Approximating the Crossing Num. of Toroidal Graphs

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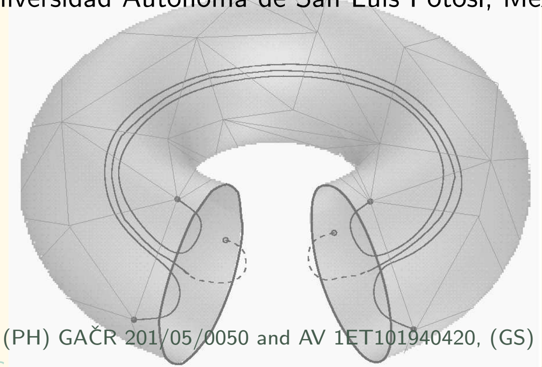

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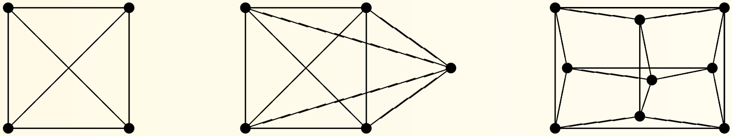
Overview

1 Drawings and the Crossing Number

Definition. *Drawing of a graph* G:

- The vertices of G are distinct points,
  and every edge $e = uv \in E(G)$ is a simple curve joining u to v.
- No edge passes through another vertex,
  and no three edges intersect in a common point.



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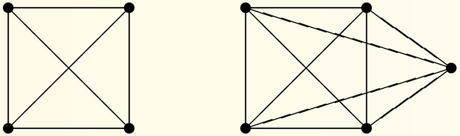

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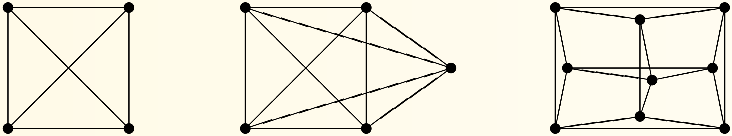

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Warning. There are slight variations of the definition of crossing number, some giving different numbers! (Like counting odd-crossing pairs of edges.)

Computational complexity

Remark. It is practically very hard to determine the crossing number.

Observation. The problem $\textsc{CrossingNumber}(\leq k)$ is in NP:
Guess a suitable drawing of G, then replace crossings with new vertices, and test planarity. . .

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Theorem 3. [**PH**, 2004]
$\textsc{CrossingNumber}$ *is* NP*-hard even on simple* 3*-connected cubic graphs.*

Corollary 4. *The minor-monotone version of c.n. is also* NP*-hard.*

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Question 9. Can we get any reasonable FPT algorithm for (approximating, at least?) CrossingNumber based on "how far" the graph is from planarity?

The next step — **Toroidal graphs**. . .

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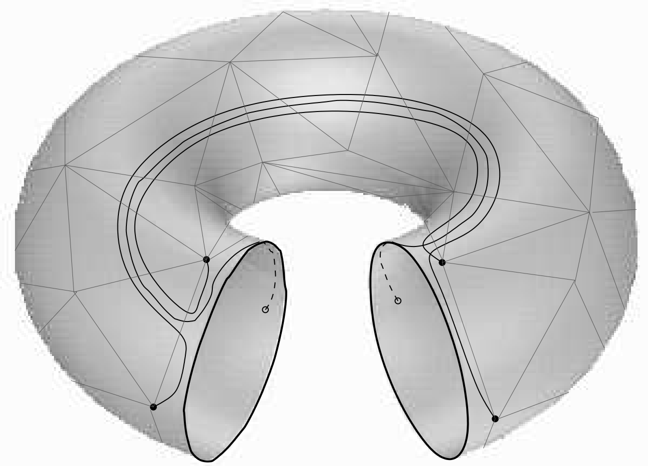
Approximation?

Unfortunately, the above constructions in no way provide approximation algorithms.

The reason — lack of a corresponding *lower bound* on the crossing number...

Cut–and–redraw a toroidal graph



- We embed G on the torus (linear time by [Mohar 1999]).
- We find a "shortest nonseparating" loop of length k on the torus, using an $O(n \log n)$ algorithm of [Kutz 2006]. ($k =$ dual *edge-width* of G.)
- Cutting the torus into a cylinder, we "reconnect" the cut edges along a shortest length-ℓ dual path, producing $\leq k\ell + k^2/4$ crossings.

3 Lower-bounding the Crossing Number of Toroidal Graphs

For the rest we have k the dual edge-width of G on the torus, and ℓ the "dual length" of the cylindrical embedding of G we cut out from our torus.

Lemma 10.

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- If H is a minor of G, and H has maximum degree at most 4, then $\mathrm{cr}(G) \geq \frac{1}{4}\,\mathrm{cr}(H)$.
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Actually, without asymptotic terms our lower bound reads $\mathrm{cr}(G) \geq \frac{1}{4\Delta^2} \cdot k\ell$, provided that $k \geq 16\lfloor \Delta/2 \rfloor$.

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Hence we need to prove:

Theorem 11. *G contains a minor isomorphic to the toroidal grid of size*

$$\max\left(\left\lfloor \frac{2}{3}\frac{k}{\lfloor \Delta/2\rfloor}\right\rfloor, \left\lceil \frac{\ell}{\lfloor \Delta/2\rfloor}\right\rceil\right) \times \left\lfloor \frac{2}{3}\frac{k}{\lfloor \Delta/2\rfloor}\right\rfloor .$$

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- We obtain another collection of $\left\lceil \frac{\ell}{\lfloor\Delta/2\rfloor}\right\rceil$ pairwise disjoint cycles of G on our cylinder, using a network-flow duality argument.
- We will then combine one collection of $\left\lfloor \frac{2}{3}\frac{k}{\lfloor\Delta/2\rfloor}\right\rfloor$ cycles in G with the latter collection to form a new toroidal grid minor of the required size.

Our main theoretical contribution actually is the following:

Theorem 12. *Suppose a toroidal graph* H *contains a collection* $\mathcal{C}$ *of* p *pairwise disjoint pairwise freely homotopic cycles, and an analogous collection* $\mathcal{D}$ *of* q *cycles, such that* $\mathcal{D}$ *is not homotopic to an iteration of* $\mathcal{C}$.

Then H *contains a* $p \times q$ *toroidal grid minor.*

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Then H contains a $p \times q$ toroidal grid minor.

Unfortunately, the two cycle collections can interact in really nasty ways on the torus, and the proof requires a detailed technical analysis (proceedings).

4 Conclusion and Future Steps

Main result. We have got an $O(n \log n)$ time algorithm that approximates CrossingNumber on toroidal graphs up to a

factor of $\mathbf{6\Delta(G)^2}$,

provided that the graph embeds with dual edge-width at least $8\Delta(G)$.

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Possible extensions. For graphs embedded on a higher orientable surface Σ_g. (Assume bounded g and Δ.)

- Repeat the algorithm of Section 2 for g steps until Σ_g is cut down to a plane. Denote by k_i and ℓ_i the "dual lengths" obtained at step i.
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  but we need to prove $\mathrm{cr}(G) \geq \Omega(\max_{i=1,\dots,g} k_i \cdot \ell_i)$, which is still open (work in progress), and it does not seem easy to finish. . .