcal{D}}_{17}$ can be obtained from $\boldsymbol{G}$ by a sequence of $Y\Delta$-transformations; and let $\Phi = \Phi' \cup \{\boldsymbol{K}_{4,4}-e, \boldsymbol{K}_{3,5}, \boldsymbol{K}_{4,5}-\boldsymbol{M}_4\}$. Notice that Φ' includes only finitely many nonisomorphic graphs, because a $Y\Delta$-transformation preserves the number of edges.

Lemma 6.3.1. *Suppose that $\boldsymbol{G}'$ is a simple graph obtained from a graph $\boldsymbol{G}$ by a $Y\Delta$-transformation, and that a simple graph $\boldsymbol{F}'$ is a minor of $\boldsymbol{G}'$. Then $\boldsymbol{G}$ has an $\boldsymbol{F}$ minor, where $\boldsymbol{F} = \boldsymbol{F}'$ or $\boldsymbol{F}$ can be $Y\Delta$-transformed to $\boldsymbol{F}'$.*

Proof. Let $\boldsymbol{G}' = \boldsymbol{G}\ {\scriptstyle Y\Delta}\ \{w\}$ be obtained by a $Y\Delta$-transformation carrying a cubic vertex w of $\boldsymbol{G}$ to a triangle τ of $\boldsymbol{G}'$. If all three edges of τ are in $E(\boldsymbol{F}')$, then they form a triangle in $\boldsymbol{F}'$ since $\boldsymbol{F}'$ is simple. Then $\boldsymbol{F}$ is constructed from $\boldsymbol{F}'$ by adding a new vertex adjacent to the vertices of τ and deleting the edges of τ. Clearly, $\boldsymbol{F}$ is a minor of $\boldsymbol{G}$.

If an edge e of τ is not in $E(\boldsymbol{F}')$, let v be the vertex of τ not incident with e. Then $\boldsymbol{G}' - e$ has an $\boldsymbol{F}'$ minor, and $\boldsymbol{G}' - e$ is obtained from $\boldsymbol{G}$ by contracting the edge $\{v, w\}$. Thus $\boldsymbol{G}$ has an $\boldsymbol{F} = \boldsymbol{F}'$ minor. ∎

Lemma 6.3.2. *If $\boldsymbol{G} \in \Phi'$, then $\boldsymbol{G}$ has a minor isomorphic to some member of Λ_0.*

Proof. This statement follows from the arguments in [14], even though it is not explicitly stated there. Let $\boldsymbol{J}_0 = \boldsymbol{G}, \boldsymbol{J}_1, \ldots, \boldsymbol{J}_t$ be a sequence of graphs such that $\boldsymbol{J}_t \in \{\boldsymbol{K}_7-\boldsymbol{C}_4, \boldsymbol{\mathcal{D}}_3, \boldsymbol{\mathcal{D}}_{17}\}$, and for $i = 1, \ldots, t$, $\boldsymbol{J}_i$ is obtained from $\boldsymbol{J}_{i-1}$ by a $Y\Delta$-transformation. It is easy to check, using Lemma 6.3.1, that if $\boldsymbol{J}_t = \boldsymbol{K}_7-\boldsymbol{C}_4$ or $\boldsymbol{J}_t = \boldsymbol{\mathcal{D}}_3$, then $\boldsymbol{J}_i$, $i = 0, 1, \ldots, t-1$ has a minor isomorphic to one of $\boldsymbol{\mathcal{D}}_3, \boldsymbol{K}_{3,5}, \boldsymbol{\mathcal{E}}_5, \boldsymbol{\mathcal{F}}_1$; and if $\boldsymbol{J}_t = \boldsymbol{\mathcal{D}}_{17}$, then $\boldsymbol{J}_i$, $i = 0, 1, \ldots, t-1$ has a minor isomorphic to one of $\boldsymbol{\mathcal{E}}_{20}, \boldsymbol{\mathcal{G}}_1, \boldsymbol{\mathcal{F}}_4$. (See Appendix A for pictures of these graphs.) ∎

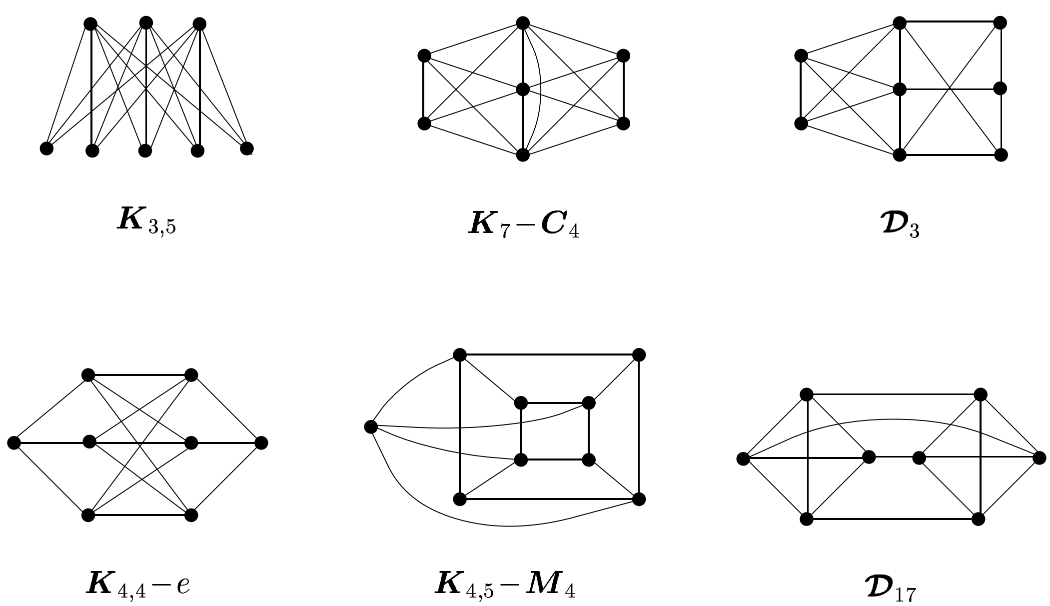


Figure 6.3:

Lemma 6.3.3. *Let $\boldsymbol{G}$ be a graph, and let a simple graph $\boldsymbol{G}'$ be obtained from $\boldsymbol{G}$ by a sequence of $Y\Delta$-transformations. If $\boldsymbol{G}'$ has a minor isomorphic to some member of Φ, then so does $\boldsymbol{G}$. Consequently, $\boldsymbol{G}$ has a minor isomorphic to some member of Λ_0.*

Proof. Notice that each of the graphs $\boldsymbol{K}_{4,4}-e, \boldsymbol{K}_{3,5}, \boldsymbol{K}_{4,5}-\boldsymbol{M}_4$ is triangle-free. If $\boldsymbol{G}'$ has a minor isomorphic to any one of them, then so does $\boldsymbol{G}$ by Lemma 6.3.1. Otherwise, $\boldsymbol{G}'$ has an $\boldsymbol{F}'$ minor for some $\boldsymbol{F}' \in \Phi'$. By Lemma 6.3.1, the graph $\boldsymbol{G}$ has a minor isomorphic to a member $\boldsymbol{F}''$ of Φ'. By Lemma 6.3.3, the graph $\boldsymbol{F}''$ has a minor isomorphic to a member of Λ_0, and hence so does $\boldsymbol{G}$. ∎

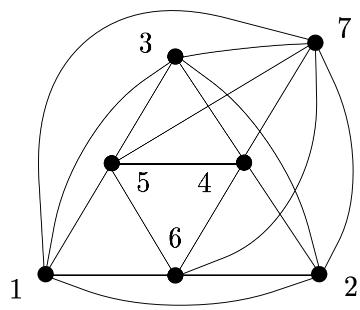
Let the vertices of the graphs $\boldsymbol{K}_{1,2,2,2}, \boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{D}}_2$ be numbered as in Fig. 6.4.

Lemma 6.3.4. *Let $\{\boldsymbol{F}_i\}_{i=1}^4$ be a sequence of graphs defined by $\boldsymbol{F}_1 = \boldsymbol{K}_{1,2,2,2}$, $\boldsymbol{F}_2 = \boldsymbol{\mathcal{B}}_7$, $\boldsymbol{F}_3 = \boldsymbol{\mathcal{C}}_3$, $\boldsymbol{F}_4 = \boldsymbol{\mathcal{D}}_2$. For $i = 1, 2, 3, 4$, the following statements hold.*
a) Suppose that $\boldsymbol{F}' = \boldsymbol{F}_i + e$ is obtained from $\boldsymbol{F}_i$ by adding an edge e joining two distinct nonadjacent vertices of $\boldsymbol{F}_i$. If e is not violating in $\boldsymbol{F}'$, then $\boldsymbol{F}'$ has a minor isomorphic to a member of Λ_0, unless $i = 2$ ($\boldsymbol{F}_i = \boldsymbol{\mathcal{B}}_7$) and $e = \{7, 8\}$.
b) Suppose that $\boldsymbol{F}'$ is obtained by splitting a vertex w in $\boldsymbol{F}_i$. Then either $\boldsymbol{F}'$ has a minor isomorphic to a member of Λ_0, or $i \leq 3$ and $w \neq 7$ and $\boldsymbol{F}'$ has a subgraph $\boldsymbol{F}_{i+1}$.

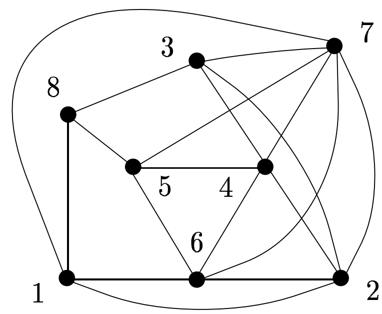
Proof. The proof proceeds along the sequence $\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3, \boldsymbol{F}_4$, considering parts (a) and (b) together. Up to symmetry, there is only one possibility to add an edge to $\boldsymbol{F}_1 = \boldsymbol{K}_{1,2,2,2}$.

- $\boldsymbol{F}' = \boldsymbol{K}_{1,2,2,2} + \{1, 4\}$ has a $\boldsymbol{K}_7 - \boldsymbol{C}_4$ subgraph.

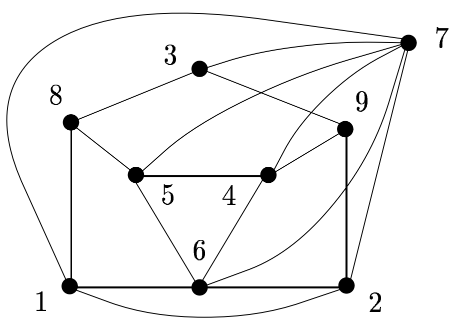
The four possibilities to split vertex 7, up to symmetry, are discussed as follows:



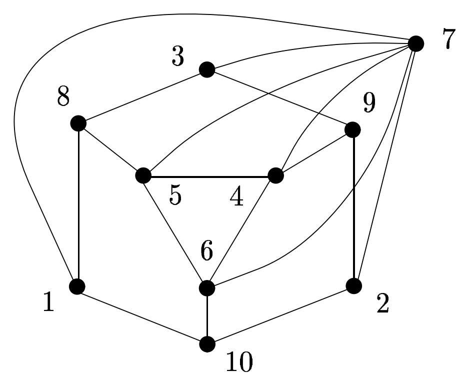
$\boldsymbol{K}_{1,2,2,2}$



$\mathcal{B}_7$



$\mathcal{C}_3$



$\mathcal{D}_2$

Figure 6.4:

– $\boldsymbol{F}' = \boldsymbol{K}_{1,2,2,2} \angle 7 \left\{{1,2,3 \atop 4,5,6}\right\}$ has a $\boldsymbol{\mathcal{D}}_{17}$ subgraph.

– $\boldsymbol{F}' = \boldsymbol{K}_{1,2,2,2} \angle 7 \left\{{1,2,4 \atop 3,5,6}\right\}$ has a $\boldsymbol{K}_{4,4}-e$ subgraph.

– $\boldsymbol{F}' = \boldsymbol{K}_{1,2,2,2} \angle 7 \left\{{1,2 \atop 3,4,5,6}\right\}$ has a $\boldsymbol{\mathcal{D}}_{3}$ subgraph.

– $\boldsymbol{F}' = \boldsymbol{K}_{1,2,2,2} \angle 7 \left\{{1,4 \atop 2,3,5,6}\right\}$ has a $\boldsymbol{K}_{3,5}$ subgraph.

All vertices other than 7 are symmetric in $\boldsymbol{K}_{1,2,2,2}$, so it suffices to consider the three possible nonsymmetric splittings of vertex 1.

– $\boldsymbol{F}' = \boldsymbol{K}_{1,2,2,2} \angle 1 \left\{{3,6 \atop 2,5,7}\right\}$ and $\boldsymbol{F}' = \boldsymbol{K}_{1,2,2,2} \angle 1 \left\{{2,7 \atop 3,5,6}\right\}$ have $\boldsymbol{\mathcal{D}}_{3}$ subgraphs.

– $\boldsymbol{F}' = \boldsymbol{K}_{1,2,2,2} \angle 1 \left\{{3,5 \atop 2,6,7}\right\}$ has a $\boldsymbol{F}_2 = \boldsymbol{\mathcal{B}}_7$ subgraph.

Since the graphs $\boldsymbol{K}_7-\boldsymbol{C}_4$, $\boldsymbol{\mathcal{D}}_{17}$, $\boldsymbol{K}_{4,4}-e$, $\boldsymbol{\mathcal{D}}_3$, $\boldsymbol{K}_{3,5}$ are members of Λ_0, the statement is proved for $\boldsymbol{F}_1$.

If $\boldsymbol{F}' = \boldsymbol{\mathcal{B}}_7 + e$ where e is one of $\{1,4\}$, $\{2,5\}$, $\{3,6\}$, then the graph $\boldsymbol{F}' \,\mathrm{Y}\!\Delta\, \{8\}$ equals to the graph $\boldsymbol{K}_{1,2,2,2} + e$. So, using Lemma 6.3.3, the arguments in the previous paragraph imply that $\boldsymbol{F}'$ has a minor isomorphic to a graph from Λ_0. The remaining possible edge addition, up to symmetry, is covered next.

– $\boldsymbol{F}' = \boldsymbol{\mathcal{B}}_7 + \{2,8\}$ has a $\boldsymbol{K}_7-\boldsymbol{C}_4$ minor via contracting $\{5,8\}$.

Let a *step splitting* be the splitting operation $\boldsymbol{F}_i \angle w \left\{{u_1,u_2 \atop u_3,\ldots,u_k}\right\}$, $i \in \{1,2,3,4\}$ such that $w, u_1, u_2 \in \{1,2,3,4,5,6\}$, and $\{w, u_1, u_2\}$ does not contain any one of the pairs $\{1,4\}, \{2,5\}, \{3,6\}$. Notice that all non-step splittings in $\boldsymbol{F}_1$ produce members of Φ. The vertex 8 cannot be split in $\boldsymbol{\mathcal{B}}_7$, so if $\boldsymbol{F}'$ results by a non-step splitting in $\boldsymbol{\mathcal{B}}_7$, then the graph $\boldsymbol{F}' \,\mathrm{Y}\!\Delta\, \{8\}$ results by a non-step splitting in $\boldsymbol{K}_{1,2,2,2}$; hence $\boldsymbol{F}'$ has a Λ_0 minor by Lemma 6.3.3. All possible step splittings in $\boldsymbol{\mathcal{B}}_7$ are discussed as follows.

– $\boldsymbol{F}' = \boldsymbol{\mathcal{B}}_7 \angle 3 \left\{{2,4 \atop 7,8}\right\}$ and $\boldsymbol{F}' = \boldsymbol{\mathcal{B}}_7 \angle 2 \left\{{3,4 \atop 1,6,7}\right\}$ have $\boldsymbol{F}_3 = \boldsymbol{\mathcal{C}}_3$ subgraphs.

- $\boldsymbol{F}' = \boldsymbol{\mathcal{B}}_7 \angle 2\,\{^{1,3}_{4,6,7}\}$ has a $\boldsymbol{K}_{4,5}-\boldsymbol{M}_4$ subgraph.

- $\boldsymbol{F}' = \boldsymbol{\mathcal{B}}_7 \angle 2\,\{^{4,6}_{1,3,7}\}$ has a $\boldsymbol{\mathcal{C}}_4$ subgraph.

The discussion is continued in the same manner for $\boldsymbol{F}_3 = \boldsymbol{\mathcal{C}}_3$. If $\boldsymbol{F}' = \boldsymbol{\mathcal{C}}_3 + e$ and e is not violating in $\boldsymbol{F}'$, then e is incident with at most one of the vertices $8, 9$, say 8, and it is not $\{7, 8\}$. Thus $\boldsymbol{F}'\ \mathrm{Y}\!\triangle\,\{9\} = \boldsymbol{\mathcal{B}}_7 + e$ has a Λ_0 minor by the previous analysis, and so does $\boldsymbol{F}'$ by Lemma 6.3.3. Similarly, if $\boldsymbol{F}'$ results by a non-step splitting in $\boldsymbol{\mathcal{C}}_3$, then $\boldsymbol{F}'\ \mathrm{Y}\!\triangle\,\{9\}$ results by a non-step splitting in $\boldsymbol{\mathcal{B}}_7$, so $\boldsymbol{F}'$ has a Λ_0 minor again. The possible step splittings in $\boldsymbol{\mathcal{C}}_3$ are covered next.

- $\boldsymbol{F}' = \boldsymbol{\mathcal{C}}_3 \angle 1\,\{^{2,6}_{7,8}\}$ and $\boldsymbol{F}' = \boldsymbol{\mathcal{C}}_3 \angle 6\,\{^{1,2}_{4,5,7}\}$ have $\boldsymbol{F}_4 = \boldsymbol{\mathcal{D}}_2$ subgraphs.

- $\boldsymbol{F}' = \boldsymbol{\mathcal{C}}_3 \angle 6\,\{^{1,5}_{2,4,7}\}$ has a $\boldsymbol{K}_{4,5}-\boldsymbol{M}_4$ minor via contracting $\{3, 9\}$.

Finally, the same arguments as above apply for the cases of an edge addition or a non-step splitting in $\boldsymbol{F}_4 = \boldsymbol{\mathcal{D}}_2$. So there is only one step splitting remaining to be checked.

- $\boldsymbol{J}' = \boldsymbol{\mathcal{D}}_2 \angle 5\,\{^{4,6}_{7,8}\}$ has an $\boldsymbol{\mathcal{E}}_2$ subgraph. ∎

The following statement about planar embeddings of graphs appears more or less explicitly in [33, 35, 37, 39].

Theorem 6.3.5. (N. Robertson, P.D. Seymour, Y. Shiloach, C. Thomassen) *Let $\boldsymbol{G}$ be a 3-connected graph, and let u, v, w be three distinct vertices of $\boldsymbol{G}$. If $\boldsymbol{G}$ has no planar embedding in which u, v and w are all incident with the outer face, then $\boldsymbol{G}$ has an $\boldsymbol{F}$ minor such that $\boldsymbol{F}$ is isomorphic to $\boldsymbol{K}_{2,3}$, and the vertices u, v, w are contracted into three distinct vertices u', v', w' which form the part of size three in $\boldsymbol{F}$.*

Suppose that $\boldsymbol{G}$ is a graph, and that $v_1, v_2, v_3 \in V(\boldsymbol{G})$ are three distinct vertices of $\boldsymbol{G}$. Let $\boldsymbol{G} \wr \{v_1, v_2, v_3\}$ denote the graph $\boldsymbol{H}$ defined as follows: If there exists a cubic vertex $w \in V(\boldsymbol{G})$ with the neighbors v_1, v_2, v_3, then $\boldsymbol{H}$ results from $\boldsymbol{G}$ by adding one new vertex t adjacent to all three vertices v_1, v_2, v_3. Otherwise, $\boldsymbol{H}$ results from $\boldsymbol{G}$ by adding two new vertices s, t both adjacent to all three vertices v_1, v_2, v_3, and by deleting all edges with both ends in $\{v_1, v_2, v_3\}$.

Lemma 6.3.6. *Let $\boldsymbol{G}$ be a 3-connected graph, and let (A, B) be a non-flat separation of order three in $\boldsymbol{G}$. Let $\boldsymbol{F}_0$ be a simple 3-connected graph. Suppose that $\boldsymbol{F} \subseteq \boldsymbol{G}$ is a subgraph of $\boldsymbol{G}$ isomorphic to a subdivision of $\boldsymbol{F}_0$, and that $W \subseteq V(\boldsymbol{F})$ is the subset of vertices that have degree more than 2 in $\boldsymbol{F}$. If $|W \cap (B - A)| \leq 1$, then $\boldsymbol{G}$ contains a minor isomorphic to the graph $\boldsymbol{F}_0 \wr \{w_1, w_2, w_3\}$ for some three vertices $w_1, w_2, w_3 \in V(\boldsymbol{F}_0)$.*

Proof. Let $A \cap B = \{b_1, b_2, b_3\}$. By Theorem 6.3.5, there is a minor $\boldsymbol{G}'$ of $\boldsymbol{G}$, and a 3-separation (A', B') in $\boldsymbol{G}'$, such that $A = A'$, $\boldsymbol{G}' \upharpoonright A' = \boldsymbol{G} \upharpoonright A$, and $B' - A' = \{s, t\}$ where each of s, t is adjacent to all three vertices b_1, b_2, b_3. (Hence $\boldsymbol{F} \upharpoonright A$ is a subgraph of $\boldsymbol{G}'$.)

Suppose that $|W \cap B| = 1$. Let $W \cap B = \{w\}$, and let w' be the vertex of $\boldsymbol{F}_0$ corresponding to w. Let Q_e denote the path in $\boldsymbol{F}$ that corresponds to an edge $e \in E(\boldsymbol{F}_0)$. Since $\boldsymbol{F}_0$ is 3-connected, there are at least three edges incident with w' in $\boldsymbol{F}_0$. On the other hand, the vertex $w \in B$ can be connected by at most three disjoint paths with vertices in $W - \{w\} \subset A$. Hence w' is a cubic vertex in $\boldsymbol{F}_0$, and the edges incident with w' can be denoted by $e_1, e_2, e_3 \in E(\boldsymbol{F}_0)$ so that $b_i \in V(Q_{e_i})$ for $i = 1, 2, 3$. Let $\boldsymbol{G}''$ be the graph obtained from $\boldsymbol{G}'$ by contracting each of the paths $Q_{e_i} \upharpoonright A$, $i = 1, 2, 3$, into one vertex. Then $\boldsymbol{G}''$ contains a subgraph isomorphic to a subdivision of $\boldsymbol{F}_0 \wr \{v_1, v_2, v_3\}$ where v_1, v_2, v_3 are the neighbors of w in $\boldsymbol{F}_0$.

So suppose that $W \cap B = \emptyset$. Since $\boldsymbol{G}$ is 3-connected, there exist, by Menger's theorem, three vertices $d_1, d_2, d_3 \in W$, and three vertex-disjoint paths P_1, P_2, P_3 such that P_i has ends b_i and d_i for $i = 1, 2, 3$. For a path P, let $P[u, v]$ denote the subpath of P connecting the vertices $u, v \in V(P)$. Let c_i be the vertex of $V(P_i) \cap V(\boldsymbol{F})$ closest to b_i in P_i, and let $P'_i = P_i[b_i, c_i]$, for $i = 1, 2, 3$. (It may happen that $c_i = b_i$.)

First suppose the case that not all c_1, c_2, c_3 belong to the same path Q_e, $e \in E(\boldsymbol{F}_0)$. Then, for $i = 1, 2, 3$, there exists an edge $e_i \in E(\boldsymbol{F}_0)$ such that $c_i \in V(Q_{e_i})$; and one of the ends of the path Q_{e_i} can be denoted by x_i, so that the path $Q'_i = Q_{e_i}[x_i, c_i]$ does not intersect the set $\{c_1, c_2, c_3\} - \{c_i\}$. Also, x_1, x_2, x_3 can be chosen all distinct since $\boldsymbol{F}_0$ has no multiple edges. Let $\boldsymbol{G}''$ denote the graph obtained from $\boldsymbol{G}'$ by contracting each of the paths $Q'_1 \cup P'_1$, $Q'_2 \cup P'_2$, and $Q'_3 \cup P'_3$ into one vertex. One can easily check that $\boldsymbol{G}''$ has a subgraph isomorphic to a subdivision of $\boldsymbol{F}_0 \between \{x'_1, x'_2, x'_3\}$, where x'_1, x'_2, x'_3 are the vertices of $\boldsymbol{F}_0$ corresponding to x_1, x_2, x_3. (If for some $e \in E(\boldsymbol{F}_0)$ there exists a path Q_e in $\boldsymbol{G}$ intersecting $B - A$, then both ends of Q_e are in $\{x_1, x_2, x_3\}$. Thus, if there is no cubic vertex adjacent to x'_1, x'_2, x'_3 in $\boldsymbol{F}_0$, the edge e is not present in $\boldsymbol{F}_0 \between \{x'_1, x'_2, x'_3\}$; and otherwise, the path Q_e can be replaced by a path Q'_e that uses one of the vertices s, t in $\boldsymbol{G}'$.)

Next, suppose that there is an edge $e \in E(\boldsymbol{F}_0)$ such that $c_1, c_2, c_3 \in V(Q_e)$. Let x, y be the ends of the path Q_e, and let $U = V(Q_e) - \{x, y\}$. For $i = 1, 2, 3$, let a_i be the vertex of $V(P_i) \cap (V(\boldsymbol{F}) - U)$ closest to b_i in P_i, and let $P''_i = P_i[b_i, a_i]$. Assume, without loss of generality, that, for some vertex $v \in V(P''_1) \cap V(Q_e)$, one of the paths $Q_e[v, x], Q_e[v, y]$ is disjoint from $P''_2 \cup P''_3$. Let $v_1 \in V(P''_1) \cap V(Q_e)$ be such a vertex which is the one closest to b_1 in P''_1; and assume that the path $Q_e[v_1, x]$ is disjoint from $P''_2 \cup P''_3$. Further, let $v_2 \in V(Q_e) \cap (V(P''_2) \cup V(P''_3))$ be the vertex closest to y in Q_e, and assume that $v_2 \in V(P''_2)$. Then the path $P^o_1 = P''_1[b_1, v_1] \cup Q_e[v_1, x]$

is disjoint from P_2'' and P_3'', and the path $P_2^o = P_2''[b_2, v_2] \cup Q_e[v_2, y]$ is disjoint from P_1^o and P_3''. In particular, $a_3 \notin \{x, y\}$. Let $e' \in E(\boldsymbol{F}_0)$ be the edge such that $a_3 \in V(Q_{e'})$; and let z be an end of $Q_{e'}$ such that $z \notin \{x, y\}$, and that $z = a_3$ if a_3 is an end of $Q_{e'}$. Let $P_3^o = P_3'' \cup Q_{e'}[a_3, z]$. Let $\boldsymbol{G}''$ denote the graph obtained from $\boldsymbol{G}'$ by contracting each of the paths P_1^o, P_2^o, and P_3^o into one vertex. Then $\boldsymbol{G}''$ has a subgraph isomorphic to a subdivision of $(\boldsymbol{F}_0 - e) \wr \{x', y', z'\} = \boldsymbol{F}_0 \wr \{x', y', z'\}$, where x', y', z' are the vertices of $\boldsymbol{F}_0$ corresponding to x, y, z. ∎

Finally, the postponed proof of Lemma 6.1.5 about flat separations from Section 6.1 is presented.

Proof of Lemma 6.1.5. Assume first that $k = 1$, and let $\boldsymbol{G}_A = \boldsymbol{G} \restriction A$, $\boldsymbol{G}_B = \boldsymbol{G} \restriction B$. If neither (A, B) nor (B, A) are flat, then both graphs $\boldsymbol{G}_A$ and $\boldsymbol{G}_B$ are nonplanar, and thus by the Kuratowski theorem they contain subgraphs $\boldsymbol{F}_A \subseteq \boldsymbol{G}_A$ and $\boldsymbol{F}_B \subseteq \boldsymbol{G}_B$ isomorphic to subdivisions of $\boldsymbol{K}_5$ or $\boldsymbol{K}_{3,3}$. Since $\boldsymbol{G}$ is connected, it is easy to contract $\boldsymbol{F}_A$ and $\boldsymbol{F}_B$ into a minor isomorphic to one of $\boldsymbol{K}_5 \cdot \boldsymbol{K}_5, \boldsymbol{K}_5 \cdot \boldsymbol{K}_{3,3}, \boldsymbol{K}_{3,3} \cdot \boldsymbol{K}_{3,3} \in \Lambda_0$, a contradiction.

The case of $k = 2$ is settled similarly. Let $A \cap B = \{u, v\}$, and let $\boldsymbol{G}_A = (\boldsymbol{G} \restriction A) + \{u, v\}$, $\boldsymbol{G}_B = (\boldsymbol{G} \restriction B) + \{u, v\}$. If neither (A, B) nor (B, A) are flat, then both graphs $\boldsymbol{G}_A$ and $\boldsymbol{G}_B$ are nonplanar, and thus by the Kuratowski theorem they contain subgraphs $\boldsymbol{F}_A \subseteq \boldsymbol{G}_A$ and $\boldsymbol{F}_B \subseteq \boldsymbol{G}_B$ isomorphic to subdivisions of $\boldsymbol{K}_5$ or $\boldsymbol{K}_{3,3}$. Without loss of generality, let us focus on the graph $\boldsymbol{G}_A$. Since $\boldsymbol{G}$ is 2-connected in this case, there exist two disjoint paths P_u, P_v between $\{u, v\}$ and $V(\boldsymbol{F}_A)$. Clearly, the graph $\boldsymbol{F}_A \cup P_u \cup P_v \cup \{uv\}$ has a minor $\boldsymbol{F}'_A$ isomorphic to $\boldsymbol{K}_5$ or $\boldsymbol{K}_{3,3}$ such that u, v are two distinct vertices of $\boldsymbol{F}'_A$. The graph $\boldsymbol{F}'_B$ is found in the same way as a minor of $\boldsymbol{G}_B$. So the graph $\boldsymbol{G}' = \boldsymbol{F}'_A \cup \boldsymbol{F}'_B - uv$ is a minor of $\boldsymbol{G}$. It is easy to show that $\boldsymbol{G}'$ is isomorphic to some member of Λ_0: If $\boldsymbol{F}'_A \simeq \boldsymbol{K}_5$, there

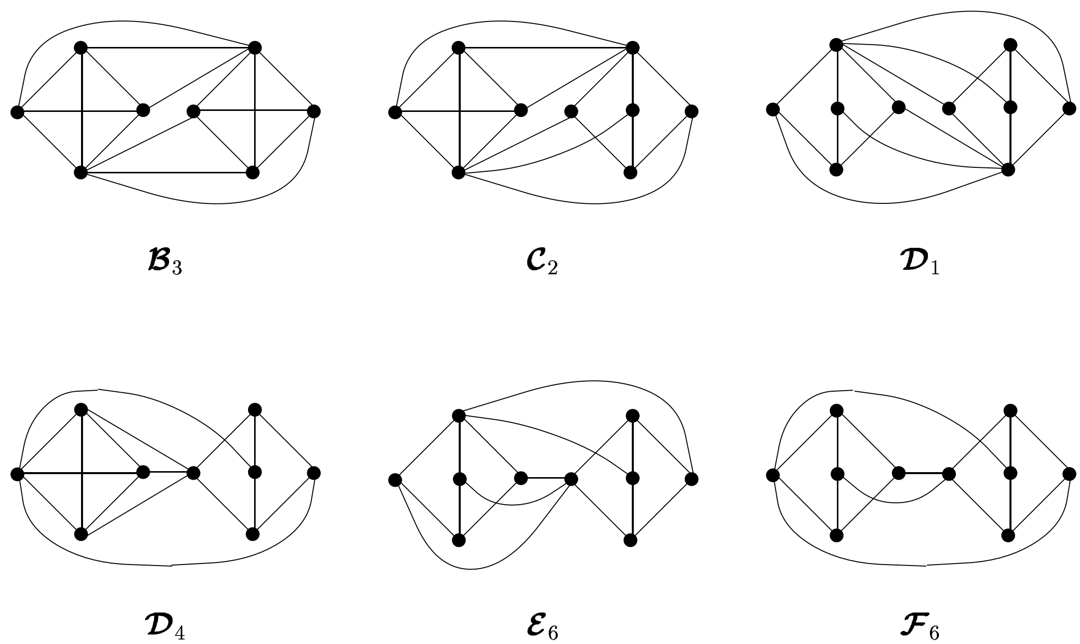

$\mathcal{B}_3$
$\mathcal{C}_2$
$\mathcal{D}_1$
$\mathcal{D}_4$
$\mathcal{E}_6$
$\mathcal{F}_6$

Figure 6.5:

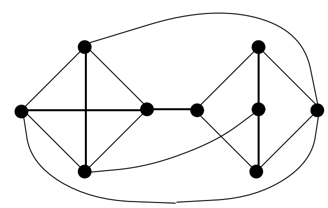
is only one nonsymmetric choice of $u, v \in V(\boldsymbol{F}'_A)$. If $\boldsymbol{F}'_A \simeq \boldsymbol{K}_{3,3}$, there are two nonsymmetric choices of $u, v \in V(\boldsymbol{F}'_A)$ (either from the same part, or from different ones). Therefore $\boldsymbol{G}'$ is isomorphic to $\boldsymbol{\mathcal{B}}_3 \in \Lambda_0$ for $\boldsymbol{F}'_A \simeq \boldsymbol{F}'_B \simeq \boldsymbol{K}_5$, $\boldsymbol{G}'$ is isomorphic to one of $\boldsymbol{\mathcal{D}}_1, \boldsymbol{\mathcal{E}}_6, \boldsymbol{\mathcal{F}}_6 \in \Lambda_0$ for $\boldsymbol{F}'_A \simeq \boldsymbol{F}'_B \simeq \boldsymbol{K}_{3,3}$, and $\boldsymbol{G}'$ is isomorphic to one of $\boldsymbol{\mathcal{C}}_2, \boldsymbol{\mathcal{D}}_4 \in \Lambda_0$ for $\boldsymbol{F}'_A \simeq \boldsymbol{K}_5$, $\boldsymbol{F}'_B \simeq \boldsymbol{K}_{3,3}$, which is a contradiction to our assumption. (See Fig. 6.5 for these graphs.)

Next, let $k = 3$, and suppose for a contradiction that neither (A, B) nor (B, A) are flat. Notice that the assumptions guarantee that $\boldsymbol{G}$ is 3-connected in this case. By Corollary 6.1.4, $\boldsymbol{G}$ has a minor isomorphic to one of the graphs $\boldsymbol{K}_{1,2,2,2}, \boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{D}}_2$. It follows from Lemma 6.3.4 that $\boldsymbol{G}$ actually contains a subgraph $\boldsymbol{M}'$ isomorphic to a subdivision of some $\boldsymbol{M} \in \{\boldsymbol{K}_{1,2,2,2}, \boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{D}}_2\}$. Let $W \subseteq V(\boldsymbol{M}')$ be the set of the vertices that have degree more than two in $\boldsymbol{M}'$. Since $\boldsymbol{M}$ is internally 4-connected, one of the sets $A - B, B - A$, say $B - A$, contains at most one vertex of W. Thus, by Lemma 6.3.6, $\boldsymbol{G}$ has a minor isomorphic to $\boldsymbol{N} = \boldsymbol{M} \nmid \{b_1, b_2, b_3\}$ for some three distinct vertices $b_1, b_2, b_3 \in V(\boldsymbol{M})$.

It is shown that the graph $\boldsymbol{N}$ (and hence also $\boldsymbol{G}$) has a minor isomorphic to some member of Λ_0 for any choice of $\{b_1, b_2, b_3\}$ from $V(\boldsymbol{M})$. Let the vertices of $\boldsymbol{M}$ be numbered as in Fig. 6.4. Suppose first that no cubic vertex in $\boldsymbol{M}$ has the neighbors b_1, b_2, b_3.

- If the vertices b_1, b_2, b_3 disconnect the graph $\boldsymbol{M}$, then $\boldsymbol{N}$ has a $\boldsymbol{K}_{3,5} \in \Lambda_0$ minor since one side of the separation induced by $\{b_1, b_2, b_3\}$ in $\boldsymbol{M}$ is always non-flat.
- If $\{b_1, b_2, b_3\}$ includes any one of the pairs $\{1, 4\}$ or $\{2, 5\}$ or $\{3, 6\}$, say $\{b_1, b_2\}$, then $\boldsymbol{N}$ contains an $\boldsymbol{M} + b_1b_2$ minor. Since b_1b_2 is not an edge of $\boldsymbol{M}$, and it is not violating in $\boldsymbol{M} + b_1b_2$, it follows from Lemma 6.3.4(a) that $\boldsymbol{M} + b_1b_2$ has a minor isomorphic to a member of Λ_0.

– If $\boldsymbol{M} - \{b_1, b_2, b_3\}$ contains a subgraph $\boldsymbol{G}_0$ isomorphic to $\boldsymbol{K}_4$, then, by the 3-connectivity of $\boldsymbol{M}$, for some $v \in V(\boldsymbol{G}_0)$ there exist three disjoint paths between the sets $V(\boldsymbol{G}_0) - \{v\}$ and $\{b_1, b_2, b_3\}$ in $\boldsymbol{M} - v$. Moreover, since $\boldsymbol{M}$ is internally 4-connected, there exists a path between v and some vertex of $\{b_1, b_2, b_3\}$ avoiding $V(\boldsymbol{G}_0) - \{v\}$. Thus $\boldsymbol{N}$ contains an $\boldsymbol{\mathcal{E}}_{19} \in \Lambda_0$ minor, see Fig. 6.6.



$\boldsymbol{\mathcal{E}}_{19}$

Figure 6.6:

Let $\boldsymbol{N}'$ ($\boldsymbol{M}'$) be the graph obtained from $\boldsymbol{N}$ ($\boldsymbol{M}$) by $Y\Delta$-transformations of the vertices from the set $(\{8, 9, 10\} \cap V(\boldsymbol{M})) - \{b_1, b_2, b_3\}$. If $\boldsymbol{M}' \simeq \boldsymbol{K}_{1,2,2,2}$, and none of the three above general cases apply, then there is just one possibility, up to symmetry.

– For $\{b_1, b_2, b_3\} = \{1, 2, 7\}$, the resulting graph $\boldsymbol{N}'$ has a $\boldsymbol{\mathcal{D}}_3$ minor via contracting the edge $\{2, 3\}$.

If $\boldsymbol{M}' \simeq \boldsymbol{\mathcal{B}}_7$, then $|\{8, 9, 10\} \cap \{b_1, b_2, b_3\}| = 1$, and so $8 \in \{b_1, b_2, b_3\}$ by symmetry. Then all three remaining nonsymmetric possibilities are as follows.

– For $\{b_1, b_2, b_3\} = \{4, 6, 8\}$ or $\{b_1, b_2, b_3\} = \{6, 7, 8\}$, the graph $\boldsymbol{N}'$ has a $\boldsymbol{K}_{3,5}$ minor via contracting the edges $\{2, 6\}$ and $\{4, 7\}$.

- For $\{b_1, b_2, b_3\} = \{5, 7, 8\}$, the graph $\boldsymbol{N}'$ has a $\boldsymbol{\mathcal{D}}_3$ minor via contracting the edges $\{1, 8\}$ and $\{3, 8\}$.

If $\boldsymbol{M}' \simeq \boldsymbol{\mathcal{C}}_3$, then $8, 9 \in \{b_1, b_2, b_3\}$ by symmetry. There are two nonsymmetric possibilities remaining to be checked.

- For $\{b_1, b_2, b_3\} = \{6, 8, 9\}$ or $\{b_1, b_2, b_3\} = \{7, 8, 9\}$, the graph $\boldsymbol{N}'$ has a $\boldsymbol{K}_{3,5}$ minor via contracting the edges $\{1, 2\}$, $\{4, 5\}$, $\{6, 7\}$.

And if $\boldsymbol{M}' \simeq \boldsymbol{\mathcal{D}}_2$, then $\{b_1, b_2, b_3\} = \{8, 9, 10\}$, but the graph $\boldsymbol{M}' - \{8, 9, 10\}$ has a $\boldsymbol{K}_4$ subgraph, so this case was already covered above. Since $\boldsymbol{N}'$ has a Λ_0 minor, so does $\boldsymbol{N}$ by Lemma 6.3.3, a contradiction.

Finally, consider the case that b_1, b_2, b_3 are the neighbors of a cubic vertex w in $\boldsymbol{M}$. Similarly as above, let $\boldsymbol{N}'$ be the graph obtained from $\boldsymbol{N}$ by $Y\Delta$-transformations of the vertices $(\{8, 9, 10\} \cap V(\boldsymbol{M})) - \{w, b_1, b_2, b_3\}$. It is easy to see that there are only two nonsymmetric possibilities to consider.

- The graph $\boldsymbol{N}'$ constructed from $\boldsymbol{\mathcal{B}}_7$ by adding a new vertex t adjacent to the neighbors of $s = 8$ has a $\boldsymbol{\mathcal{D}}_3$ minor via contracting the edge $\{2, 4\}$.

- The graph $\boldsymbol{N}'$ constructed from $\boldsymbol{\mathcal{C}}_3$ by adding a new vertex t adjacent to the neighbors of $s = 3$ has a $\boldsymbol{K}_{3,5}$ minor via contracting the edges $\{1, 8\}$, $\{4, 9\}$.

So it follows from Lemma 6.3.3 that $\boldsymbol{N}$ (and hence also $\boldsymbol{G}$) has a minor isomorphic to some member of Λ_0, a contradiction. ∎

6.4 The Splitting Process

Our objective is to prove that if an internally 4-connected graph $\boldsymbol{H}$ has a minor isomorphic to one of $\boldsymbol{K}_{1,2,2,2}, \boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{D}}_2$, then either $\boldsymbol{H}$ itself is isomorphic to one of the 16 specific graphs defined later in this section (see also Appendix B), or $\boldsymbol{H}$ has a minor isomorphic to a graph from Λ_0. For the readers' convenience, the proof is divided into four steps in Lemmas 6.4.1, 6.4.2, 6.4.3, and 6.4.4.

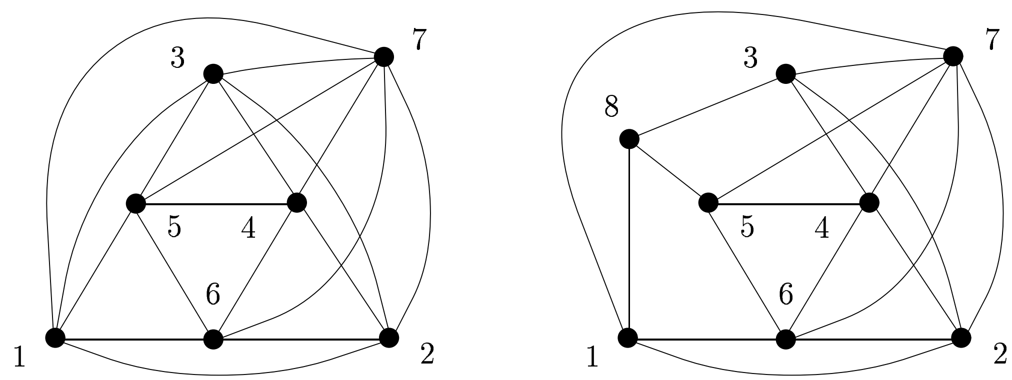


Figure 6.7: Numbering of vertices in the graphs $\boldsymbol{K}_{1,2,2,2}$ and $\boldsymbol{\mathcal{B}}_7$.

Lemma 6.4.1. *Let $\boldsymbol{H}$ be an internally 4-connected graph having no minor isomorphic to a graph from Λ_0. If $\boldsymbol{H}$ contains a $\boldsymbol{K}_{1,2,2,2}$ minor, then either $\boldsymbol{H}$ is isomorphic to $\boldsymbol{K}_{1,2,2,2}$ or it contains a $\boldsymbol{\mathcal{B}}_7$ minor.*

Proof. Let $\boldsymbol{J}_0 = \boldsymbol{K}_{1,2,2,2}, \boldsymbol{J}_1, \ldots, \boldsymbol{J}_k \simeq \boldsymbol{H}$ be a sequence of simple graphs as described by Theorem 6.2.1. If $k = 0$, the statement holds. Otherwise, since $\boldsymbol{J}_0 = \boldsymbol{K}_{1,2,2,2}$ has no violating edge, $\boldsymbol{J}_1$ is obtained from $\boldsymbol{J}_0$ by adding an edge or by splitting a vertex. But $\boldsymbol{J}_1$ is not allowed to have a minor isomorphic to a member of Λ_0, thus $\boldsymbol{J}_1$, and hence also $\boldsymbol{H}$, have a $\boldsymbol{\mathcal{B}}_7$ minor by Lemma 6.3.4. ∎

Suppose that the vertices of $\boldsymbol{\mathcal{B}}_7$ are numbered as in Fig. 6.7. Let $\boldsymbol{\mathcal{B}}'_7$ denote the graph obtained from $\boldsymbol{\mathcal{B}}_7$ by adding an edge $\{7,8\}$, and let $\boldsymbol{\mathcal{B}}''_7$ denote the graph obtained from $\boldsymbol{\mathcal{B}}'_7$ by adding an edge $\{1,5\}$. Notice that $\boldsymbol{\mathcal{B}}_7 \,{\scriptstyle Y\Delta}\, \{8\} = \boldsymbol{K}_{1,2,2,2}$.

Lemma 6.4.2. *Let $\boldsymbol{H}$ be an internally 4-connected graph having no minor isomorphic to a graph from Λ_0. If $\boldsymbol{H}$ contains a $\boldsymbol{\mathcal{B}}_7$ minor, then either $\boldsymbol{H}$ is isomorphic to one of $\boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{B}}'_7, \boldsymbol{\mathcal{B}}''_7$, or it contains a $\boldsymbol{\mathcal{C}}_3$ minor.*

Proof. As in the previous proof, let $\boldsymbol{J}_0 = \boldsymbol{\mathcal{B}}_7, \boldsymbol{J}_1, \dots, \boldsymbol{J}_k \simeq \boldsymbol{H}$ be a sequence of simple graphs as described by Theorem 6.2.1, and let $k > 0$. Since $\boldsymbol{J}_0 = \boldsymbol{\mathcal{B}}_7$ has no violating edge, $\boldsymbol{J}_1$ is obtained from $\boldsymbol{J}_0$ by adding an edge or by splitting a vertex. So the statement follows from Lemma 6.3.4, unless $\boldsymbol{J}_1 = \boldsymbol{J}_0 + e$ where e is one of $\{1,3\}, \{1,5\}, \{3,5\}$ or $\{7,8\}$. These possibilities reduce to two nonsymmetric cases.

- $\boldsymbol{J}_1 = \boldsymbol{\mathcal{B}}_7 + \{1,5\}$ is not internally 4-connected.
- $\boldsymbol{J}_1 = \boldsymbol{\mathcal{B}}_7 + \{7,8\}$ is the graph $\boldsymbol{\mathcal{B}}'_7$.

Consider $\boldsymbol{J}_1 = \boldsymbol{\mathcal{B}}'_7$, and $k \geq 2$. If $\boldsymbol{J}_2 = \boldsymbol{J}_1 \angle\, w \left\{{N_1 \atop N_2}\right\}$ is obtained by a vertex splitting where $w \neq 7, 8$, or where $w = 7$ and $|N_1 - \{8\}| \geq 2$ and $|N_2 - \{8\}| \geq 2$, then the same splitting can be applied to the graph $\boldsymbol{\mathcal{B}}_7$, producing a subgraph of $\boldsymbol{J}_2$. In such a case the statement follows from Lemma 6.3.4(b). Otherwise, the following possibilities, up to symmetry, are checked.

- $\boldsymbol{J}_2 = \boldsymbol{\mathcal{B}}'_7 \angle\, 8 \left\{{1,7 \atop 3,5}\right\}$ and $\boldsymbol{J}_2 = \boldsymbol{\mathcal{B}}'_7 \angle\, 7 \left\{{1,8 \atop 2,3,4,5,6}\right\}$ have $\boldsymbol{\mathcal{C}}_3$ subgraphs.
- $\boldsymbol{J}_2 = \boldsymbol{\mathcal{B}}'_7 \angle\, 7 \left\{{2,8 \atop 1,3,4,5,6}\right\}$ has a $\boldsymbol{K}_7 - \boldsymbol{C}_4$ minor via contracting $\{2,7\}, \{5,8\}$.

If $\boldsymbol{J}_2 = \boldsymbol{J}_1 + e$ where e is not one of $\{1,3\}, \{1,5\}, \{3,5\}$, then Lemma 6.3.4(a) applies to $\boldsymbol{J}_0 + e \subset \boldsymbol{J}_1$. So only the following possibility needs to be checked.

- $\boldsymbol{J}_2 = \boldsymbol{\mathcal{B}}'_7 + \{1,5\}$ is the graph $\boldsymbol{\mathcal{B}}''_7$.

Next, let $\boldsymbol{J}_1 = \boldsymbol{\mathcal{B}}_7 + \{1,5\}$. Since $\boldsymbol{J}_1$ has a violating edge $\{1,5\}$ in this case, it cannot be the last graph $\boldsymbol{H}$ in the sequence, and hence $k \geq 2$. By Theorem 6.2.1, the graph $\boldsymbol{J}_2$ is obtained from $\boldsymbol{J}_1$ by adding an edge, or by splitting a vertex, or by a triad addition, or by a triangle explosion, so that the edge $\{1,5\}$ is not violating in $\boldsymbol{J}_2$. Since a triad addition or a triangle explosion cannot be applied to $\boldsymbol{J}_1$, either an edge incident with the vertex 8 is added, or one of the vertices $1,5$ is suitably split in $\boldsymbol{J}_1$. Again, some of the possibilities are already covered by the analysis of $\boldsymbol{J}_0$, and the remaining nonsymmetric cases are as follows.

- $\boldsymbol{J}_2 = \boldsymbol{\mathcal{B}}_7 + \{1,5\} + \{7,8\}$ is the graph $\boldsymbol{\mathcal{B}}''_7$.

- $\boldsymbol{J}_2 = \boldsymbol{\mathcal{B}}_7 + \{1,5\} \angle\, 5\, \{^{1,8}_{4,6,7}\}$ still has the same violating edge $\{1,5\}$.

- $\boldsymbol{J}_2 = \boldsymbol{\mathcal{B}}_7 + \{1,5\} \angle\, 5\, \{^{1,6}_{4,7,8}\}$ has a $\boldsymbol{K}_{4,5} - \boldsymbol{M}_4$ subgraph.

- $\boldsymbol{J}_2 = \boldsymbol{\mathcal{B}}_7 + \{1,5\} \angle\, 5\, \{^{1,7}_{4,6,8}\}$ has a $\boldsymbol{\mathcal{D}}_3$ minor via contracting $\{5,8\}$.

- $\boldsymbol{J}_2 = \boldsymbol{\mathcal{B}}_7 + \{1,5\} \angle\, 5\, \{^{1,4}_{6,7,8}\}$ has a $\boldsymbol{\mathcal{D}}_3$ minor via contracting $\{3,8\}$.

Finally, it remains to analyze the next step for $\boldsymbol{J}_2 = \boldsymbol{\mathcal{B}}''_7$ if $k \geq 3$. There are only two possibilities that have not yet been resolved in the proof.

- $\boldsymbol{J}_3 = \boldsymbol{\mathcal{B}}''_7 + \{1,3\}$ has a $\boldsymbol{K}_7 - \boldsymbol{C}_4$ minor via contracting $\{3,4\}$.

- $\boldsymbol{J}_3 = \boldsymbol{\mathcal{B}}''_7 \angle\, 5\, \{^{1,8}_{4,6,7}\}$ has a $\boldsymbol{\mathcal{C}}_3$ subgraph.

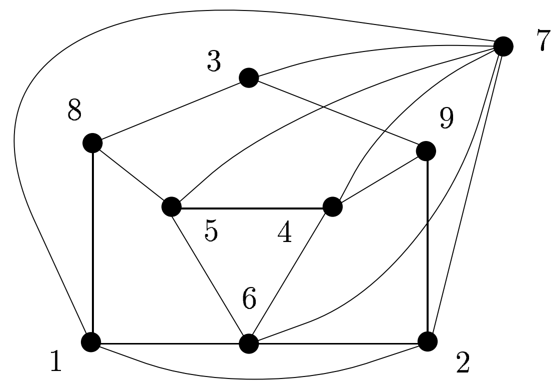


Figure 6.8: Numbering of vertices in the graph $\mathcal{C}_3$.

Since all internally 4-connected graphs arising in our case analysis are either one of $\mathcal{B}_7, \mathcal{B}'_7, \mathcal{B}''_7$, or have a minor isomorphic to one of Λ_0 (which contradicts the hypothesis), or have a $\mathcal{C}_3$ minor, the statement is proved. ∎

Suppose that the vertices of $\mathcal{C}_3$ are numbered as in Fig. 6.8. Let $\mathcal{C}'_3$ be the graph obtained from $\mathcal{C}_3$ by adding edges $\{7,8\}$, $\{3,5\}$, let $\mathcal{C}''_3$ be the graph obtained from $\mathcal{C}'_3$ by adding an edge $\{7,9\}$, and let $\mathcal{C}^\circ_3$, $\mathcal{C}^\bullet_3$ be the graphs obtained from $\mathcal{C}''_3$ by adding edges $\{2,4\}$, $\{2,3\}$, respectively. (See Appendix B for all pictures.)

Lemma 6.4.3. *Let $\boldsymbol{H}$ be an internally 4-connected graph having no minor isomorphic to a graph from Λ_0. If $\boldsymbol{H}$ contains a $\mathcal{C}_3$ minor, then either $\boldsymbol{H}$ is isomorphic to one of $\mathcal{C}_3, \mathcal{C}'_3, \mathcal{C}''_3, \mathcal{C}^\bullet_3, \mathcal{C}^\circ_3$, or it contains a $\mathcal{D}_2$ minor.*

Proof. Suppose that $\boldsymbol{J}_0 = \mathcal{C}_3, \boldsymbol{J}_1, \ldots, \boldsymbol{J}_n, \boldsymbol{J}_{n+1}, \ldots, \boldsymbol{J}_k \simeq \boldsymbol{H}$ is a sequence of simple graphs as described by Theorem 6.2.1, such that $\boldsymbol{J}_i = \boldsymbol{J}_{i-1} + e_i$ is constructed by an edge addition for $i = 1, \ldots, n$, and that $\boldsymbol{J}_{n+1}$ (if $n < k$) is obtained from $\boldsymbol{J}_n$ by an operation other than an edge addition. The case analysis gets more complicated for this graph than it was for $\boldsymbol{\mathcal{B}}_7$, and so it is divided into the following sequence of claims. Notice that it is enough to show that some graph $\boldsymbol{J}_i$ of the above sequence has a minor isomorphic to a member of Λ_0 or to $\boldsymbol{\mathcal{D}}_2$, in order to finish this proof.

Claim 1. For $i = 1, \dots, n$, e_i is one of $\{1,3\}, \{1,5\}, \{3,5\}, \{2,3\}, \{2,4\}, \{3,4\}$, $\{7,8\}, \{7,9\}$.

Proof. Since $\boldsymbol{J}_i \supseteq \boldsymbol{J}_0 + e_i$, the claim follows from Lemma 6.3.4(a), provided it is shown that $e_i \neq \{8,9\}$ for all $i \in \{1, \dots, n\}$. Suppose, for a contradiction, that $e_i = \{8,9\}$ for some $i \leq n$. There are two possibilities – either e_i is violating, or it is not violating in $\boldsymbol{J}_i$. For the second possibility, there must be another edge $e_j, j < i$ incident with the vertex 3 in $\boldsymbol{J}_i$, and hence $e_j = \{1,3\}$ up to symmetry, which is a case covered below.

For the first possibility, none of the edges $\{7,8\}, \{7,9\}$ is present in $\boldsymbol{J}_i$ since it would be another violating edge. Moreover, the operation $\boldsymbol{J}_i \to \boldsymbol{J}_{i+1}$ must "repair" the violating edge $e_i = \{8,9\}$. Using Lemma 6.3.4(a), the vertices $7, 8, 9$ have the same neighbors in $\boldsymbol{J}_i$ as in $\boldsymbol{J}_0 + e_i$. Therefore the operation next to adding e_i (whichever one of the four considered operations it is) can be applied to $\boldsymbol{J}_0 + e_i$, also, so that the resulting graph $\boldsymbol{J}'$ is a subgraph of $\boldsymbol{J}_{i+1}$. If $\boldsymbol{J}'$ results by a triangle explosion of the vertex 3 in $\boldsymbol{J}_0 + e_i$, then $\boldsymbol{J}'$ has a subgraph obtained by splitting vertex 7 in $\boldsymbol{\mathcal{C}}_3$, and hence this case is covered by Lemma 6.3.4(b). Other operations that can possibly "repair" the violating pair $(3, \{8,9\})$ follow.

- $\boldsymbol{J}' = \boldsymbol{\mathcal{C}}_3 + \{8,9\} + \{1,3\}$ has a $\boldsymbol{K}_7 - \boldsymbol{C}_4$ minor via contracting $\{2,9\}, \{8,5\}$.

- $\boldsymbol{J}' = \boldsymbol{\mathcal{C}}_3 + \{8,9\} \dashv 6\, \{8,9\}$ has a $\boldsymbol{K}_{4,5} - \boldsymbol{M}_4$ minor via contracting $\{1,2\}$.

– $\boldsymbol{J}' = \boldsymbol{\mathcal{C}}_3 + \{8,9\} \angle 8 \{{}^{1,9}_{3,5}\}$ has a $\boldsymbol{\mathcal{D}}_3$ minor via contracting $\{1,8\}$ and $\{4,9\}$.

– $\boldsymbol{J}' = \boldsymbol{\mathcal{C}}_3 + \{8,9\} \angle 8 \{{}^{3,9}_{1,5}\}$ still has the violating edge $\{8,9\}$. □

Claim 2. At most one of the edges $\{1,3\}, \{1,5\}, \{3,5\}$, and at most one of the edges $\{2,3\}, \{2,4\}, \{3,4\}$ occurs in the sequence $e_1, \dots, e_n$.

Proof. Suppose that there are two $e_i, e_j \in \{\{1,3\}, \{1,5\}, \{3,5\}\}$, $1 \le i < j \le n$. Since the graph $\boldsymbol{J}_n$ cannot have two violating edges, there is an edge e_l, $1 \le l \le n$ incident with vertex 8, and hence $e_l = \{7,8\}$ by Claim 1. The graph $\boldsymbol{J}_n$ has a subgraph $\boldsymbol{J}' = \boldsymbol{J}_0 + e_i + e_j + e_l$, so up to symmetry:

– $\boldsymbol{J}' = \boldsymbol{\mathcal{C}}_3 + \{1,3\} + \{1,5\} + \{7,8\}$ has a $\boldsymbol{K}_7 - \boldsymbol{C}_4$ minor via contracting $\{3,9\}$ and $\{4,9\}$.

The situation is symmetric for $\{2,3\}, \{2,4\}, \{3,4\}$. □

Claim 3. $\boldsymbol{J}_{n+1}$ is not obtained by a triangle explosion.

Proof. From definition it follows that there must be a violating pair (w, e_i) in $\boldsymbol{J}_n$ in order to apply a triangle explosion of w. (Clearly, no edge of $\boldsymbol{J}_0$ is violating in $\boldsymbol{J}_n$.) Claim 1 then implies that e_i is one of $\{1,3\}, \{1,5\}, \{7,8\}$, up to symmetry. However, if $e_i = \{1,3\}$, then vertex 5 has degree four by Claims 1,2, and so the triangle explosion of vertex 8 cannot be applied. The same happens if $e_i = \{1,5\}$, since at most one of the edges $\{3,2\}, \{3,4\}$ may be in $\boldsymbol{J}_n$ by Claim 2. Finally, if $e_i = \{7,8\}$, then the triangle explosion of 3 is not allowed since at most one other edge may be incident with vertex 9 by Claim 1. □

Claim 4. $\boldsymbol{J}_{n+1}$ is not obtained by a triad addition.

Proof. Similarly as in Claim 3, the edge e_i of $\boldsymbol{J}_n$ for which a triad addition is applied must be one of $\{1,3\}, \{1,5\}, \{7,8\}$, up to symmetry. However, if a triad addition can be applied in $\boldsymbol{J}_n$, it can also be applied in the subgraph $\boldsymbol{J}_0 + e_i \subseteq \boldsymbol{J}_n$. Notice that $\boldsymbol{J}_0 + \{7,8\}$ admits no triad addition, and the two other possibilities lead to the following discussion.

- $\boldsymbol{J}' = \boldsymbol{\mathcal{C}}_3 + \{1,3\} \dashv 4\,\{1,3\}$ has a $\boldsymbol{\mathcal{D}}_3$ minor via contracting $\{4,5\}, \{4,9\}$.
- $\boldsymbol{J}' = \boldsymbol{\mathcal{C}}_3 + \{1,5\} \dashv 9\,\{1,5\}$ has a $\boldsymbol{\mathcal{D}}_3$ minor via contracting $\{2,6\}, \{3,9\}$. □

Claim 5. $\boldsymbol{J}_{n+1}$ is not obtained by a vertex splitting.

Proof. Let a vertex u of $\boldsymbol{J}_n$ be split into two vertices u_1, u_2 of $\boldsymbol{J}_{n+1}$. If both of u_1, u_2 have at least two neighbors among the vertices adjacent to u in $\boldsymbol{J}_0$, then a corresponding splitting operation can be applied to $\boldsymbol{J}_0$, producing a graph $\boldsymbol{J}' \subseteq \boldsymbol{J}_{n+1}$. So $\boldsymbol{J}'$ has a Λ_0 minor or a $\boldsymbol{\mathcal{D}}_2$ subgraph by Lemma 6.3.4(b).

If u_1 has exactly one neighbor among the vertices adjacent to u in $\boldsymbol{J}_0$, then some edge e_i, $1 \le i \le n$ is incident with u_1; so a corresponding splitting operation can be applied to $\boldsymbol{J}_0 + e_i$, producing a graph $\boldsymbol{J}' \subseteq \boldsymbol{J}_{n+1}$. Using Claim 1 and symmetry, e_i is assumed to be one of $\{1,3\}, \{1,5\}$ or $\{7,8\}$, and hence u is one of $1,3,5,7,8$. So the graph $\boldsymbol{J}' \,{\scriptstyle Y\Delta}\, \{9\}$ is isomorphic to a graph $\boldsymbol{J}''$ obtained by the corresponding splitting of u in $\boldsymbol{\mathcal{B}}_7'$ or in $\boldsymbol{\mathcal{B}}_7 + \{1,5\}$. Such splittings have been already considered in the proof of Lemma 6.4.2; if $\boldsymbol{J}''$ has a Λ_0 minor, then so does $\boldsymbol{J}'$ by Lemma 6.3.3, a contradiction. Thus it remains to check those splittings of u in $\boldsymbol{J}_0 + e_i$ that do not correspond to splittings in $\boldsymbol{\mathcal{B}}_7'$ or $\boldsymbol{\mathcal{B}}_7 + \{1,5\}$ producing Λ_0 minors. First, those cases in which e_i is not violating in $\boldsymbol{J}'$ are finished.

- $\boldsymbol{J}' = \boldsymbol{\mathcal{C}}_3 + \{7,8\} \angle\, 8\,\{^{1,7}_{5,3}\}$ and $\boldsymbol{\mathcal{C}}_3 + \{7,8\} \angle\, 7\,\{^{1,8}_{2,3,4,5,6}\}$ have $\boldsymbol{\mathcal{D}}_2$ subgraphs.
- $\boldsymbol{J}' = \boldsymbol{\mathcal{C}}_3 + \{1,3\} \angle\, 3\,\{^{1,9}_{7,8}\}$ has a $\boldsymbol{\mathcal{D}}_3$ minor via contracting $\{4,5\}, \{4,9\}$.

The splittings of u such that e_i is violating in $\boldsymbol{J}'$, i.e. there is a violating pair (w, e_i) in $\boldsymbol{J}'$, are considered now. If w is a cubic vertex in $\boldsymbol{J}_n$, then it is cubic in $\boldsymbol{J}_{n+1}$ as well, and hence (w, e_i) is a violating pair in $\boldsymbol{J}_{n+1}$, which is not allowed by Theorem 6.2.1. So the degree of w in $\boldsymbol{J}_n$ is at least 4, and hence e_i is not violating in $\boldsymbol{J}_{n+1} \supset \boldsymbol{J}'$. In order to apply the splitting of u, also no edge not incident with u may be violating in $\boldsymbol{J}_n$. Thus the following edges must be present in $\boldsymbol{J}_n$ in order to meet these requirements: If e_i is one of $\{1,3\}, \{1,5\}$, then there is some $e_j = \{7,8\}$, $1 \leq j \leq n$. In particular, for $e_i = \{1,5\}$ and $e_j = \{7,8\}$, $\boldsymbol{J}_n$ must have two more edges $e_{j'} = \{3,4\}, e_{j''} = \{7,9\}$, $1 \leq j', j'' \leq n$, up to symmetry. Similarly, if $e_i = \{7,8\}$, then there is either $e_j = \{1,3\}$, $1 \leq j \leq n$, or $e_j = \{3,4\}$ and $e_{j'} = \{7,9\}$, $1 \leq j, j' \leq n$. The complete case analysis follows.

- $\boldsymbol{J}' = \boldsymbol{\mathcal{C}}_3 + \{1,3\} + \{7,8\} \angle\, 3\,\{^{1,8}_{7,9}\}$ and $\boldsymbol{J}' = \boldsymbol{\mathcal{C}}_3 + \{1,3\} + \{7,8\} \angle\, 1\,\{^{3,8}_{2,6,7}\}$ have $\boldsymbol{\mathcal{D}}_2$ subgraphs.

- $\boldsymbol{J}' = \boldsymbol{\mathcal{C}}_3 + \{1,5\} + \{7,8\} + \{3,4\} + \{7,9\} \angle\, 1\,\{^{5,8}_{2,6,7}\}$ has a $\boldsymbol{\mathcal{D}}_2$ subgraph.

- $\boldsymbol{J}' = \boldsymbol{\mathcal{C}}_3 + \{7,8\} + \{1,3\} \angle\, 8\,\{^{1,5}_{3,7}\}$ and $\boldsymbol{J}' = \boldsymbol{\mathcal{C}}_3 + \{7,8\} + \{1,3\} \angle\, 7\,\{^{3,8}_{1,2,4,5,6}\}$ have $\boldsymbol{\mathcal{D}}_3$ minors via contracting $\{3,9\}, \{4,9\}$.

- $\boldsymbol{J}' = \boldsymbol{\mathcal{C}}_3 + \{7,8\} + \{3,4\} + \{7,9\} \angle\, 8\,\{^{1,5}_{3,7}\}$ and $\boldsymbol{J}' = \boldsymbol{\mathcal{C}}_3 + \{7,8\} + \{3,4\} + \{7,9\} \angle\, 7\,\{^{3,8}_{1,2,4,5,6,9}\}$ have $\boldsymbol{\mathcal{D}}_2$ subgraphs.

Finally, if u_1 has no neighbor among the vertices adjacent to u in $\boldsymbol{J}_0$, then there are (at least) two edges e_i, e_j, $1 \leq i < j \leq n$ incident with u_1. The graph $\boldsymbol{J}_n$ has a subgraph $\boldsymbol{J}'$ obtained by the corresponding splitting in $\boldsymbol{J}_0 + e_i + e_j$. Using Claims 1,2 and symmetry, the following three cases are checked.

- $\boldsymbol{J}' = \boldsymbol{\mathcal{C}}_3 + \{1,3\} + \{3,2\} \angle\, 3\,\{^{1,2}_{7,8,9}\}$ has a $\boldsymbol{K}_{4,5} - \boldsymbol{M}_4$ minor via contracting the edge $\{5,8\}$.

– $\boldsymbol{J}' = \boldsymbol{\mathcal{C}}_3 + \{1,3\} + \{3,4\} \angle 3\,\{^{1,4}_{7,8,9}\}$ has a $\boldsymbol{\mathcal{D}}_3$ minor via contracting $\{4,5\}$ and $\{2,9\}$.

– $\boldsymbol{J}' = \boldsymbol{\mathcal{C}}_3 + \{7,8\} + \{7,9\} \angle 7\,\{^{8,9}_{1,2,3,4,5,6}\}$ has a $\boldsymbol{K}_{3,5}$ minor via contracting $\{1,8\}$ and $\{4,9\}$. □

Claims 1–5 can be summarized as follows.

Claim 6. In the sequence of graphs $\boldsymbol{J}_0 = \boldsymbol{\mathcal{C}}_3, \boldsymbol{J}_1, \ldots, \boldsymbol{J}_k \simeq \boldsymbol{H}$ defined above, $n = k$ holds. The graph $\boldsymbol{H}$ is obtained from $\boldsymbol{\mathcal{C}}_3$ by adding at most one of the edges $\{1,3\}, \{1,5\}, \{3,5\}$, at most one of the edges $\{2,3\}, \{2,4\}, \{3,4\}$, and an arbitrary choice of the edges $\{7,8\}, \{7,9\}$. □

Notice that by Claim 6, $\boldsymbol{H}$ is obtained by adding at most four edges. Moreover, if one of $\{1,3\}, \{1,5\}, \{3,5\}$ is added, then $\{7,8\}$ must be added to keep $\boldsymbol{H}$ internally 4-connected. Similarly, if $\{7,8\}$ is added, then one of $\{1,3\}, \{5,3\}, \{2,3\}, \{4,3\}$ must be added, too. So the possibilities are as follows, up to symmetry.

– $\boldsymbol{\mathcal{C}}_3 + \{3,5\} + \{7,8\}$ is the graph $\boldsymbol{\mathcal{C}}_3'$.

– $\boldsymbol{\mathcal{C}}_3 + \{3,5\} + \{7,8\} + \{7,9\}$ is the graph $\boldsymbol{\mathcal{C}}_3''$.

– $\boldsymbol{\mathcal{C}}_3 + \{3,5\} + \{3,2\} + \{7,8\} + \{7,9\}$ is the graph $\boldsymbol{\mathcal{C}}_3^{\bullet}$.

– $\boldsymbol{\mathcal{C}}_3 + \{3,5\} + \{2,4\} + \{7,8\} + \{7,9\}$ is the graph $\boldsymbol{\mathcal{C}}_3^{\circ}$.

– $\boldsymbol{\mathcal{C}}_3 + \{3,5\} + \{3,4\} + \{7,8\} + \{7,9\}$ is isomorphic to the graph $\boldsymbol{\mathcal{C}}_3^{\circ}$. ∎

Suppose that the vertices of $\boldsymbol{\mathcal{D}}_2$ are numbered as in Fig. 6.9. Let $\boldsymbol{\mathcal{D}}_2'$ be the graph obtained from $\boldsymbol{\mathcal{D}}_2$ by adding edges $\{7,8\}$, $\{1,3\}$, let $\boldsymbol{\mathcal{D}}_2''$ be obtained from

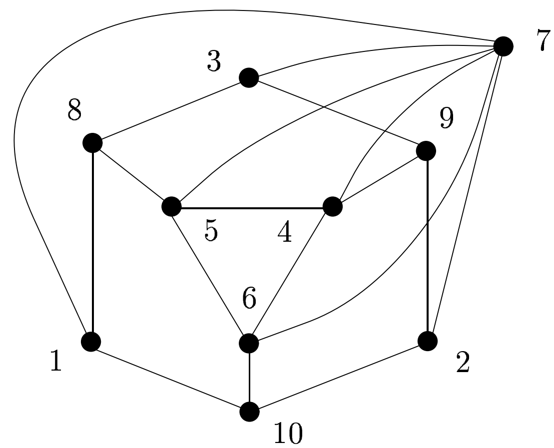


Figure 6.9: Numbering of vertices in the graph $\boldsymbol{\mathcal{D}}_2$.

$\boldsymbol{\mathcal{D}}_2'$ by adding edges $\{7,9\}$, $\{2,3\}$, and let $\boldsymbol{\mathcal{D}}_2'''$ be obtained from $\boldsymbol{\mathcal{D}}_2''$ by adding an edge $\{7,10\}$. Let $\boldsymbol{\mathcal{D}}_2^{\bullet}$, $\boldsymbol{\mathcal{D}}_2^{\circ}$ be the graphs obtained from $\boldsymbol{\mathcal{D}}_2'''$ by adding edges $\{1,2\}$, $\{1,6\}$, respectively. Let $\boldsymbol{\mathcal{D}}_2^{\star}$ be the graph obtained from $\boldsymbol{\mathcal{D}}_2$ by adding edges $\{1,5\},\{3,4\},\{2,6\},\{7,8\},\{7,9\},\{7,10\}$. (See Appendix B for all pictures.)

Lemma 6.4.4. *Let $\boldsymbol{H}$ be an internally 4-connected graph having no minor isomorphic to a graph from Λ_0. If $\boldsymbol{H}$ contains a $\boldsymbol{\mathcal{D}}_2$ minor, then $\boldsymbol{H}$ is isomorphic to one of $\boldsymbol{\mathcal{D}}_2, \boldsymbol{\mathcal{D}}_2', \boldsymbol{\mathcal{D}}_2'', \boldsymbol{\mathcal{D}}_2'''$, $\boldsymbol{\mathcal{D}}_2^{\bullet}, \boldsymbol{\mathcal{D}}_2^{\circ}, \boldsymbol{\mathcal{D}}_2^{\star}$.*

Proof. Suppose that $\boldsymbol{J}_0 = \boldsymbol{\mathcal{D}}_2, \boldsymbol{J}_1, \ldots, \boldsymbol{J}_n, \boldsymbol{J}_{n+1}, \ldots, \boldsymbol{J}_k \simeq \boldsymbol{H}$ is a sequence of simple graphs as described by Theorem 6.2.1, such that $\boldsymbol{J}_i = \boldsymbol{J}_{i-1} + e_i$ is constructed by an edge addition for $i = 1, \ldots, n$, and that $\boldsymbol{J}_{n+1}$ (if $n < k$) is obtained from $\boldsymbol{J}_n$ by an operation other than an edge addition. This proof follows the same arguments as the proof of Lemma 6.4.3.

Claim 1. For $i = 1, \dots, n$, e_i is one of $\{1,3\}, \{1,5\}, \{3,5\}, \{2,3\}, \{2,4\}, \{3,4\}$, $\{1,2\}, \{1,6\}, \{2,6\}, \{7,8\}, \{7,9\}, \{7,10\}$.

Proof. Since $\boldsymbol{J}_i \supseteq \boldsymbol{J}_0 + e_i$, the claim follows from Lemma 6.3.4(a), unless some e_i, $i \in \{1, \dots, n\}$ is one of $\{8,9\}, \{9,10\}, \{8,10\}$. (In particular, none of $e_1, \dots, e_n$ equals $\{3,10\}$.) So assume, by symmetry, that $e_i = \{8,9\}$ for some $i \in \{1, \dots, n\}$, and that none of $e_1, \dots, e_{i-1}$ is $\{8,10\}$ or $\{9,10\}$.

If $e_i = \{8,9\}$ is violating in $\boldsymbol{J}_i$, then it follows that the vertices $8, 9$ have the same neighbors in $\boldsymbol{J}_i$ as in $\boldsymbol{J}_0 + e_i$. Hence if the next operation $\boldsymbol{J}_i \to \boldsymbol{J}_{i+1}$ (which "repairs" e_i) is not a triangle explosion, then it can be applied to $\boldsymbol{J}_0 + e_i$, also, so that the resulting graph $\boldsymbol{J}'$ is a subgraph of $\boldsymbol{J}_{i+1}$. If e_i is not violating in $\boldsymbol{J}_i$, then some edge $e_j, j < i$ is incident with the vertex 3, so $\boldsymbol{J}' = \boldsymbol{J}_0 + e_j + e_i$ is considered. In both situations, if 10 is a cubic vertex in $\boldsymbol{J}'$, then since the graph $\boldsymbol{J}' \,\mathrm{Y}\!\triangle\, \{10\}$ has been shown to have a Λ_0 minor in Claim 6.4.3.1, $\boldsymbol{J}'$ has a Λ_0 minor by Lemma 6.3.3. Otherwise, the following case remains to analyze.

- $\boldsymbol{J}' = \boldsymbol{\mathcal{D}}_2 + \{8,9\} \dashv 10\,\{8,9\}$ has a $\boldsymbol{K}_{4,5} - \boldsymbol{M}_4$ minor via contracting $\{4,5\}$ and $\{4,6\}$.

Finally, consider the case that $e_i = \{8,9\}$ is violating in $\boldsymbol{J}_i$, and $\boldsymbol{J}_{i+1} = \boldsymbol{J}_i \lhd 3\, \{^{N_1}_{N_2}\}$ results by a triangle explosion. If none of $e_1, \dots, e_{i-1}$ is $\{7,10\}$, or if $|N_1 - \{10\}| \geq 2$ and $|N_2 - \{10\}| \geq 2$, then $\boldsymbol{J}_{i+1}$ has a subgraph $\boldsymbol{J}'$ obtained by splitting 7 in $\boldsymbol{\mathcal{D}}_2$. Thus the claim follows from Lemma 6.3.4(b). Otherwise, let $N_1 = \{u, 10\}$, $u \in \{1,2,4,5,6\}$. In such a case $\boldsymbol{J}_{i+1}$ has a $\boldsymbol{\mathcal{D}}_2 + e_i + \{3, u\}$ minor, and hence it has a Λ_0 minor by the previous discussion. $\square$

Claim 2. At most one edge from each of the triples $\{1,3\},\{1,5\},\{3,5\}$; $\{2,3\},\{2,4\},\{3,4\}$; and $\{1,2\},\{1,6\},\{2,6\}$ occurs in the sequence $e_1,\ldots,e_n$.

Proof. The claim easily follows from Claim 6.4.3.2 using Lemma 6.3.3. $\square$

Claim 3. $\boldsymbol{J}_{n+1}$ is not obtained by a triangle explosion, nor by a triad addition.

Proof. Let e_i be the violating edge in $\boldsymbol{J}_n$ for which a triangle explosion or a triad addition is applied. (Clearly, no edge of $\boldsymbol{J}_0$ is violating in $\boldsymbol{J}_n$.) By Claim 1, e_i is one of $\{1,3\},\{1,5\},\{7,8\}$, up to symmetry. It follows in the same way as in Claim 6.4.3.3 that no triangle explosion is allowed for these edges.

If $\boldsymbol{J}_{n+1}$ is obtained by a triad addition, then the same triad addition may be applied to $\boldsymbol{J}_0 + e_i$, producing a graph $\boldsymbol{J}' \subseteq \boldsymbol{J}_{n+1}$. If the corresponding operation $\boldsymbol{\mathcal{C}}_3 + e_i \to \boldsymbol{J}' \,{\scriptstyle Y\Delta}\, \{10\}$ is a valid triad addition for $\boldsymbol{\mathcal{C}}_3 + e_i$, then $\boldsymbol{J}'$ has a Λ_0 minor by Claim 6.4.3.4 and Lemma 6.3.3. Otherwise, the case analysis follows.

- $\boldsymbol{J}' = \boldsymbol{\mathcal{D}}_2 + \{1,3\} \dashv 2\,\{1,3\}$ has a $\boldsymbol{K}_{4,5}-\boldsymbol{M}_4$ minor via contracting $\{4,5\}$ and $\{4,6\}$.
- $\boldsymbol{J}' = \boldsymbol{\mathcal{D}}_2 + \{1,5\} \dashv 2\,\{1,5\}$ has a $\boldsymbol{K}_{4,4}-e$ minor via contracting $\{3,8\},\{3,9\}$ and $\{6,10\}$.
- $\boldsymbol{J}' = \boldsymbol{\mathcal{D}}_2 + \{7,8\} \dashv 10\,\{7,8\}$ has a $\boldsymbol{K}_{3,5}$ minor via contracting $\{2,9\},\{2,10\}$ and $\{5,8\}$. $\square$

Claim 4. $\boldsymbol{J}_{n+1}$ is not obtained by a vertex splitting, unless $\boldsymbol{J}_n \simeq \boldsymbol{\mathcal{D}}_2^\star$.

Proof. Let a vertex u of $\boldsymbol{J}_n$ be split into two vertices u_1,u_2 of $\boldsymbol{J}_{n+1}$. If both of u_1,u_2 have at least two neighbors among the vertices adjacent to u in

$\boldsymbol{J}_0$, then a corresponding splitting operation can be applied to $\boldsymbol{J}_0$, producing a graph $\boldsymbol{J}' \subseteq \boldsymbol{J}_{n+1}$. So $\boldsymbol{J}'$ has a Λ_0 minor by Lemma 6.3.4(b).

If u_1 has exactly one neighbor among the vertices adjacent to u in $\boldsymbol{J}_0$, then some edge e_i, $1 \leq i \leq n$ is incident with u_1; so a corresponding splitting operation can be applied to $\boldsymbol{J}_0 + e_i$, producing a graph $\boldsymbol{J}' \subseteq \boldsymbol{J}_{n+1}$. Using Claim 1 and symmetry, we may assume that e_i is one of $\{1,3\}, \{1,5\}$ or $\{7,8\}$, and hence u is one of $1,3,5,7,8$. Thus the graph $\boldsymbol{J}' \,\mathrm{Y}\!\triangle\, \{10\}$ results by splitting a vertex in $\boldsymbol{\mathcal{C}}_3 + e_i$. If e_i is not violating in $\boldsymbol{J}'$, then it is not violating in $\boldsymbol{J}' \,\mathrm{Y}\!\triangle\, \{10\}$. Hence the analysis in Claim 6.4.3.5 implies that $\boldsymbol{J}' \,\mathrm{Y}\!\triangle\, \{10\}$ has a Λ_0 minor, unless one of the next cases happen.

- $\boldsymbol{J}' = \boldsymbol{\mathcal{D}}_2 + \{7,8\} \angle 8 \{^{1,3}_{5,7}\}$ and $\boldsymbol{\mathcal{D}}_2 + \{7,8\} \angle 7 \{^{5,8}_{1,2,3,4,6}\}$ have $\boldsymbol{\mathcal{E}}_2$ subgraphs.

Therefore assume that $\boldsymbol{J}'$ is produced by such a splitting of u in $\boldsymbol{J}_0 + e_i$ that e_i is violating in $\boldsymbol{J}'$. Similarly as in Claim 6.4.3.5, it can be deduced from Theorem 6.2.1 that e_i is not violating in $\boldsymbol{J}_n$, and also no edge not incident with u is violating in $\boldsymbol{J}_n$. This fact is used to state the following cases of our analysis.

(1) Let $\{1,3\}, \{7,8\} \in E(\boldsymbol{J}_n)$, and let e_i be one of $\{1,3\}, \{7,8\}$. Observe that all relevant splittings of u in $\boldsymbol{J}_0 + \{1,3\} + \{7,8\}$ (i.e. those for which e_i is violating in $\boldsymbol{J}'$) reduce to valid splittings in $\boldsymbol{\mathcal{D}}_2 + \{1,3\} + \{7,8\} - \{4,6\} - \{5,7\} \simeq \boldsymbol{\mathcal{D}}_2$. (The isomorphism is given by the permutation $(1,6)(3,4)(5,8)$.) Hence the claim follows from the previous arguments.

(2) Let $\{1,5\}, \{7,8\}, \{2,3\}, \{7,9\} \in E(\boldsymbol{J}_n)$, and let e_i be one of $\{1,5\}, \{7,8\}$. Then $\boldsymbol{J}^\circ = \boldsymbol{\mathcal{D}}_2 + \{1,5\} + \{7,8\} + \{2,3\} + \{7,9\} \subseteq \boldsymbol{J}_n$, and $\boldsymbol{J}^\circ - \{2,9\} - \{3,7\} - \{4,7\} - \{5,6\} \simeq \boldsymbol{\mathcal{D}}_2$ (via the isomorphism $(1,6,2)(3,8,5,4,9)$). It can be checked that all relevant splittings of u in $\boldsymbol{J}_n$ reduce to the discussion above, except for one case when the splitting is not valid without the edge $\{3,7\}$:

– $\boldsymbol{J}' = \boldsymbol{J}^\circ \angle 7 \{^{3,8}_{1,2,4,5,6,9}\}$ has an $\boldsymbol{\mathcal{E}}_2$ subgraph.

(3) Let $\{1,5\},\{7,8\},\{3,4\},\{7,9\}\{1,2\},\{7,10\} \in E(\boldsymbol{J}_n)$, and let e_i be one of $\{1,5\},\{7,8\}$. Then $\boldsymbol{J}^\circ = \boldsymbol{\mathcal{D}}_2 + \{1,5\} + \{7,8\} + \{3,4\} + \{7,9\} + \{1,2\} + \{7,10\} \subseteq \boldsymbol{J}_n$, and $\boldsymbol{J}^\circ - \{1,10\} - \{2,7\} - \{3,7\} - \{4,5\} - \{4,9\} - \{6,7\} \simeq \boldsymbol{\mathcal{D}}_2$ (via the isomorphism $(1,4)(2,9,3,8,5,6,10)$). Again, all relevant splittings of u in $\boldsymbol{J}_n$ reduce to the discussion above, except for:

– $\boldsymbol{J}' = \boldsymbol{J}^\circ \angle 7 \{^{3,8}_{1,2,4,5,6,9,10}\}$ has an $\boldsymbol{\mathcal{E}}_2$ subgraph.

(4) Thus, using Claims 1,2 and symmetry, the remaining possibilities are either $\boldsymbol{J}_n = \boldsymbol{\mathcal{D}}_2 + \{1,5\} + \{7,8\} + \{3,4\} + \{7,9\} + \{2,6\} + \{7,10\} = \boldsymbol{\mathcal{D}}_2^\star$ (which is postponed until the next claim), or $\boldsymbol{J}_n = \boldsymbol{J}^\circ = \boldsymbol{\mathcal{D}}_2 + \{7,8\} + \{2,3\} + \{7,9\} + \{1,2\} + \{7,10\} \simeq \boldsymbol{\mathcal{D}}_2''' \simeq \boldsymbol{\mathcal{D}}_2 + \{7,8\} + \{3,4\} + \{7,9\} + \{1,2\} + \{7,10\}$. So the following relevant splittings in $\boldsymbol{J}^\circ$ have to be checked.

– $\boldsymbol{J}' = \boldsymbol{J}^\circ \angle 8 \{^{1,5}_{3,7}\}$ and $\boldsymbol{J}' = \boldsymbol{J}^\circ \angle 7 \{^{3,8}_{1,2,4,5,6,9,10}\}$ have $\boldsymbol{\mathcal{E}}_2$ subgraphs.

Finally, if u_1 has no neighbor among the vertices adjacent to u in $\boldsymbol{J}_0$, then there are (at least) two edges e_i, e_j, $1 \leq i < j \leq n$ incident with u_1. By symmetry, it may be supposed that e_i, e_j are not incident with 10, and hence a corresponding splitting in $\boldsymbol{J}_0 + e_i + e_j \, \mathrm{Y}\!\Delta \, \{10\}$ has been covered in the case analysis in Claim 6.4.3.5. Thus the claim follows by Lemma 6.3.3. □

Claim 5. If $\boldsymbol{J}_n = \boldsymbol{\mathcal{D}}_2^\star$, then $\boldsymbol{J}_{n+1}$ is not obtained by a vertex splitting.

Proof. Notice that the vertices 1 and 8 are symmetric in $\boldsymbol{\mathcal{D}}_2^\star$. The case analysis in Claim 4 clearly covers all splittings in $\boldsymbol{J}_n$, except for the following possibilities.

– $\boldsymbol{J}_{n+1} = \boldsymbol{\mathcal{D}}_2^\star \angle 8 \{^{1,7}_{3,5}\}$ and $\boldsymbol{J}_{n+1} = \boldsymbol{\mathcal{D}}_2^\star \angle 7 \{^{1,8}_{2,3,4,5,6,9,10}\}$ have $\boldsymbol{\mathcal{D}}_3$ minor via contracting $\{1,10\},\{2,10\},\{2,9\}$.

– $\boldsymbol{J}_{n+1} = \boldsymbol{\mathcal{D}}_2^\star \angle 8 \{^{3,7}_{1,5}\}$ and $\boldsymbol{J}_{n+1} = \boldsymbol{\mathcal{D}}_2^\star \angle 1 \{^{5,8}_{7,10}\}$ have two violating edges.

– $\boldsymbol{J}_{n+1} = \boldsymbol{\mathcal{D}}_2^\star \angle 5 \{^{1,8}_{4,6,7}\}$ has the violating edge $\{1, 8\}$.

– $\boldsymbol{J}_{n+1} = \boldsymbol{\mathcal{D}}_2^\star \angle 7 \{^{3,8}_{1,2,4,5,6,9,10}\}$ has the violating edge $\{3, 8\}$.

Suppose now that $\boldsymbol{J}_{n+1} = \boldsymbol{\mathcal{D}}_2^\star \angle 5 \{^{1,8}_{4,6,7}\}$. Since this graph is not internally 4-connected, there must be at least one more graph $\boldsymbol{J}_{n+2}$ in the sequence, so that the edge $\{1, 8\}$ is not violating in $\boldsymbol{J}_{n+2}$. Let 11 denote the new vertex in $\boldsymbol{J}_{n+1}$ such that $(11, \{1, 8\})$ is a violating pair. A triangle explosion of 11 is not allowed in $\boldsymbol{J}_{n+1}$. Possible vertex splittings of 1 or 8 are $\boldsymbol{J}_{n+1} \angle 8 \{^{1,7}_{3,11}\}$ or $\boldsymbol{J}_{n+1} \angle 8 \{^{1,3}_{7,11}\}$ (since $1, 8$ are symmetric), but both of them reduce to the previous cases of splittings in $\boldsymbol{\mathcal{D}}_2^\star$ via contracting $\{5, 11\}$. If $\boldsymbol{J}_{n+2} = \boldsymbol{J}_{n+1} + \{u, 11\}$ is obtained by adding an edge incident with 11, then the edge $\{5, 11\}$ for $u = 2, 3, 9, 10$, or $\{8, 11\}$ for $u = 4, 6$, may be contracted to obtain a graph that has a Λ_0 minor by the discussion in Claims 1,2, except for:

– $\boldsymbol{J}_{n+2} = \boldsymbol{J}_{n+1} + \{7, 11\}$ has an $\boldsymbol{\mathcal{E}}_2$ subgraph.

Similarly, both graphs obtained by possible triad additions $\boldsymbol{J}_{n+1} \dashv 6 \{1, 8\}$, $\boldsymbol{J}_{n+1} \dashv 2 \{1, 8\}$ have minors $\boldsymbol{\mathcal{D}}_2 + \{6, 8\}$, $\boldsymbol{\mathcal{D}}_2 + \{2, 8\}$ respectively, and hence also Λ_0 minors by Claim 1.

Suppose further that $\boldsymbol{J}_{n+1} = \boldsymbol{\mathcal{D}}_2^\star \angle 7 \{^{3,8}_{1,2,4,5,6,9,10}\}$. Let 11 denote the new vertex in $\boldsymbol{J}_{n+1}$ such that $(11, \{3, 8\})$ is a violating pair. Again, possible vertex splittings of 3 or 8 in $\boldsymbol{J}_{n+1}$ reduce to the above cases of splittings in $\boldsymbol{\mathcal{D}}_2^\star$ via contracting $\{7, 11\}$. If $\boldsymbol{J}_{n+2} = \boldsymbol{J}_{n+1} + \{u, 11\}$, then for $u = 1, 2, 5, 6, 10$, contracting the edge $\{3, 11\}$ produces a graph that has a Λ_0 minor by the discussion in Claims 1,2. The remaining cases are:

– $\boldsymbol{J}_{n+2} = \boldsymbol{J}_{n+1} + \{4, 11\}$ has a $\boldsymbol{\mathcal{D}}_3$ minor via contracting $\{1, 5\}, \{1, 8\}, \{2, 9\}$.

- $\boldsymbol{J}_{n+2} = \boldsymbol{J}_{n+1} + \{9,11\}$ has a $\boldsymbol{\mathcal{D}}_3$ minor via contracting $\{1,3\}, \{1,8\}, \{1,10\}$.

The graphs obtained by possible triad additions $\boldsymbol{J}_{n+1} \dashv 6\,\{3,8\}$, $\boldsymbol{J}_{n+1} \dashv 10\,\{3,8\}$ have minors $\boldsymbol{\mathcal{D}}_2 + \{3,6\}$, $\boldsymbol{\mathcal{D}}_2 + \{3,10\}$ respectively, and hence also Λ_0 minors by Claim 1. If $\boldsymbol{J}_{n+2}$ is obtained by a triangle explosion of the vertex 7 in $\boldsymbol{J}_{n+1}$, then it contains a minor obtained by a splitting of 7 in $\boldsymbol{\mathcal{D}}_2^\star$. Thus $\boldsymbol{J}_{n+2}$ has a Λ_0 minor by the above analysis and symmetry, except for the following case.

- $\boldsymbol{J}_{n+2} = \boldsymbol{J}_{n+1} \lhd 7\,\{^{2,9}_{1,4,5,6,10,11}\}$ has a $\boldsymbol{\mathcal{D}}_3$ minor via contracting $\{1,8\}, \{1,10\}$, $\{2,10\}$, and $\{3,11\}$. □

Claims 1–5 can be summarized as follows.

Claim 6. In the sequence of graphs $\boldsymbol{J}_0 = \boldsymbol{\mathcal{D}}_2, \boldsymbol{J}_1, \ldots, \boldsymbol{J}_k \simeq \boldsymbol{H}$ defined above, $n = k$ holds. The graph $\boldsymbol{H}$ is obtained from $\boldsymbol{\mathcal{D}}_2$ by adding at most one of the edges $\{1,3\}, \{1,5\}, \{3,5\}$, at most one of the edges $\{2,3\}, \{2,4\}, \{3,4\}$, at most one of the edges $\{1,2\}, \{1,6\}, \{2,6\}$, and an arbitrary choice of the edges $\{7,8\}, \{7,9\}, \{7,10\}$. □

Notice that by Claim 6, $\boldsymbol{H}$ is obtained from $\boldsymbol{\mathcal{D}}_2$ by adding at most six edges. Moreover, if one of $\{1,3\}, \{1,5\}, \{3,5\}$ is added, then $\{7,8\}$ must be added to keep $\boldsymbol{H}$ internally 4-connected. Similarly, if $\{7,8\}$ is added, then either the single edge $\{1,3\}$, or two other edges – one incident with 1 and one incident with 3, must be added to $\boldsymbol{\mathcal{D}}_2$, too. Therefore, using symmetry, the possibilities for the graph $\boldsymbol{H}$ are as follows, ordered by the number of edges.

- $\boldsymbol{\mathcal{D}}_2 + \{1,3\} + \{7,8\}$ is the graph $\boldsymbol{\mathcal{D}}_2'$.
- $\boldsymbol{\mathcal{D}}_2' + \{2,3\} + \{7,9\}$ is the graph $\boldsymbol{\mathcal{D}}_2''$.
- $\boldsymbol{\mathcal{D}}_2' + \{2,4\} + \{7,9\}$ is isomorphic to the graph $\boldsymbol{\mathcal{D}}_2''$.

– $\mathcal{D}_2'' + \{7, 10\}$ is the graph $\mathcal{D}_2'''$.

– $\mathcal{D}_2''' + \{1, 2\}$ is the graph $\mathcal{D}_2^\bullet$.

– $\mathcal{D}_2''' + \{1, 6\}$ is the graph $\mathcal{D}_2^\circ$.

– $\mathcal{D}_2 + \{1, 5\} + \{3, 4\} + \{2, 6\} + \{7, 8\} + \{7, 9\} + \{7, 10\}$ is the graph $\mathcal{D}_2^\star$. ∎

Proof of Theorem 6.1.3. Setting $\Pi = \{\boldsymbol{K}_{1,2,2,2},\ \mathcal{B}_7, \mathcal{B}_7', \mathcal{B}_7'',\ \mathcal{C}_3, \mathcal{C}_3', \mathcal{C}_3'',\ \mathcal{C}_3^\circ, \mathcal{C}_3^\bullet,\ \mathcal{D}_2, \mathcal{D}_2', \mathcal{D}_2'', \mathcal{D}_2''',\ \mathcal{D}_2^\circ, \mathcal{D}_2^\bullet, \mathcal{D}_2^\star\}$, the statement of the theorem immediately follows from Corollary 6.1.4, Corollary 6.1.6, and Lemmas 6.4.1, 6.4.2, 6.4.3, 6.4.4. ∎

6.5 Final Remarks

Theorem 6.1.3 describes all possible counterexamples to Conjecture 2.2.1. Furthermore, Lemma 2.1.5 and Proposition 2.1.6 allow us to deduce certain implications among members of Π. Let us write $\boldsymbol{G} \xrightarrow[\text{NC}]{} \boldsymbol{H}$ to mean that "if $\boldsymbol{G}$ has no planar cover, then neither has $\boldsymbol{H}$".

Proposition 6.5.1. *The following relations hold among the members of* Π.

$$\boldsymbol{K}_{1,2,2,2} \xrightarrow[\text{NC}]{} \mathcal{B}_7 \xrightarrow[\text{NC}]{} \mathcal{B}_7' \xrightarrow[\text{NC}]{} \mathcal{B}_7'' \xrightarrow[\text{NC}]{} \mathcal{C}_3 \xrightarrow[\text{NC}]{} \mathcal{C}_3' \xrightarrow[\text{NC}]{} \mathcal{C}_3''$$

$$\begin{array}{ccccccccccccc}
 & \nearrow_{\text{NC}} & \rightarrow\ \mathcal{C}_3^\bullet\ \rightarrow & & & \searrow^{\text{NC}} & & & & & & \nearrow_{\text{NC}} & \mathcal{D}_2^\bullet \\
\mathcal{C}_3'' & \xrightarrow[\text{NC}]{} & \mathcal{C}_3^\circ & \xrightarrow[\text{NC}]{} & \mathcal{D}_2 & \xrightarrow[\text{NC}]{} & \mathcal{D}_2' & \xrightarrow[\text{NC}]{} & \mathcal{D}_2'' & \xrightarrow[\text{NC}]{} & \mathcal{D}_2''' & \xrightarrow[\text{NC}]{} & \mathcal{D}_2^\circ \\
 & & & & \updownarrow{\scriptstyle\text{NC}} & & & & & & & & \\
 & & & & \mathcal{D}_2^\star & & & & & & & &
\end{array}$$

Proof. The relations in the first row of the diagram follow immediately from Lemma 2.1.5, or from the next argument:

$\boldsymbol{\mathcal{B}}_7'' \xrightarrow[\text{NC}]{} \boldsymbol{\mathcal{C}}_3$: Notice that $\boldsymbol{\mathcal{B}}_7'' = \boldsymbol{\mathcal{C}}_3 + \{1,5\} + \{7,8\} \,{\scriptstyle Y\Delta}\, \{9\}$, and that the edges $\{1,5\}$ and $\{7,8\}$ are added between neighbors of cubic vertices. Thus if $\boldsymbol{\mathcal{C}}_3$ had a planar cover, then so would have $\boldsymbol{\mathcal{B}}_7''$ by Proposition 2.1.6.

The relations in the second row of the diagram follow similarly, except for:

$\boldsymbol{\mathcal{D}}_2^\star \xrightarrow[\text{NC}]{} \boldsymbol{\mathcal{D}}_2$: As in the previous case, $\boldsymbol{\mathcal{D}}_2^\star$ can be obtained from $\boldsymbol{\mathcal{D}}_2$ by subdividing the edges $\{1,10\}$, $\{2,9\}$, $\{3,8\}$ with new vertices $11, 12, 13$ respectively, then adding the edges $\{11,7\}, \{12,7\}, \{13,7\}$, and finally $Y\Delta$-transforming the vertices $8, 9, 10$. Thus if $\boldsymbol{\mathcal{D}}_2$ had a planar cover, then so would have $\boldsymbol{\mathcal{D}}_2^\star$ by Proposition 2.1.6. ∎

It is possible that more relations between the graphs from Π can be derived in a similar fashion, but this possibility is not pursued in the paper.

By Lemma 2.1.5 and the Graph Minor Theorem of Robertson and Seymour [34], there exists a finite set Σ of graphs such that a connected graph has no planar cover if and only if it has a minor isomorphic to a member of Σ. Since it is not known if Conjecture 2.2.1 holds, it is not even known whether $\boldsymbol{K}_{1,2,2,2} \in \Sigma$. However, the lemmas developed in this paper imply the following.

Theorem 6.5.2. *The set Σ is the union of Λ_0 and some set $\Pi_0 \subseteq \Pi - \{\boldsymbol{\mathcal{D}}_2^\star\}$ such that no graph in Π_0 is a subgraph of another graph in Π_0.*

Proof. Clearly, no graph in Σ is a subgraph of another member of Σ, and all graphs in Σ are connected and have no embedding in the projective plane. Moreover, all graphs in Λ_0 are minor-minimal with respect to the property of having a planar cover, since all of their proper minors are projective-planar, and hence have double planar covers.

Let $\boldsymbol{G} \in \Sigma$. If $\boldsymbol{G}$ happens to be one of the graphs in Λ_0, the proof is done, otherwise $\boldsymbol{G}$ contains one of $\boldsymbol{K}_{1,2,2,2}, \boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{D}}_2$ as a minor by Theorem 2.2.2. In such a case it follows from Corollary 6.1.6 that $\boldsymbol{G}$ is internally 4-connected. The statement then follows from Lemmas 6.4.1, 6.4.2, 6.4.3, 6.4.4. Moreover, by Proposition 6.5.1, $\boldsymbol{\mathcal{D}}_2^\star$ is not a minor-minimal graph having no planar cover, since it has a planar cover if and only if so does $\boldsymbol{\mathcal{D}}_2 \subset \boldsymbol{\mathcal{D}}_2^\star$. ∎

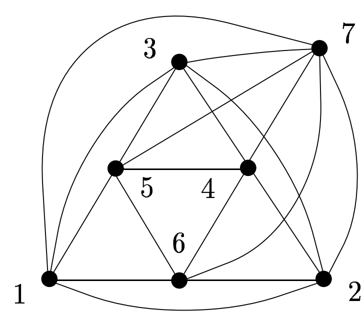
(Notice that Theorem 6.5.2 does *not* seem to be a consequence of Theorem 6.1.3.)

CHAPTER VII

FUTURE RESEARCH

7.1 Towards a Proof for $\boldsymbol{K}_{1,2,2,2}$

Recall that it is enough to prove that the graph $\boldsymbol{K}_{1,2,2,2}$ has no planar cover in order to prove Negami's planar cover conjecture. The aim of this section is to review some of the ideas or methods that R. Thomas and the author considered when they were looking for a solution of this last case.



$\boldsymbol{K}_{1,2,2,2}$

Figure 7.1: Numbering of vertices in the graph $\boldsymbol{K}_{1,2,2,2}$.

Ordinary Discharging

If a plane graph $\boldsymbol{H}$ is a cover of $\boldsymbol{K}_{1,2,2,2}$, then $\boldsymbol{H}$ is simple. Let the vertices in $\boldsymbol{K}_{1,2,2,2}$ be denoted as in Fig. 7.1, and let $\varphi : V(\boldsymbol{H}) \to V(\boldsymbol{K}_{1,2,2,2})$ be the projection. Then every vertex of $\varphi^{-1}(7)$ has degree 6 in $\boldsymbol{H}$, and every vertex of $\varphi^{-1}(1) \cup \ldots \cup \varphi^{-1}(6)$ has degree 5 in $\boldsymbol{H}$. A vertex $v \in \varphi^{-1}(7)$ is adjacent to vertices labeled $1, 2, 3, 4, 5, 6$. A vertex $v \in \varphi^{-1}(1) \cup \ldots \cup \varphi^{-1}(6)$ is adjacent to one vertex labeled 7, and to four vertices labeled $l \in \{1, 2, 3, 4, 5, 6\}$ such that $3 \not| (l - \varphi(v))$. R. Thomas suggested to consider the following partial discharging rules.

Initial charges. Each vertex of degree d in $\boldsymbol{H}$ starts with a charge of $6 - d$, and each face of length k in $\boldsymbol{H}$ starts with a charge of $2(3 - k)$. The edges of $\boldsymbol{H}$ have no charge.

Discharging rules. If f is a face of length k bounded by a cycle $v_1 \ldots v_k$, then one of the following rules apply.

- ($k = 4$): If $\varphi(v_i) = 7$ for some $i \in \{1, 2, 3, 4\}$, then the vertices v_{i-1}, v_{i+1} send each a charge of 1 to the face f. (Indices are considered modulo d.) If there is one pair of vertices v_i, v_{i+2} such that $|\varphi(v_i) - \varphi(v_{i+2})| = 3$, then each of v_i, v_{i+2} sends a charge of 1 to f. If there are two such pairs, then the vertices of an arbitrary one of these pairs send a charge of 1 to f.

- ($k = 5$): If $\varphi(v_i) = 7$ for some $i \in \{1, 2, 3, 4, 5\}$, then all remaining vertices incident with the face f send a charge of 1 to f. Otherwise, a vertex v_i, $i \in \{1, 2, 3, 4, 5\}$ sends a charge of 1 to f if and only if there exists $j \in \{1, 2, 3, 4, 5\}$ such that $|\varphi(v_i) - \varphi(v_j)| = 3$.

- ($k \geq 6$): In this case, each of the vertices $v_1, \ldots, v_k$ sends a charge of 1 to the face f.

It is easy to compute that the total initial charge of $\boldsymbol{H}$ is $6|V(\boldsymbol{H})| - 2|E(\boldsymbol{H})| + 6|F(\boldsymbol{H})| - 4|E(\boldsymbol{H})| = 12 > 0$ by Euler's formula. Thus if it was proved that all charges at the end of the discharging process are nonpositive, the contradiction would show that the graph $\boldsymbol{K}_{1,2,2,2}$ had no planar cover. That clearly cannot be achieved with the above partial discharging rules — all faces of $\boldsymbol{H}$ end up with a nonpositive charge, but some vertices may still have charges of 1, as the example in Fig. 7.2 shows. (Applications of the discharging rules are demonstrated by dotted lines, and the over-charged vertices are dark-shaded.)

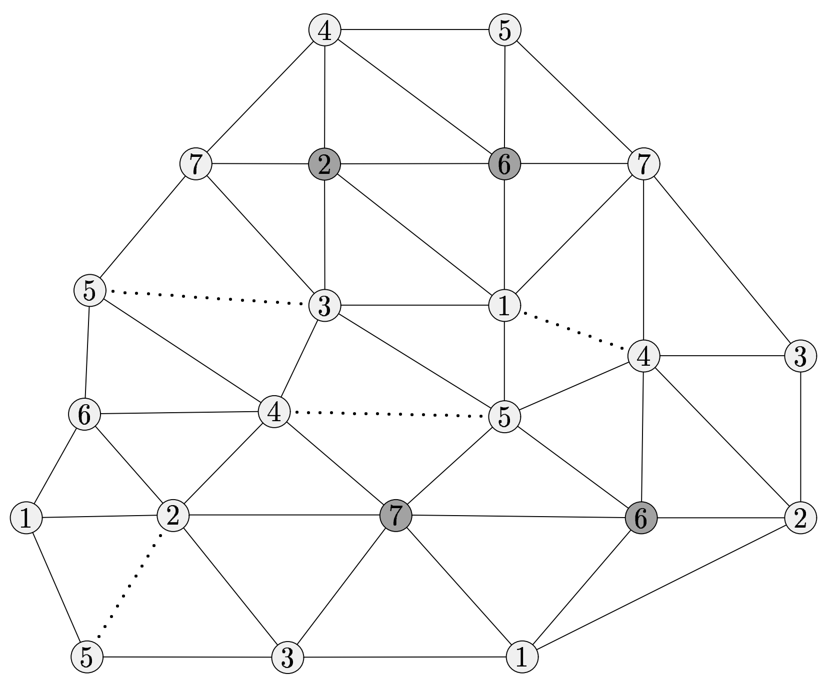


Figure 7.2:

The goal was to design more discharging rules based on properties of the cover of $\boldsymbol{K}_{1,2,2,2}$ in addition to the above presented rules, in order to get the desired contradiction. (Compare with Chapters III,IV.) Unfortunately, no reasonable way to handle "locally dense" configurations similar to the one presented in Fig. 7.2 was found. Attempts were made to extend such configurations towards producing a counterexample to Negami's conjecture, but they also failed.

Star Discharging

Another discharging method was inspired by the proof in Chapter III. Let $\boldsymbol{H}'$ be the graph obtained from $\boldsymbol{H}$ by contracting all edges incident with the vertices of $\varphi^{-1}(7)$, by deleting all loops, and by replacing with a single edge each bunch of edges forming faces of size 2. Let the subgraph of $\boldsymbol{H}$ corresponding to a single edge e of $\boldsymbol{H}'$ be called the *arm* corresponding to e. Let the union of all arms corresponding to edges incident with a vertex x of $\boldsymbol{H}'$ be called the *star* centered at x. The degree of the star is the degree of x in $\boldsymbol{H}'$.

The graph $\boldsymbol{H}'$ is a plane graph without loops or faces of size 2. It can be shown that each vertex of $\boldsymbol{H}'$ has degree at least 4. Let the initial charges be set up similarly as above.

Initial charges. Each vertex of degree d in $\boldsymbol{H}'$ starts with a charge of $6-d$, and each face of length k in $\boldsymbol{H}'$ starts with a charge of $2(3-k)$. The edges of $\boldsymbol{H}'$ have no charge.

Again, the goal was to design discharging rules such that the final charges of all elements in $\boldsymbol{H}'$ are nonpositive.

One problem occurred that the number of all possible stars was enormous. (When rotation symmetries and label permutations were factored out, there were 79 possible stars of degree 4, 7548 stars of degree 5, 388081 stars of degree 6, and 13687595 stars of degree 7.) Nevertheless, efficient computer programs for handling local configurations of stars were developed. However, no usable discharging rule

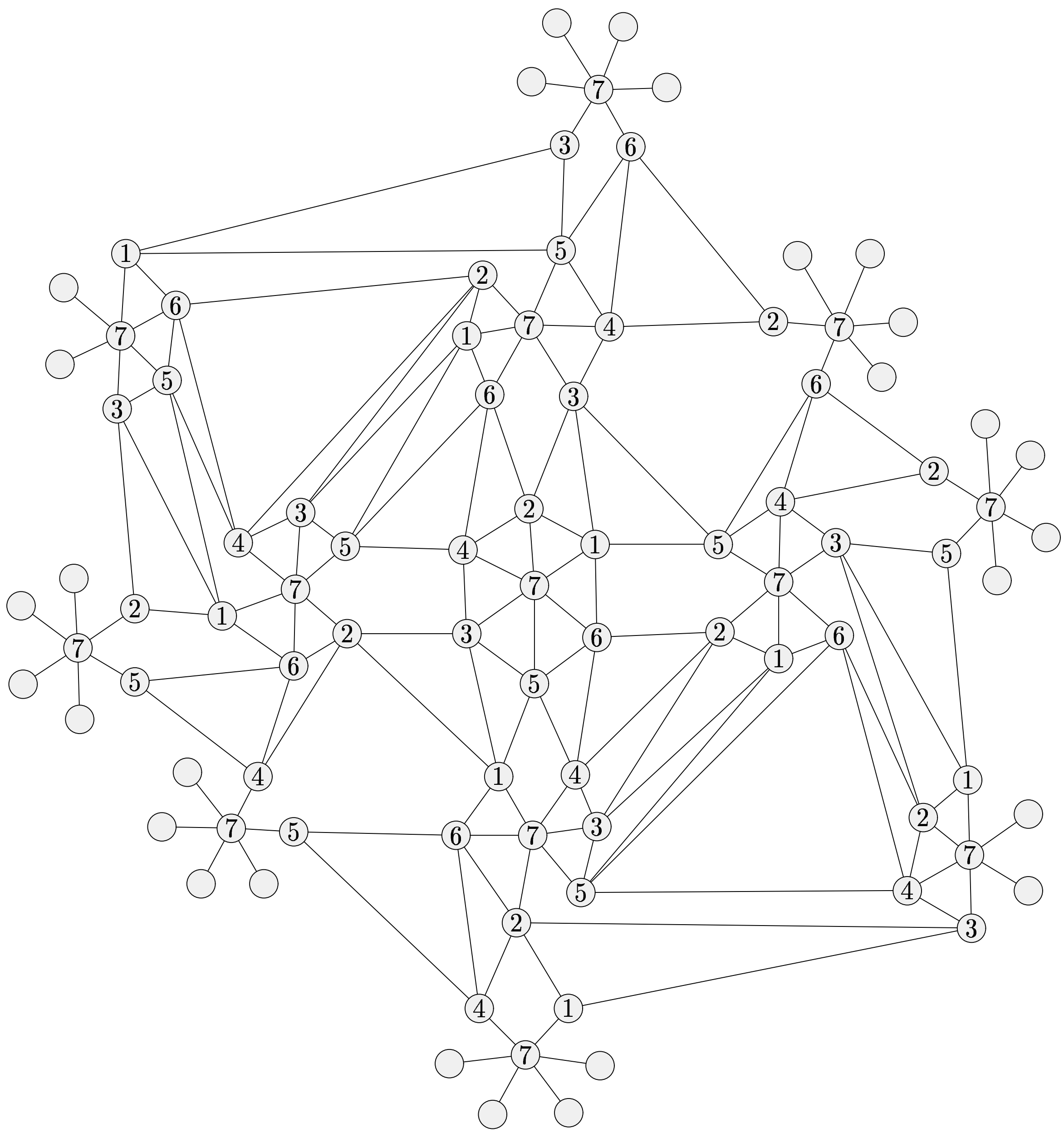


Figure 7.3:

was found for the configuration which is presented in Fig. 7.3. (There is no obvious destination where to send a charge from the central star of degree 4.) Notice also similarities between some parts of of the configuration from Fig. 7.3, and the graph in Fig. 7.2.

Remark. The discharging method used in Chapter IV is so special that it does not seem to be usable at all in the case of $\boldsymbol{K}_{1,2,2,2}$. However, it is possible that it has an extension towards the graph $\boldsymbol{\mathcal{D}}_2$.

Necklace-Like Argument

A different attempt was based on the ideas used in Chapter V. We tried to find a necklace-like argument showing that the graph $\boldsymbol{K}_{1,2,2,2}$ had no planar cover, or at least to find such an argument for the graph $\boldsymbol{\mathcal{C}}_3$ which seems to be very similar to $\boldsymbol{\mathcal{C}}_4$. However, this attempt was also unsuccessful. (The major complication seems to be in the facts that there are no two disjoint cycles longer than 3 in the graphs $\boldsymbol{K}_{1,2,2,2} - \{7\}$ or $\boldsymbol{\mathcal{C}}_3 - \{7\}$, and that the complement of any cycle in $\boldsymbol{K}_{1,2,2,2}$ or $\boldsymbol{\mathcal{C}}_3$ containing the vertex 7 is an outerplanar graph.)

It is possible that a final solution can be obtained using a clever combination of the discharging method and a necklace-like argument, but it is not clear how to combine the two.

7.2 Generalization of Negami's Conjecture

If $\mathcal{S}$ is a surface, and the graph $\boldsymbol{G}$ has a cover $\boldsymbol{H}$ that embeds in $\mathcal{S}$, then $\boldsymbol{G}$ is said to have an *$\mathcal{S}$-cover.* The terms projective cover or Klein cover are used in the obvious sense. The above presented statement of Conjecture 2.2.1 probably has no direct analogue for surfaces of higher genera. However, the aim of this section is to suggest another formulation of Negami's conjecture that has a straightforward generalization to other nonorientable surfaces.

Conjecture 7.2.1. *A connected graph embeds in the projective plane if and only if it has a projective cover.*

Proof of equivalence with Conjecture 2.2.1. It is enough to prove that a graph has a finite planar cover if and only if it has a finite projective cover. Indeed, a planar cover is a projective cover, too. On the other hand, let $\boldsymbol{H}$ be a projective cover of a graph $\boldsymbol{G}$, then $\boldsymbol{H}$ has a double planar cover $\boldsymbol{F}$. One can check that the property of having a cover is a transitive relation, thus $\boldsymbol{F}$ is a planar cover of $\boldsymbol{G}$. ∎

The advantage of the latter formulation is that it speaks about one surface only, and it directly relates the properties of having a cover and of having an embedding in the surface to each other. Surprisingly, it appears that nobody has considered that formulation before. Conjecture 7.2.1 holds for no orientable surface, since projective graphs (hence having planar covers) of arbitrarily high orientable genera [6] exist. However, for nonorientable surfaces it is conjectured:

Conjecture 7.2.2. *A connected graph embeds in the Klein bottle if and only if it has a Klein cover.*

As far as we know, it is possible that Conjecture 7.2.2 holds for all nonorientable surfaces, but at the moment there is no evidence in favor of that. To provide some support for the conjecture, it is shown that three minor-minimal graphs not embeddable in the Klein bottle do not have Klein covers: The complete graph $\boldsymbol{K}_7$, the graph $\boldsymbol{K}_8 - \boldsymbol{M}_4$ (a complete graph on 8 vertices minus a perfect matching),

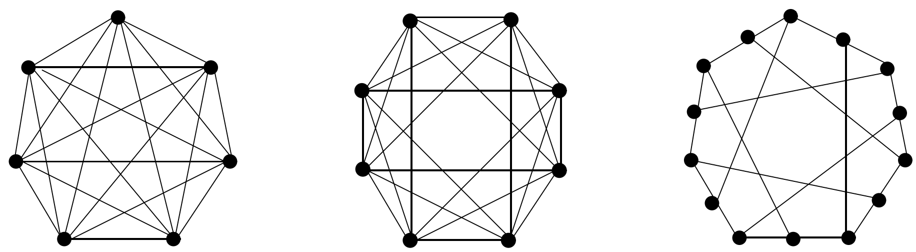


Figure 7.4: The graphs $\boldsymbol{K}_7$, $\boldsymbol{K}_8 - \boldsymbol{M}_4$, and $\boldsymbol{H}_{14}$, respectively.

and the Heawood graph $\boldsymbol{H}_{14}$ (the only cubic graph of girth 6 on 14 vertices; also, the geometrical dual of any toroidal embedding of $\boldsymbol{K}_7$). See Fig. 7.4 for pictures of these graphs.

7.3 Three Supporting Examples for the Klein Bottle

Proposition 7.3.1. *The graphs $\boldsymbol{K}_7$ and $\boldsymbol{H}_{14}$ have no Klein covers.*

Proof. Suppose that $\boldsymbol{G}$ is a cover of $\boldsymbol{K}_7$ embedded in the Klein bottle. Clearly, $\boldsymbol{G}$ is a simple graph. By Euler's formula, $\boldsymbol{G}$ is a 6-regular triangulation of the surface. The cover projection is represented as a labeling of the vertices of $\boldsymbol{G}$ by labels $1, 2, \ldots, 7$, where each label is connected with all the other six labels. In particular, two vertices of the same label are at distance of at least 3.

A *straight-ahead walk* is a walk which leaves each internal vertex through an edge opposite to the edge through which it entered. This is well-defined, because every vertex of $\boldsymbol{G}$ has an even degree. A key observation is that two vertices of the same label cannot be connected by a straight-ahead walk of length three. To prove

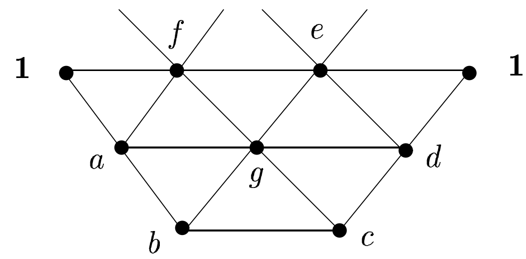


Figure 7.5: A straight-ahead walk between two vertices of the same label.

it, see Fig 7.5—the seven vertices a, b, c, d, e, f, g must have mutually distinct labels, and none of them may have the label 1, a contradiction. (The pictures presented here should be regarded as a lifting of the graph into the universal covering surface. It could happen, for example, that in the Klein bottle embedding of $\boldsymbol{G}$ the two vertices labeled 1 are actually equal.)

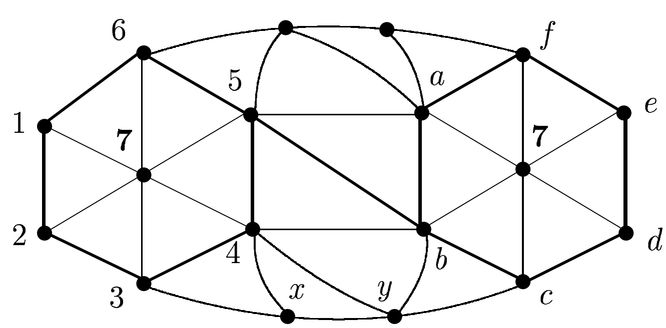


Figure 7.6: A fragment of a Klein cover of $\boldsymbol{K}_7$.

Let us now look at the components of the graph obtained from $\boldsymbol{G}$ by deleting all edges not contained in any closed neighborhood of a vertex of label 7. These components are wheels with the central label 7; and it is further shown that the rim vertices of each wheel are labeled in the same cyclic order, which contradicts the nonorientability of the Klein bottle.

It is assumed, without loss of generality, that one of the wheels W is labeled $1, 2, 3, 4, 5, 6$ in order. Let another wheel labeled a, b, c, d, e, f be connected to W by at least one edge, say $b5$ (see Fig. 7.6). The edge $b5$ is not the only edge between these two wheels, for otherwise the central vertices would be connected by a straight-ahead walk of length three. Thus the wheels are connected by a triangle, say $ab5$. Since the vertex 5 has only one more edge, there must be another edge, say $b4$, between the wheels. Let x, y denote the other two neighbors of the vertex 4, as in the figure.

The label b cannot be $3, 4, 5, 6$ since these labels already occur at distance of at most two from it; similarly a cannot be $4, 5, 6$; and c cannot be $4, 5$, and 6 since c is connected with 6 by a straight-ahead walk of length three. Then one of a, b, c is 2, so the vertex y is at distance of at most two from labels $2, 3, 4, 5, 7$, and y is connected with 1 by a straight-ahead walk of length three. Hence $y = 6$, and consequently $x = 1$, which already implies $b = 2$, $c = 3$, $a = 1$. By symmetry between the two wheels, $4 = d$, $5 = e$, $6 = f$, and the claim follows by induction.

The Heawood graph $\boldsymbol{H}_{14}$ is a bipartite cubic graph of girth 6. If each vertex in one part of the bipartition of $\boldsymbol{H}_{14}$ is replaced with a triangle on its three neighbors (a $Y\Delta$-transformation), then this results in the complete graph $\boldsymbol{K}_7$. The same transformation is applicable to any cover of $\boldsymbol{H}_{14}$; thus a Klein cover of $\boldsymbol{H}_{14}$ could be transformed to a Klein cover of $\boldsymbol{K}_7$, and this was already shown to be impossible. ∎

The proof of the next statement is very similar to the previous one.

Proposition 7.3.2. *The graph* $\boldsymbol{K}_8 - \boldsymbol{M}_4$ *has no Klein cover.*

Proof. Let the vertices of $\boldsymbol{K}_8 - \boldsymbol{M}_4$ be $1, 2, \ldots, 8$, so that the four missing edges are $12, 34, 56, 78$. Suppose that $\boldsymbol{G}$ is a Klein cover of $\boldsymbol{K}_8 - \boldsymbol{M}_4$. By Euler's formula, $\boldsymbol{G}$ is a 6-regular triangulation of the surface. The covering projection is represented as a labeling of the vertices of $\boldsymbol{G}$ by the labels $1, 2, \ldots, 8$ in the natural way.

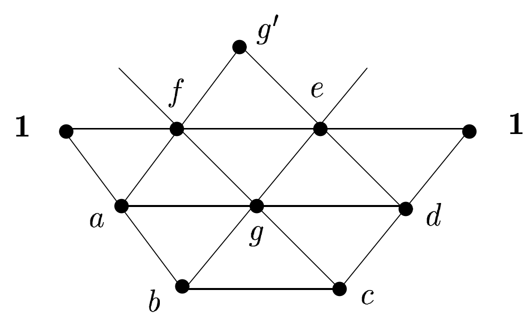


Figure 7.7: A straight-ahead walk between two vertices of the same label.

Again, two vertices of the same label cannot be connected by a straight-ahead walk of length 3. If the situation depicted in Fig. 7.7 happened, then the vertices a, b, c, d, e, f, g would get seven distinct labels other than 1, so g which is connected with all of them, would be labeled 2. So by mirror symmetry, another vertex g' labeled 2 would be at distance of two from g, a contradiction.

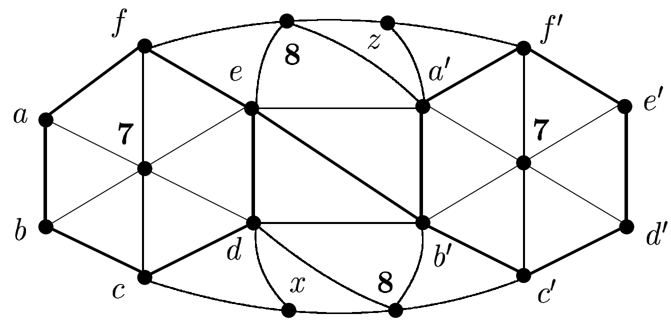


Figure 7.8: A fragment of a Klein cover of $\boldsymbol{K}_8 - \boldsymbol{M}_4$.

The supposed Klein cover $\boldsymbol{G}$ of $\boldsymbol{K}_8 - \boldsymbol{M}_4$ is partitioned into wheels centered at labels 7, and into the remaining vertices of labels 8. It can be argued in the same way as in the proof of Proposition 7.3.1 that the two neighboring wheels are connected by at least three edges, see Fig. 7.8. Then the positions of labels 8 are determined, since each vertex other than 7 must have a neighbor labeled by 8. The labels $1, \ldots, 6$ cannot be specified since they are not mutually equivalent in this case, so we denote them by a, b, c, d, e, f in this cyclic order for the first wheel, and by a', b', c', d', e', f' in this cyclic order for the second wheel, as in Fig. 7.8.

The label b' can only be a or b, and the label a' can only be a or c. (There is a straight-ahead a', b-walk of length three.) If $b' = a$, then $a' = c$, and hence the cover would contain seven edges with labels $a7, a8, ab, af, ad, ae, ac$, which is impossible. Thus $b' = b$, and consequently, the cover contains edges labeled $b7, b8, ba, bc, be, bd$, so bf is one of the missing edges of the graph $\boldsymbol{K}_8 - \boldsymbol{M}_4$. Since 78 is a missing edge, too, and cd, de are present in the cover, the remaining two missing edges are ce, ad. Consequently, $a' = a$ since there is an edge labeled $a'e$, and $c' = c$; and finally, $d = d'$, $e = e'$, $f = f'$ by symmetry. However, that means by induction, that the rim vertices of all wheels are labeled in the same cyclic order, which contradicts the nonorientability of the Klein bottle. ∎

Unfortunately, there is little hope to prove Conjecture 7.2.1 by examining all forbidden minors for the Klein bottle or for higher nonorientable surfaces. Those lists are not known, and even if they are eventually found, they will probably be too numerous to be useful. However, it is worthwhile to mention that the lists are finite by Theorem 1.1.6.

7.4 Planar Emulators

A graph $\boldsymbol{H}$ is an *emulator* of a graph $\boldsymbol{G}$ if there exist a pair of onto mappings (φ, ψ), $\varphi : V(\boldsymbol{H}) \to V(\boldsymbol{G})$, $\psi : E(\boldsymbol{H}) \to E(\boldsymbol{G})$, called an (emulator) *projection*, such that ψ maps the edges incident with each vertex v in $\boldsymbol{H}$ (surjectively) onto the edges incident with $\varphi(v)$ in $\boldsymbol{G}$. In the case of simple graphs, it is enough to specify the vertex projection φ. Informally speaking, the difference between covers and emulators is that the neighborhood of a vertex in an emulator may contain "repeated edges", unlike in a cover where the neighborhoods must be one-to-one.

It is clear that every cover is an emulator, but the converse is not true in general, as an example in Fig. 7.9 shows. Thus the notion of an emulator is a relaxation of that of a cover. However, Fellows conjectured [13]:

Conjecture 7.4.1. (M. Fellows, 1988) *A graph has a planar emulator if and only if it has a planar cover.*

Despite similarity of definitions, planar emulators are far less understood than planar covers. Fellows [13] adapted some of the solved cases of Negami's

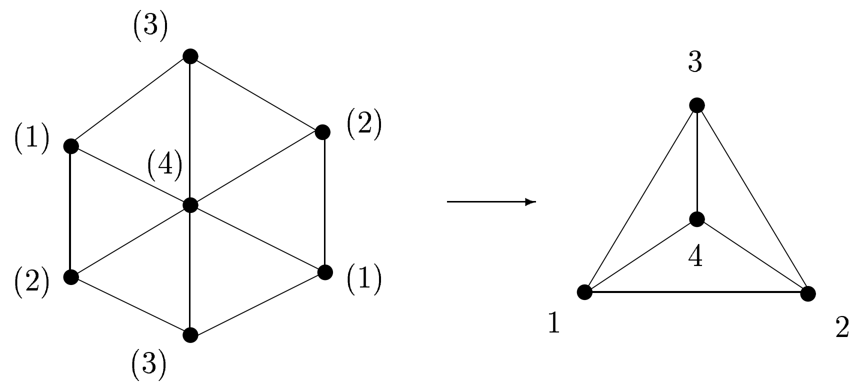


Figure 7.9: An emulator of $\boldsymbol{K}_4$ that is not a cover.

conjecture to planar emulators, but other cases completely failed here. In a fashion similar to that of Chapter II, it is enough to prove that the four graphs $\boldsymbol{K}_7 - \boldsymbol{C}_4, \boldsymbol{K}_{4,5} - \boldsymbol{M}_4, \boldsymbol{K}_{4,4} - e, \boldsymbol{K}_{1,2,2,2}$ have no planar emulators in order to prove Fellows' conjecture (compared to just one case $\boldsymbol{K}_{1,2,2,2}$ needed for planar covers). However, not much effort has been spent on solving Fellows' conjecture so far, and there is still a good chance of finding an elementary direct proof of it.

One might speculate that Conjecture 7.4.1 could hold for other surfaces as well. Unfortunately it does not, as it is now shown.

Proposition 7.4.2. *There exists a connected graph $\boldsymbol{G}$ that has an emulator embeddable in the triple-torus, but $\boldsymbol{G}$ has no finite cover (and hence no embedding) there.*

Proof. Let $\boldsymbol{T}$ be any triangulation of the double-torus, and let $\boldsymbol{P}$ be any projective graph that does not embed in the triple-torus [6]. A counterexample graph $\boldsymbol{G}$ is constructed by connecting some vertex t of $\boldsymbol{T}$ with some vertex p of $\boldsymbol{P}$.

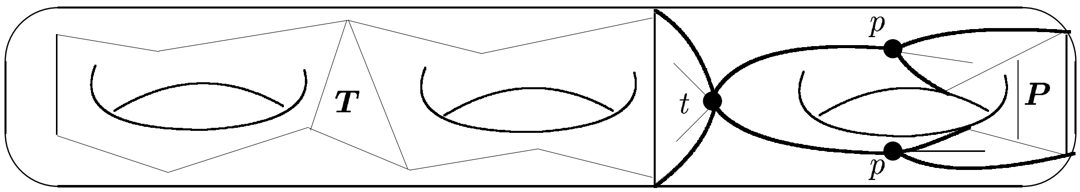


Figure 7.10: An emulator of the graph $\boldsymbol{G}$.

This graph obviously does not embed in the triple-torus. An emulator is obtained by drawing $\boldsymbol{T}$ on two of the handles of the triple-torus, putting a double-cover of $\boldsymbol{P}$ on the third handle, and connecting the vertex t with both of the vertices covering p, as shown in Fig. 7.10.

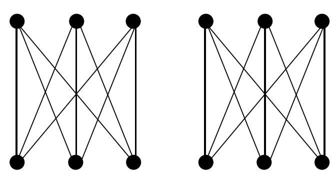
Suppose that there is a cover of the graph $\boldsymbol{G}$ that embeds in the triple-torus. Euler's formula applied to $\boldsymbol{T}$ shows that it may be at most a double-cover, and then the subcover of $\boldsymbol{T}$ triangulates the whole triple-torus. But in such case, it is impossible to connect both of the vertices covering p with the subcover of $\boldsymbol{P}$, a contradiction. ∎

Fellows' planar emulator conjecture is a nice and natural extension of Negami's planar cover conjecture, and it is open as well.

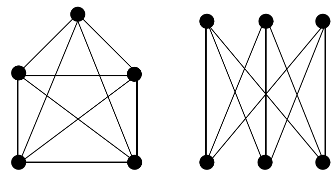
APPENDIX A

MINOR-MINIMAL NONPROJECTIVE GRAPHS

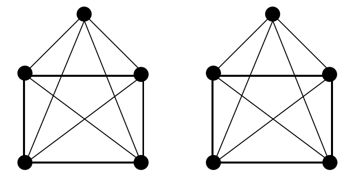
This appendix presents a list of all 35 minor minimal nonprojective graphs, as found in [14, 3]. (See Theorem 1.2.6.) The notation of these graphs mostly follows [14], except when more convenient common notation is available. The first three graphs in the list are disconnected, the remaining 32 of them (starting from $\boldsymbol{K}_{3,3} \cdot \boldsymbol{K}_{3,3}$) are connected, and they form the *family* Λ referred to throughout the thesis text.



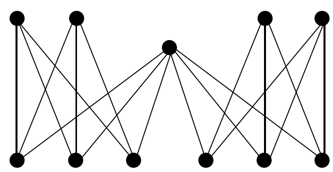
$\boldsymbol{K}_{3,3}+K_{3,3}$



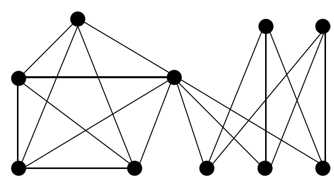
$\boldsymbol{K}_5+\boldsymbol{K}_{3,3}$



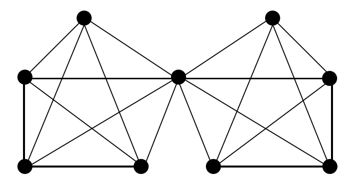
$\boldsymbol{K}_5+\boldsymbol{K}_5$



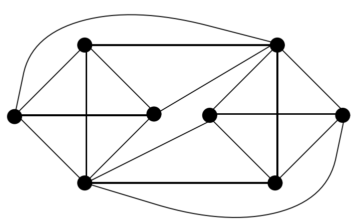
$\boldsymbol{K}_{3,3}\cdot\boldsymbol{K}_{3,3}$



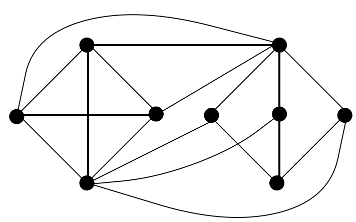
$\boldsymbol{K}_5\cdot\boldsymbol{K}_{3,3}$



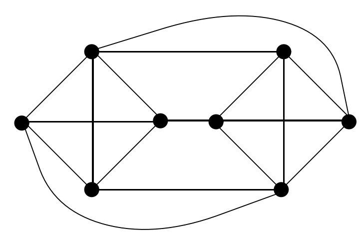
$\boldsymbol{K}_5\cdot\boldsymbol{K}_5$



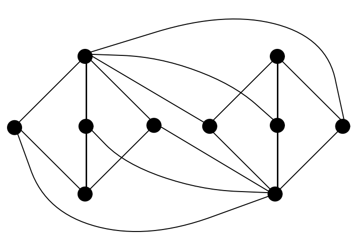
$\boldsymbol{\mathcal{B}}_3$



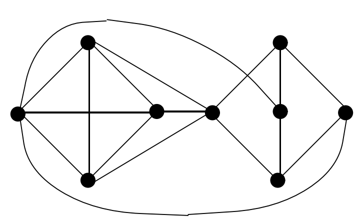
$\boldsymbol{\mathcal{C}}_2$



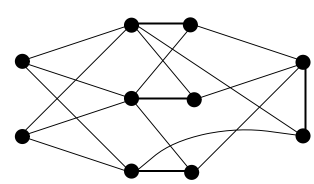
$\boldsymbol{\mathcal{C}}_7$



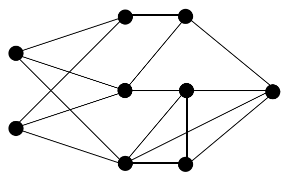
$\boldsymbol{\mathcal{D}}_1$



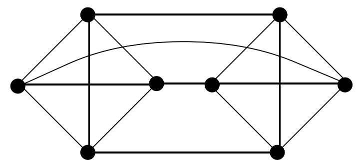
$\boldsymbol{\mathcal{D}}_4$



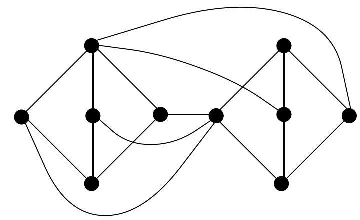
$\boldsymbol{\mathcal{D}}_9$



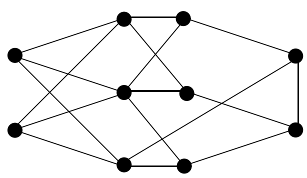
$\mathcal{D}_{12}$



$\mathcal{D}_{17}$



$\mathcal{E}_{6}$



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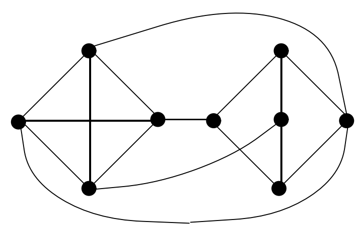

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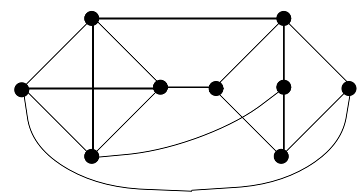

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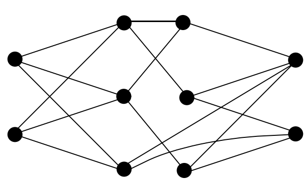

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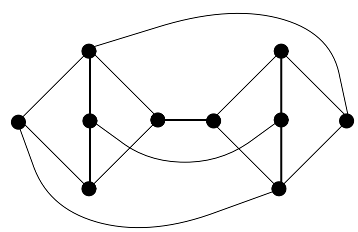
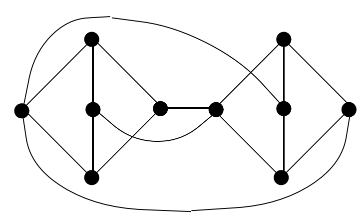
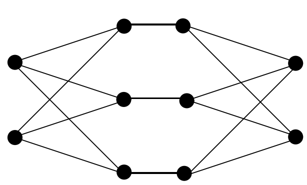


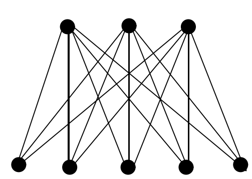
$\mathcal{F}_{4}$



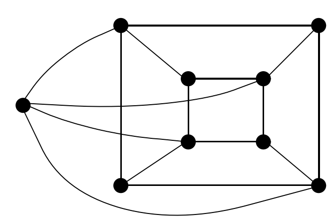
$\mathcal{F}_{6}$



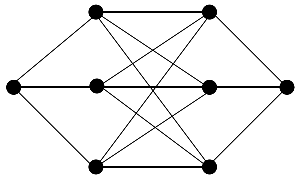
$\mathcal{G}_{1}$



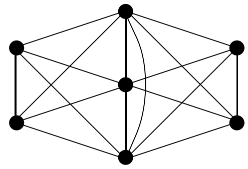
$\boldsymbol{K}_{3,5}$



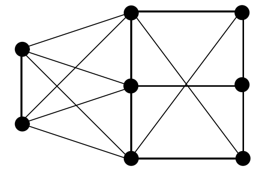
$\boldsymbol{K}_{4,5}-\boldsymbol{M}_{4}$



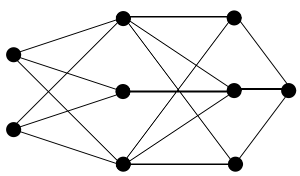
$\boldsymbol{K}_{4,4}-e$



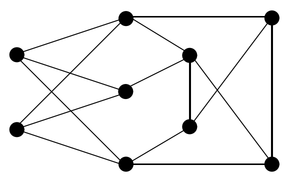
$\boldsymbol{K}_7-\boldsymbol{C}_4$



$\mathcal{D}_3$



$\mathcal{E}_5$



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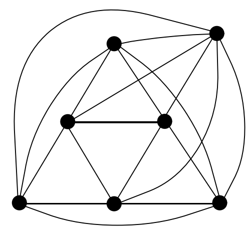
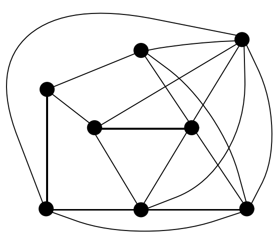
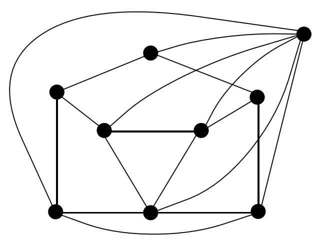


$\boldsymbol{K}_{1,2,2,2}$



$\mathcal{B}_7$



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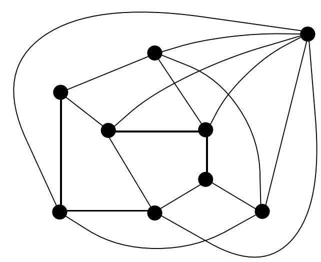

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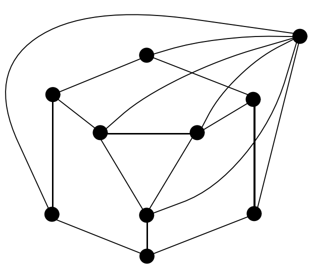
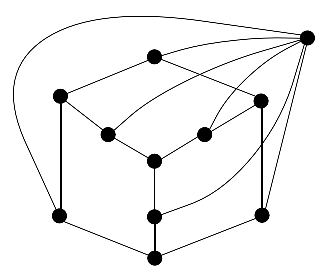


$\mathcal{D}_2$

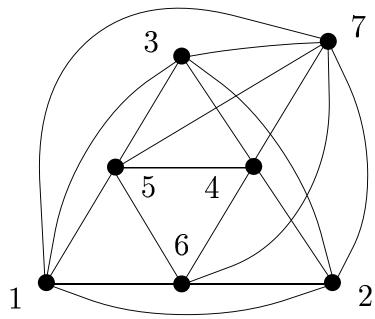


$\mathcal{E}_2$

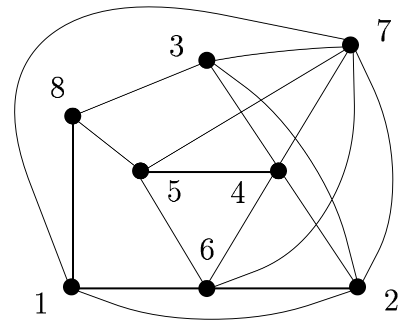
APPENDIX B

POSSIBLE INTERNALLY 4-CONNECTED COUNTEREXAMPLES

This appendix lists all 16 internally 4-connected graphs that have no embedding in the projective plane, but may possibly have a planar cover, cf. Theorem 6.1.3. These graphs form the *family* Π referred to throughout the thesis text.



$K_{1,2,2,2}$



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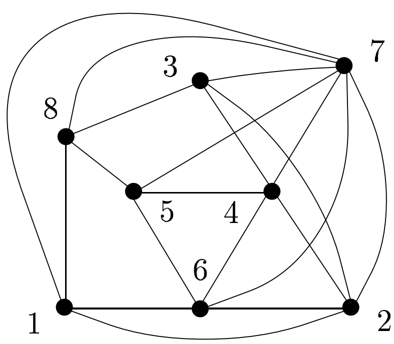
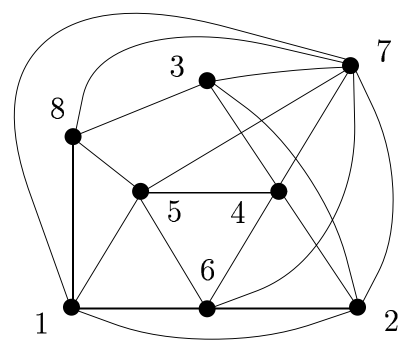
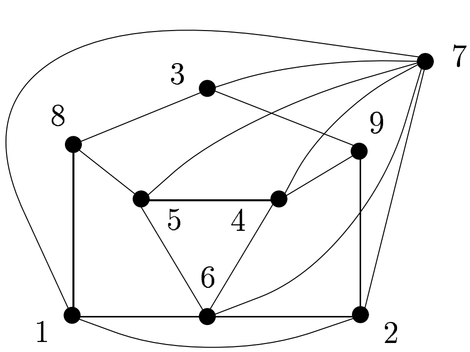


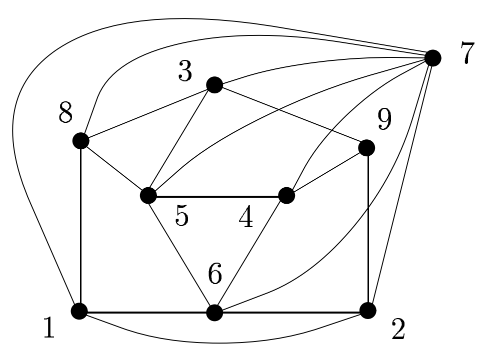
$\mathcal{B}'_7$



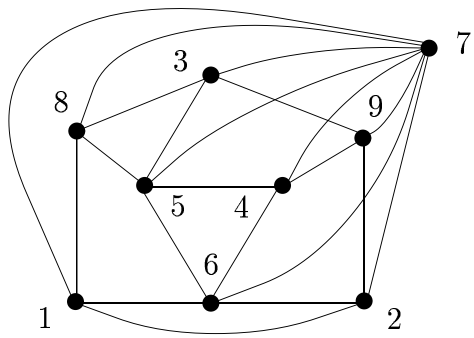
$\mathcal{B}''_7$



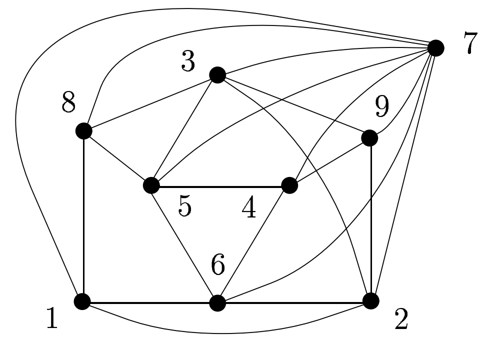
$\mathcal{C}_3$



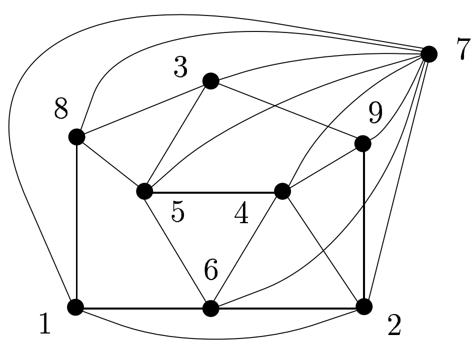
$\mathcal{C}'_3$



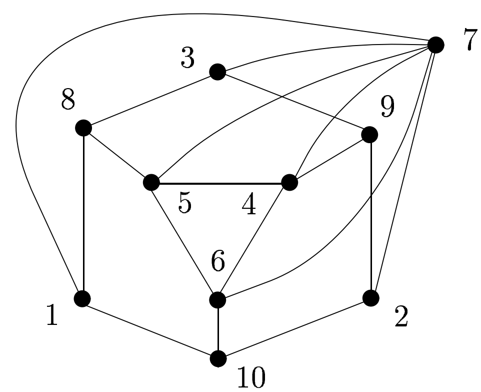
$\mathcal{C}_3''$



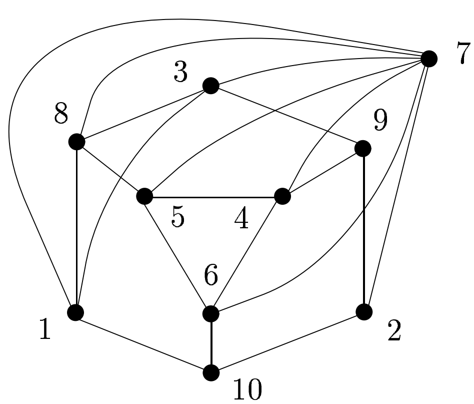
$\mathcal{C}_3^{\bullet}$



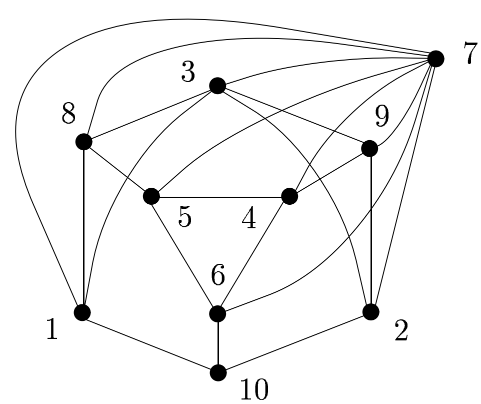
$\mathcal{C}_3^{\circ}$



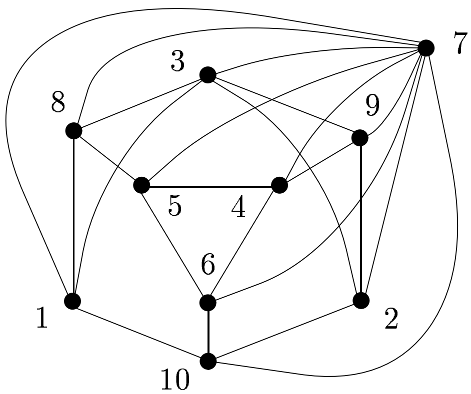
$\mathcal{D}_2$



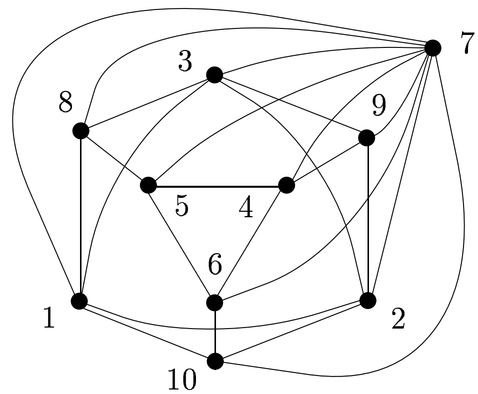
$\mathcal{D}_2'$



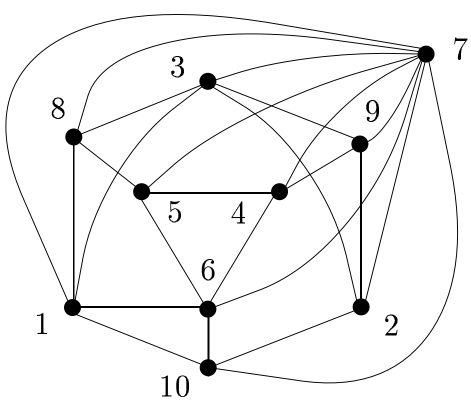
$\mathcal{D}_2''$



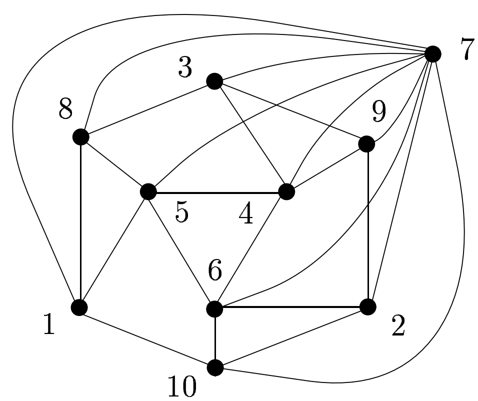
$\mathcal{D}_2'''$



$\mathcal{D}_2^\bullet$



$\mathcal{D}_2^\circ$



$\mathcal{D}_2^\star$

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VITA

Petr Hliněný was born on October 14, 1971 in Ostrava, the Czech Republic, formerly Czechoslovakia. He studied in basic schools in Ostrava, and a special math-oriented class at the high school in Bílovec. He participated three times in the International Mathematical Olympiads (Canberra 1988, Braunschweig 1989, second prize, and Beijing 1990, second prize).

In 1990, he began his studies in Computer Science in the Faculty of Mathematics and Physics at Charles University in Prague, the Czech Republic. He got a Magister degree (an equivalent of Masters) in Computer Science in 1995. A paper "Contact graphs of curves" based on his Master's thesis was awarded the Bolzano prize at Charles University.

After finishing his masters study, he enrolled in a Ph.D. program in Discrete Mathematics in the Department of Applied Mathematics, Faculty of Mathematics and Physics of Charles University, beginning in 1995. His advisor was Professor Jan Kratochvíl. In January 1998, he transferred to the Ph.D. program in Algorithms, Combinatorics, and Optimization at Georgia Institute of Technology in Atlanta, Georgia. He studied in the School of Mathematics, under advisor Professor Robin Thomas.

He has been offered and has accepted a postdoctoral position at the Fields Institute in Toronto, Canada, starting in September 1999.

Planar Covers of Graphs: Negami's Conjecture

Petr Hliněný

127 pages

Directed by Dr. Robin Thomas

Planar covers of graphs, with an extension to covers on other surfaces, were studied. A simple graph $\boldsymbol{H}$ is a cover of a simple graph $\boldsymbol{G}$ if there exists a mapping φ from $V(\boldsymbol{H})$ onto $V(\boldsymbol{G})$ such that for every vertex v of $\boldsymbol{G}$, φ maps the neighbors of v in $\boldsymbol{H}$ bijectively onto the neighbors of $\varphi(v)$ in $\boldsymbol{G}$. In 1986, S. Negami conjectured that a connected graph has a finite planar cover if and only if it embeds in the projective plane.

The "Kuratowski theorem for the projective plane" by D. Archdeacon implies that Negami's conjecture holds as long as none of the 32 connected minor-minimal nonprojective graphs has a planar cover. Results by D. Archdeacon, M. Fellows, and S. Negami from 1987–1988 stated that 25 of these graphs had no planar covers. In this thesis, the conjecture was verified for three other graphs ($\boldsymbol{K}_{4,4}-e$, $\boldsymbol{\mathcal{C}}_4$, and $\boldsymbol{\mathcal{D}}_2$) of the 32. Using those results, it was proved that, up to obvious constructions, there were at most 16 possible counterexamples to Negami's conjecture. (This was joint work with R. Thomas.) A consequence of this work is that in order to prove Negami's conjecture it suffices to prove that $\boldsymbol{K}_{1,2,2,2}$ has no planar cover. However, the conjecture is still open.

A reformulation of Negami's conjecture, which had a straightforward generalization to nonorientable surfaces, was proposed. Some support for the generalized conjecture was given in the case of the Klein bottle.