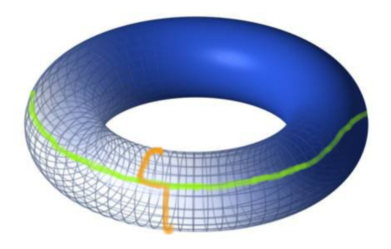
Toroidal Grid Minors, Embedding Stretch, and Crossing Number



Petr Hliněný

Faculty of Informatics, Masaryk University, Czech Rep.

based on joint work with

Sergio Cabello, **Markus Chimani** and **Gelasio Salazar**

Foreword

Purpose of the talk

To outline and promote some tools for topologically-restricted graphs which turned out very useful in our crossing-number-related research...

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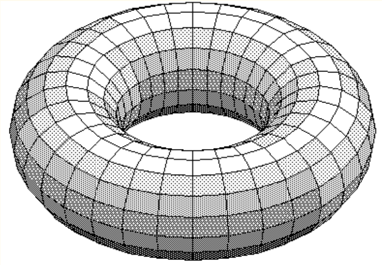
Related papers

- P. Hliněný and G. Salazar. Approximating the crossing number of toroidal graphs. In: ISAAC 2007, LNCS 4835, 148–159.
- M. Chimani and P. Hliněný. Approximating the crossing number of graphs embeddable in any orientable surface. In: SODA 2010, 918–927.
- S. Cabello, M. Chimani and P. Hliněný. Computing the stretch of an embedded graph. SIAM J. Discrete Math. 28 (2014), 1391–1401.
- M. Chimani, P. Hliněný and G. Salazar. Toroidal Grid Minors and Stretch in Embedded Graphs. Submitted (2014), 32 p.

1 Toroidal Grids



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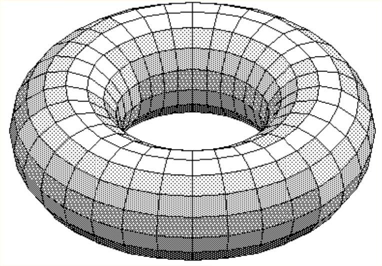

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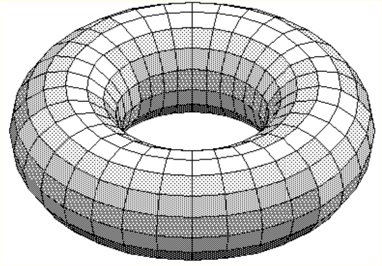

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Theorem. [de Graaf and Schrijver] Let G be a graph embedded in the torus with face-width *fw*$(G) = r \geq 5$. Then G contains the toroidal

$$\lfloor 2r/3 \rfloor \times \lfloor 2r/3 \rfloor \text{ -grid}$$

as a minor (and this is tight).

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$$\mathrm{cr}(G) \geq \frac{1}{12} \mathit{Tex}(G).$$

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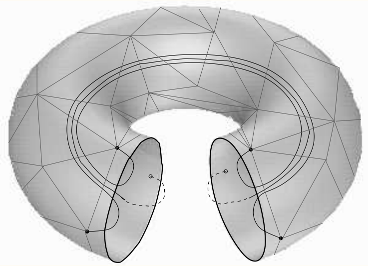
  - Specifically, on the torus;

$$\mathrm{cr}(G) = \mathcal{O}\big(\Delta(G)^2 \cdot \mathit{Tex}(G)\big).$$

On the torus



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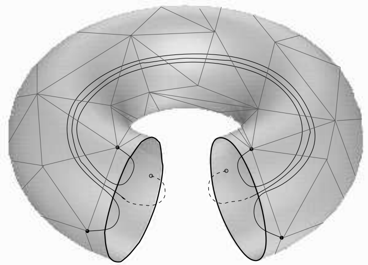

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- Re-route the k cut edges along a shortest dual path – length $\ell \geq k/2$.
- Consequently;

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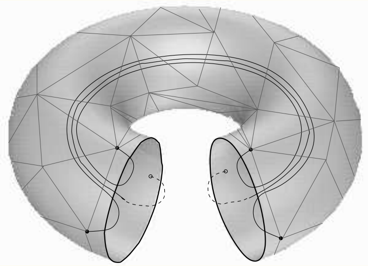

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Lemma. Assume G on the torus contains $p \geq 3$ disjoint cycles of one homotopy class and $q \geq 3$ disjoint cycles of another homotopy, then G has a $C_p \Box C_q$-minor.

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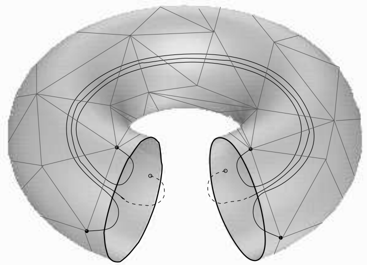

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- Consequently (only on the torus!);

$$\mathit{Tex}(G) \geq \left\lceil \frac{\ell}{\lfloor \Delta(G)/2 \rfloor} \right\rceil \cdot \left\lfloor \frac{2}{3} \left\lceil \frac{k}{\lfloor \Delta(G)/2 \rfloor} \right\rceil \right\rfloor \geq \frac{16}{7\Delta(G)^2} k\ell \geq \frac{32}{21\Delta(G)^2} \mathrm{cr}(G).$$

Beyond the torus

- Considering only orientable surf.
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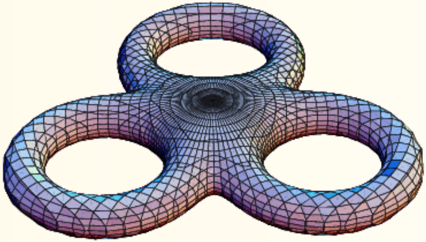

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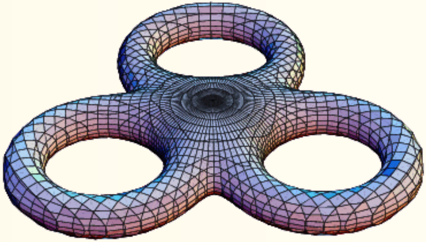

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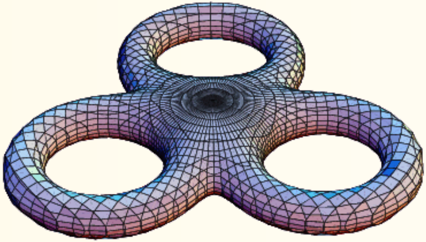

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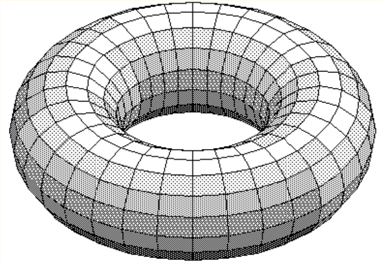
- Even for a high-genus grid, its "essential part" (note; fixed g!) can be captured by a suitable toroidal grid...
- In fact, we can prove (under suitable density assumption);

$$c_0(\Delta, g) \cdot \mathrm{cr}(G) \leq \mathit{Tex}(G) \leq c_1 \cdot \mathrm{cr}(G).$$

2 Stretch of an Embedding



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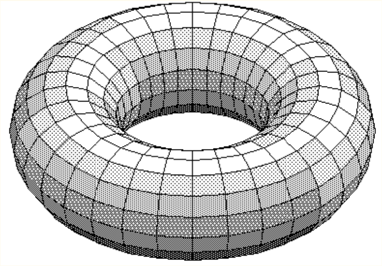

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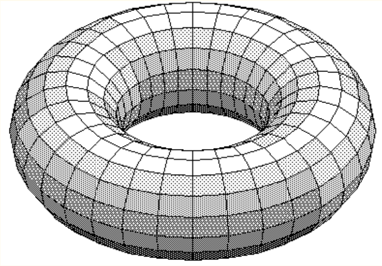

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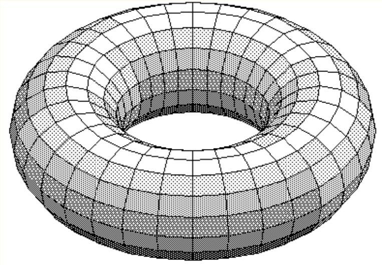

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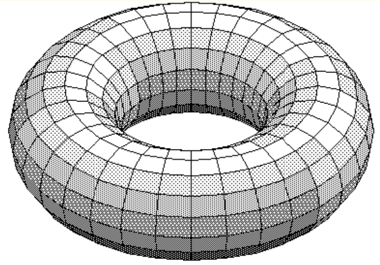

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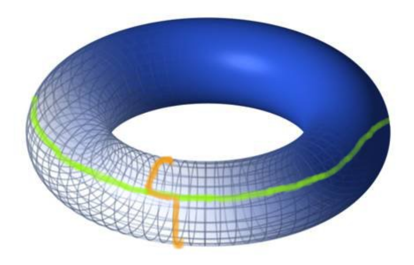
  Of course, would be nice to have the definition symmetric (in "k and ℓ").

Defining stretch



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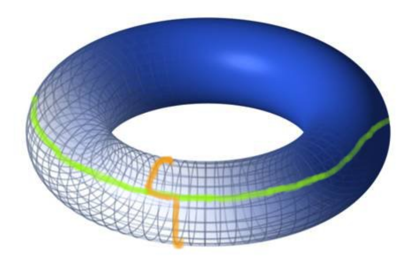

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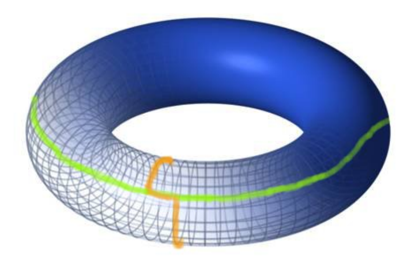

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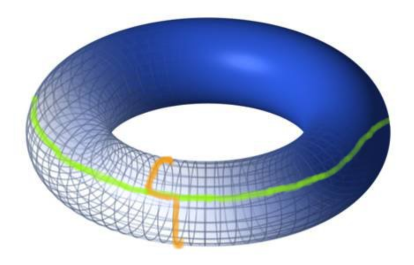

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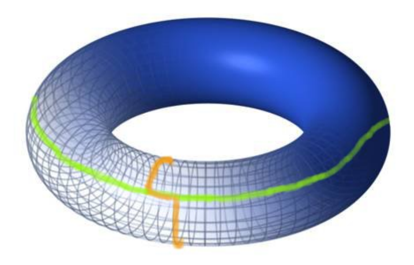

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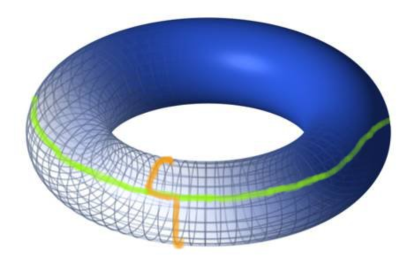
- Note; 1-crossing loops $\Rightarrow$ non-contractible & non-separating.
- Relation to the crossing number:

$$\mathrm{cr}(G) \le \mathit{Str}(G^*) \quad \text{on the torus – trivial}$$

$$\mathrm{cr}(G) \ge c_2(\Delta, g) \cdot \mathit{Str}(G^*) \quad \text{in general – not so easy}$$

Some properties of stretch



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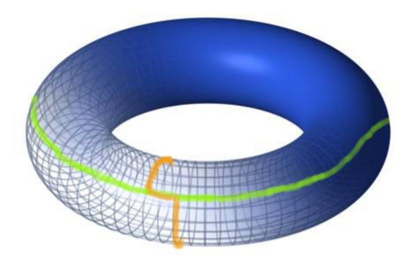

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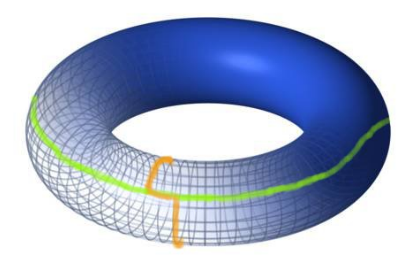

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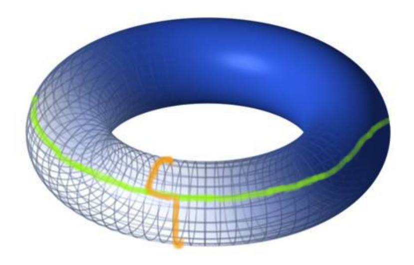

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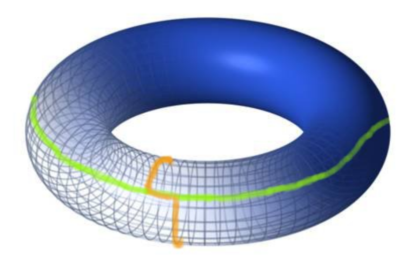

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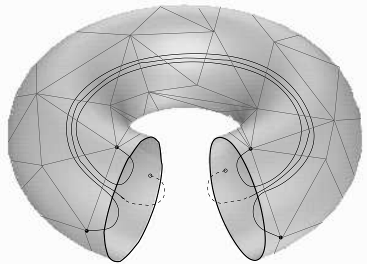
  Consequently, $\mathit{odd\text{-}Str}(G) = \mathit{Str}(G)$ (allowing odd-crossing pairs A, B).
- Note; above the torus, we cannot directly relate $\mathrm{cr}(G)$ to $\mathit{Str}(G^*)$!

3 Tie Up the Ends



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- We have already seen $\mathit{Tex}(G) \leq 12\,\mathrm{cr}(G)$. OK

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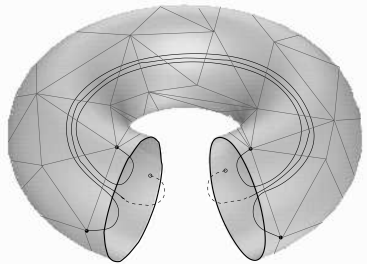

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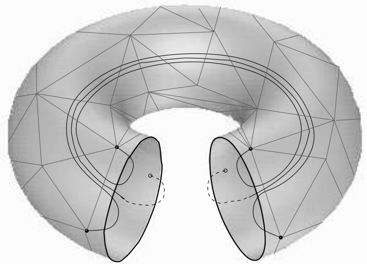

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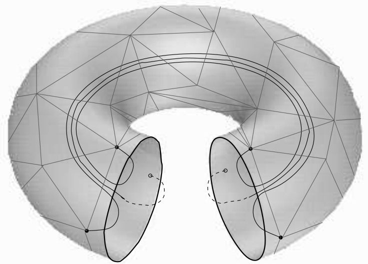

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- Continue this G_1 to G_2, ..., to G_g, getting pairs (k_i, ℓ_i) for $i = 1, \ldots, g$.

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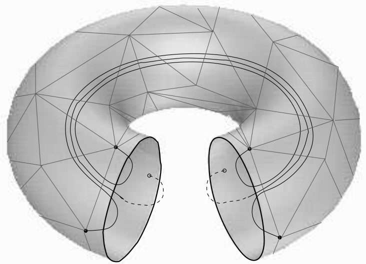

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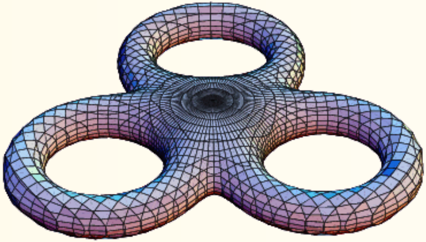
  Imagine we cut $G_0 = G$ along a shortest non-sep. dual cycle of length k_1 to G_1, with dual distance ℓ_1 between the sides in G_1.

- Continue this G_1 to G_2, ..., to G_g, getting pairs (k_i, ℓ_i) for $i = 1, \ldots, g$.

- In plane G_g, reconnect the $k_1 + \cdots + k_g$ cut edges, costing only

$$c_3(g) \cdot \max_i(k_i \ell_i) \text{ crossings.}$$

The tough guy



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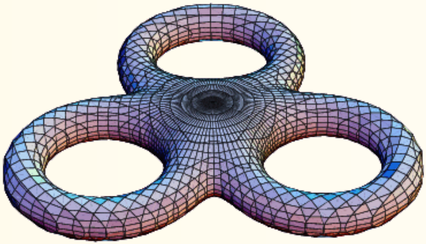

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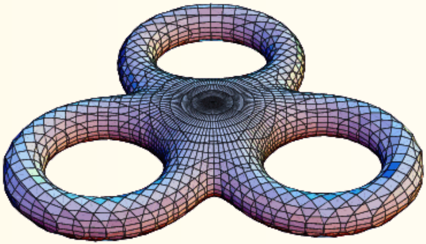

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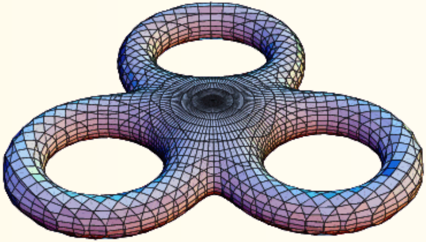

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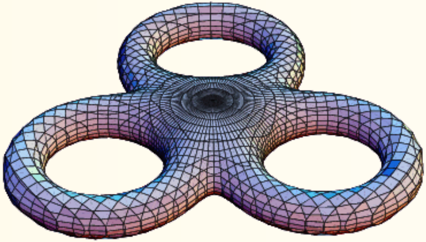

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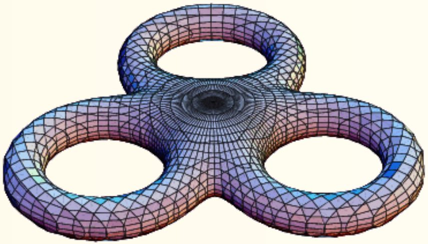

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- After all; *Tex*$(G) \sim k_1\ell_1 \geq c_0(\Delta, g) \cdot \mathrm{cr}(G)$.

Theorem. $c_0(\Delta, g) \cdot \mathrm{cr}(G) \leq$ *Tex*$(G) \leq c_1 \cdot \mathrm{cr}(G)$ for G densely embedded on an orientable surface.

4 Algorithmic Corner

Crossing Approximation Algorithm

- **Algorithm** CrossingApproximation
  Input: graph G embedded in a surface Σ of fixed genus g
  Output: a drawing of G with $c(\Delta(G), g) \cdot \mathrm{cr}(G)$ crossings
  1. $(G_0, \Sigma_0) \leftarrow (G, \Sigma)$
  2. $F \leftarrow \emptyset$
  3. **for** $i = 1, 2, \ldots, g$ **do**
  4. $\quad\gamma_i \leftarrow$ shortest non-separating dual cycle in G_{i-1}
  5. $\quad F \leftarrow F \cup E^*(\gamma_i)$
  6. $\quad(G_i, \Sigma_i) \leftarrow$ cut (G_{i-1}, Σ_{i-1}) through γ_i
  7. **for** $f = uv \in F$ **do**
  8. $\quad\pi_f \leftarrow$ shortest dual u-v path in $(G_g, \mathcal{S}_0)$
  9. $\quad$draw f along π_f (avoid multi-crossings)
  10. **return** $(G_g + F, \mathcal{S}_0)$

- Runtime $\mathcal{O}(n \log n)$

Stretch Algorithm

- **Algorithm** ComputeStretchSurgery
  Input: graph G embedded in a surface Σ of genus g
  Output: the stretch of G
  1. $i \leftarrow 1$
  2. $(G_1, \Sigma_1) \leftarrow (G, \Sigma)$
  3. str $\leftarrow \infty$
  4. **while** Σ_i not the sphere and $|V(G_i)| \leq g \cdot |V(G)|$ **do**
  5. $\quad\alpha_i \leftarrow$ shortest non-separating cycle in G_i
  6. $\quad\beta_i \leftarrow$ shortest cycle crossing α_i exactly once
  7. $\quad$str $\leftarrow \min\{$str, *len*$(\alpha_i) \cdot$ *len*$(\beta_i)\}$
  8. $\quad(G_{i+1}, \Sigma_{i+1}) \leftarrow$ cut (G_i, Σ_i) along α_i
  9. $\quad\quad$and attach disks to the boundaries
  10. $\quad i \leftarrow i + 1$
  11. **return** str

- Runtime $\mathcal{O}(g^4 n \log^2 n)$ – but watch $|V(G_i)|$ carefully...

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- Finding other applications of stretch in algorithms...

  Any suggestions?

Thank you for your attention.