The crossing number of a projective graph is quadratic in the face–width

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Abstract

We show that for each integer $g \geq 0$ there is a constant $c_g > 0$ such that every graph that embeds in the projective plane with sufficiently large face–width r has crossing number at least $c_g r^2$ in the orientable surface Σ_g of genus g. As a corollary, we give a polynomial time constant factor approximation algorithm for the crossing number of projective graphs with bounded degree.

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1 Introduction

We recall that the face–width of a graph G embedded in a surface Σ is the minimum number of intersections of G with a noncontractible curve in Σ.

Fiedler et al. [5] proved that the orientable genus of a projective graph grows linearly with the face–width. Our aim is to show that for each integer $g \geq 0$, the crossing number cr_g of projective graphs in the closed orientable surface Σ_g of genus g grows quadratically with the face–width.

Theorem 1.1 *For every integer $g \geq 0$ there are constants $c_g, r_g > 0$, such that if G embeds in the projective plane with face–width at least $r \geq r_g$, then the crossing number $\mathrm{cr}_g(G)$ of G in Σ_g is at least $c_g r^2$.*

Our strategy for proving Theorem 1.1 is to show the existence of sufficiently large grid–like structures, so called diamond grids (Theorem 2.1), in projective graphs, and then prove that diamond grids have large crossing number (Section 3, which concludes with a proof of Theorem 1.1).

No algorithm is known for approximating the crossing number of arbitrary (not even bounded–degree) graphs within a constant factor. The best result reported in this direction is by Even, Guha, and Schieber [4], who give an $O(\log^3 n)$ approximation algorithm for $\mathrm{cr}(G) + |V(G)|$ (not for $\mathrm{cr}(G)$, thus weak in the case of graphs with few crossings) on bounded-degree graphs.

Answering affirmatively to a question of P. Brass, Pach and Tóth [7] gave upper bounds for the crossing numbers (in the plane) of graphs embedded in a given orientable surface. Böröczky, Pach and Tóth then extended this results to arbitrary surfaces [2], showing that for every surface χ there is a constant c_χ such that if a graph with n vertices and maximum degree Δ embeds in χ, then its planar crossing number is at most $c_\chi \Delta\, n$. This was recently generalized by Wood and Telle to graphs with an excluded minor [9] (see also [1]).

Along a similar vein, we also give natural upper bounds for the crossing numberof a projective graph in terms of its face–width r and its maximum degree Δ, see in Section 4. Consequently, we have an approximation algorithm:

Theorem 1.2 *For every fixed Δ and orientable surface Σ_g, there is a polynomial time approximation algorithm that computes the crossing number cr_g of a projective graph with maximum degree Δ within a constant factor.*

2 Finding a large diamond projective grid

Randby [8] gave, for each integer $r > 0$, a full characterization of those projective graphs that are minor–minimal with respect to having face–width r.

He showed that all such graphs can be obtained from the $r \times r$ projective grid by $Y\Delta$– and ΔY–exchanges. Now although it is not too difficult to show that the $r \times r$ projective grid has crossing number quadratic in r for $r \geq 3$, it is not that straightforward to show that performing $Y\Delta$ and ΔY operations does not decrease the crossing number significantly. Thus our approach is to find, in projective graphs of given face–width, a related grid–like structure that better suits our purposes.

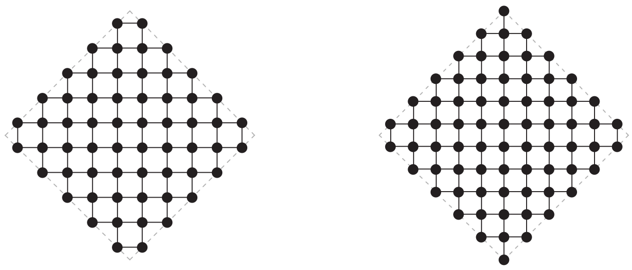

Fig. 1. Projective diamond grids of sizes 10 (left) and 11 (right).

The *diamond grid* D_r of size r is a plane graph whose vertices are all integer pairs (i, j) where $|i| + |j| \leq r$, such that j is always odd, the parity of i is the opposite of the parity of r, and an edge of D_r joins (i, j) to (i', j') iff $|i - i'| + |j - j'| = 2$. The *projective diamond grid* P_r of size r is obtained from D_r by identifying the opposite pairs of its "boundary" vertices, that is, (i, j) with $(-i, -j)$ whenever $|i| + |j| = r$. On the left (respectively right) hand side of Fig. 1 we illustrate the projective diamond grid of size 10 (respectively, 11).

Theorem 2.1 *Every graph that embeds in the projective plane with face-width r has a minor isomorphic to P_r.*

3 Crossing number of projective diamond grids

A set $\mathcal{C}$ of cycles in a graph is an *I-collection* if each two cycles in $\mathcal{C}$ have connected, nonempty intersection, and no vertex is in more than two cycles of $\mathcal{C}$. The following statement is an easy exercise (see Fig. 2).

Proposition 3.1 *The projective diamond grid P_r of size r contains an I-collection of $r - 1$ cycles.*

The first key observation is that each fixed orientable surface cannot host an arbitrarily large embedded I–collection.

Proposition 3.2 *For each nonnegative integer g there is a positive constant M_g such that if an I–collection $\mathcal{C}$ is embedded in Σ_g then $|\mathcal{C}| \leq M_g$.*

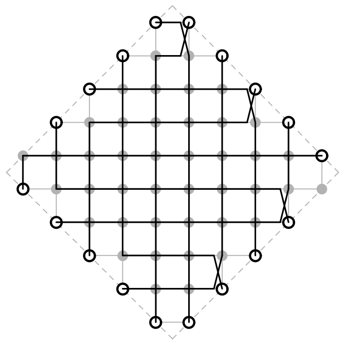

Fig. 2. Finding an I-collection of 9 cycles in P_{10}.

Secondly, we show that the crossing number of sufficiently large I-collections grows quadratically with their size, which finishes the main proof.

Theorem 3.3 *Let G be a graph that contains an I–collection of size $k > M_g$, where M_g is the constant in Proposition 3.2. Then the crossing number of G in Σ_g is at least $k(k-1)/(M_g(M_g+1))$.*

Proof of Theorem 1.1. By Theorem 2.1, G contains a P_r-minor. It is moreover obvious that if a minor of G contains an I-collection, then an I-collection of the same size is contained also in G itself. Hence it now follows from Proposition 3.1 that G contains an I-collection of $r-1$ cycles, and from Theorem 3.3 that $\mathrm{cr}_g(G) \geq (r-1)(r-2)/(M_g(M_g+1))$. Thus Theorem 1.1 follows if we set $r_g = M_g + 2$, and $c_g = 1/(M_g+2)^2$ since $M_g + 2 \leq r$. □

It is easy to see that $M_0 = 4$ (planar case) satisfies Proposition 3.1, and so a special planar ($g = 0$) version of Theorem 1.1 gives a lower bound $\frac{1}{36}r^2$.

4 Estimating the crossing number

The basic idea behind our approximation algorithm is that the crossing number of bounded degree projective graphs is bounded from above and from below by quantities that are within a constant factor of each other. The required lower bound is given in Theorem 1.1.

To obtain the upper bound we perform surgery on the projective plane: cut along an essential (noncontractible) curve that intersects the embedded graph as little as possible, then rejoin the pieces and bound the number of crossings thus obtained. This technique is presented in its full generality (applies to all surfaces) by Böröczky, J. Pach, and G. Tóth in [2]. Using these techniques, we now give a bound that explicitly involves the face–width of the embedding.

Proposition 4.1 *Suppose that G is a graph with maximum degree Δ that embeds in the projective plane with face–width r. Then the crossing number of G in the plane (and thus in any orientable surface) is at most $r^2\Delta^2/8$.*

The idea of the previous paragraph readily translates into an approximation algorithm proving Theorem 1.2: We test whether the input graph G embeds in Σ_g using the $O(n)$-time algorithm by Mohar [6], and then we use the $O(n\sqrt{n})$-time algorithm of Cabello and Mohar [3] to find a shortest noncontractible cycle in dual G^*.

Remark 4.2 In the planar case of Theorem 1.2, the described approximation algorithm yields a drawing of G within a factor $4.5\Delta^2$ of $\mathrm{cr}_0(G)$.

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