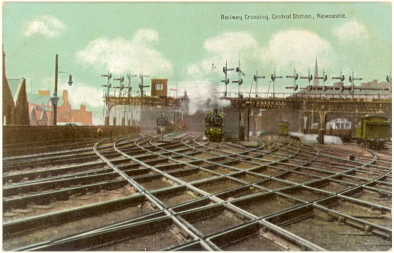
On the Crossing Number of Surface-Embedded Graphs



Petr Hliněný

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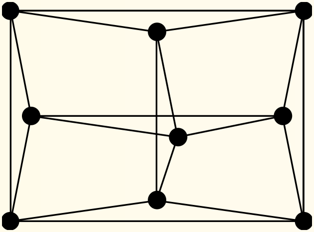
Botanická 68a, 602 00 Brno, Czech Rep.

based on joint work with **Markus Chimani** and **Gelasio Salazar**

1 Crossing Number ?

- Everybody here knows very well. . .



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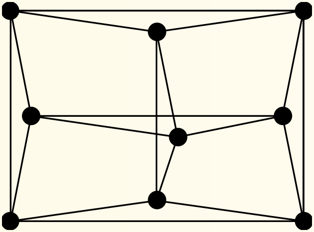

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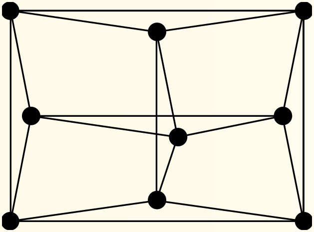

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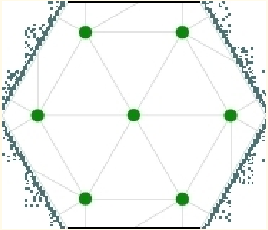
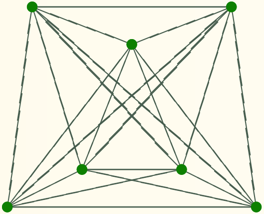
- The *minor-monote* version is of some interest as well.

Our objective

- To provide a two-way math relation between an *embedding* of a graph (on a surface) and its *crossing number*.





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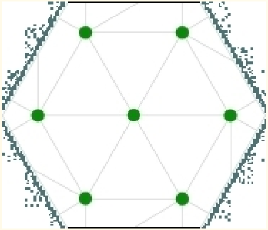

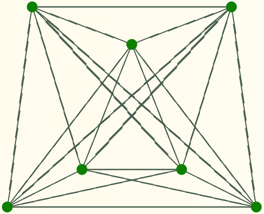

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- Although this is math, our motivation is algoritmic.

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Can anything be computed efficiently?

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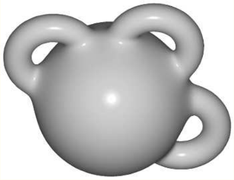
- Constant factors for surface-embedded bounded-degree graphs

[**Gitler et al**, 2007], [**PH and Salazar**, 2007], [**PH and Chimani**, 2010]

3 Crossing Number of Embedded Graphs



Related:

- Crossing number is linear, $O(|V|)$, for any graph of bounded degree and embedded in a fixed surface.
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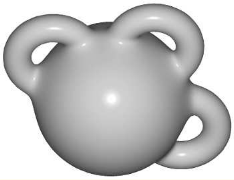

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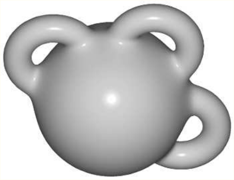

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Unfortunately,

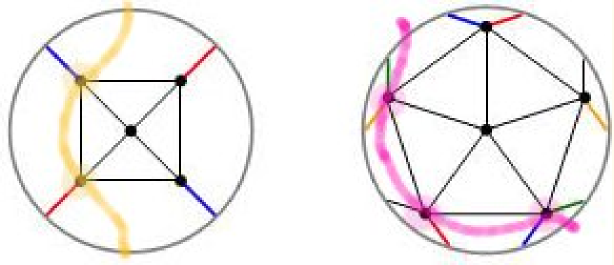
all these are only upper bounds, giving no approximation guarantees;

— we need fine-resolution measure(s) of embedded graphs!

E.g., just for the projective graphs...

Face-width

$fw(G) =$ shortest noncontractible "vertex-face cycle" in the embedding.

$fw = 2$  $fw = 3$

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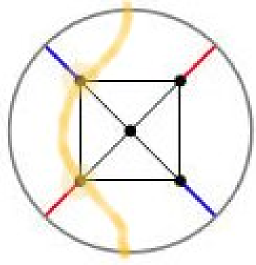
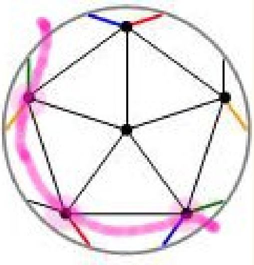
$fw = 2$   $fw = 3$

- By cutting along (close to) such a vertex-face cycle, one gets a pretty good drawing in the plane;

$$\mathrm{cr}(G) \leq \frac{1}{8}\Delta(G)^2 \cdot fw(G)^2 \,.$$

... the projective approximation

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- A lower-bound argument, prov. $\mathit{fw}(G) \geq 6$, gives

$$\frac{1}{36}\,\mathit{fw}(G)^2 \leq \mathrm{cr}(G)\,.$$

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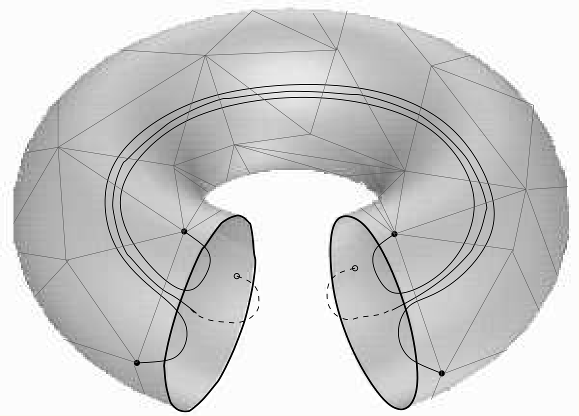
Impr. for minor crossing number

- Can do much better – removing the Δ !

$$\frac{1}{36}\, fw(G)^2 \le \mathrm{cr}(G) \le \binom{fw(G)}{2} .$$

And on the torus...



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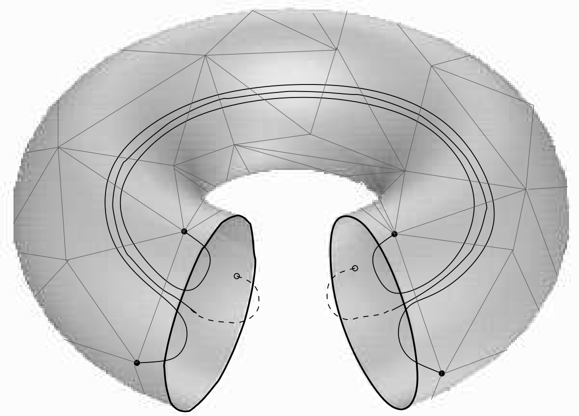

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- However, *face-width* $fw(G)$ is no longer enough to express the resulting number of crossings.

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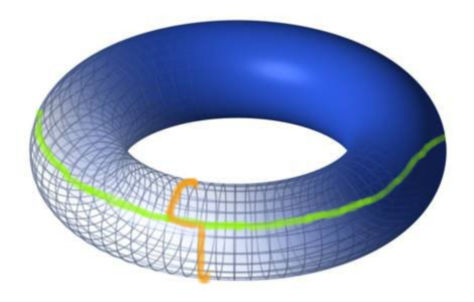


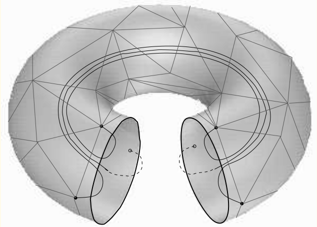
Stretch *stretch*(G) = min *len*$(\alpha)\cdot$ *len*(β) over all (α,β);

– (α,β) "*one-leaping*" pair of dual cycles in G,

– i.e, meet once and transversally.

Stretch and crossings; the toroidal case

Particularly nice, since the aforementioned easy approach;



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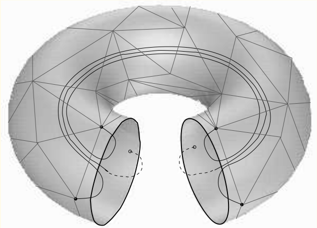

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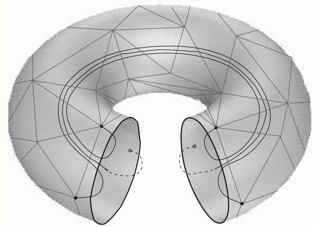

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(Even smaller bound can be given – “remove” the shared sect. length. . .)

- Furthermore, for $\mathit{stretch}(G) = \mathit{len}(\alpha) \cdot \mathit{len}(\beta)$ on the torus, one may assume $\mathit{len}(\alpha) = \mathit{ew}^*(G)$.

Lower bound on the torus

– finding a large *toroidal grid minor* in G:

- Set $k = \mathit{len}(\alpha)/\lfloor\Delta/2\rfloor$ and $\ell = (\,\mathit{len}(\beta) - \mathit{len}(\alpha)/2)/\lfloor\Delta/2\rfloor$.

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For minor crossing, again

- (Skip the above shift $\mathit{fw} \to \mathit{ew}^* \to \mathit{fw}$.)
- Analog., use "*face-width stretch*", and get rid of Δ... factor 8

Stretch and crossings; higher surfaces

Say, G on an orientable surface of genus g.

- **Lower bound** $$\mathbf{cr}(\boldsymbol{G}) \geq \frac{\mathbf{1}}{\mathbf{2}^{\mathbf{1}+\mathbf{2}\boldsymbol{g}}\boldsymbol{\Delta}^{\mathbf{2}}} \cdot \textit{stretch}(\boldsymbol{G})$$

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- **Though**, not good enough for a general approximation!

  – bad "interference" in a sequence of g cuts...

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  Need to cut "low-stretch handles" to raise the stretch value. In other words, we "hunt" for the $c \times \ell$ tor. grid minor in G.

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- Count the crossings: $\leq 3 \cdot 2^{g+1} \cdot \max\{c_i \ell_i\}$
  and apply the lower bound.

Approximation Algorithm

- Input G, (embed on $\mathcal{S}_g$).
- Separately cut and open all sparse handles:

  bounded num. of edges $\rightarrow$ separate *MEI approx.* [Chim. PH, 2011].
- Seq. cut and open $\boldsymbol{G_{i+1} = G_i/\gamma_i}$ where $c_i =$ *len*$(\gamma_i) =$ *ew*$^*(G_i)$, and $c_i \geq 2^{g+3-i}\Delta$. Store corresp. ℓ_i.
- Finally, G_g is plane!

  Insert the cut edges back optimally into G_g (recall the MEI part).
- Count the crossings: $\leq 3 \cdot 2^{g+1} \cdot \max\{c_i\ell_i\}$
  and apply the lower bound.

  $\longrightarrow$ factor $\boldsymbol{3 \cdot 2^{3g+2} \cdot \Delta^2}$.

6 Concluding Remarks

- The crossing number of a graph embedded on an orientable surface can be reasonably well approximated.

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- Other applications for stretch?