The crossing number of a projective graph is quadratic in the face–width

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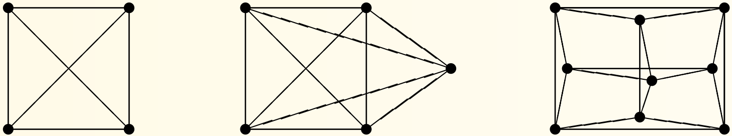
Overview

1 Drawings and the Crossing Number

Definition. *Drawing of a graph* G:

– The vertices of G are distinct points,
and every edge $e = uv \in E(G)$ is a simple curve joining u to v.

– No edge passes through another vertex,
and no three edges intersect in a common point.



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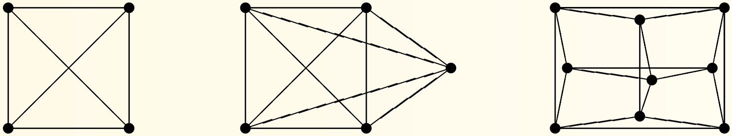

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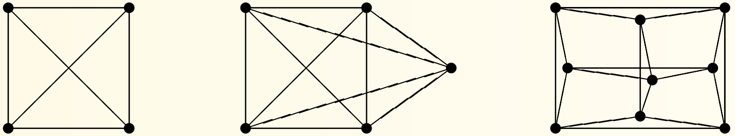

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Warning. There are slight variations of the definition of crossing number, some giving different numbers! (Like counting odd-crossing pairs of edges.)

Embedded graphs

Consider graphs embedded on a (fixed) surface Σ.

Theorem 1. [**Böröczky, Pach and Tóth** / **Djidjev and Vrt'o**, 2006]
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Definition. *Face-width* of a graph G in Σ is the smallest number of points a Σ-noncontractible loop intersects the drawing of G.

2 Projective graphs

We prove the following...

Theorem 3. *If G embeds in the projective plane with face-width at least $r \geq 6$, then the crossing number of G in the plane is at least $r^2/36$.*

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Proposition 4. *If G is a graph with maximum degree Δ that embeds in the projective plane with face-width r, then the crossing number of G in the plane is at most $r^2\Delta^2/8$.*

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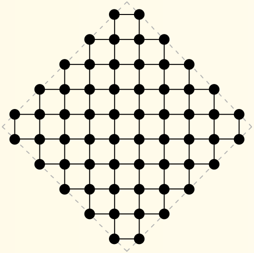
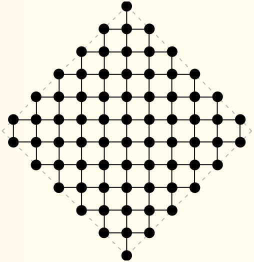
Proof. Trivially – cut the projective embedding of G at r points (and open it to the plane).

Hence there are at most $s = r\Delta/2$ affected edges, and redrawing those induces at most $s^2/2$ crossings. ∎

To prove Theorem 3, we argue...

Theorem 5. *Every graph that embeds in the projective plane with face-width* r *has a minor isomorphic to the projective diamond grid* P_r.





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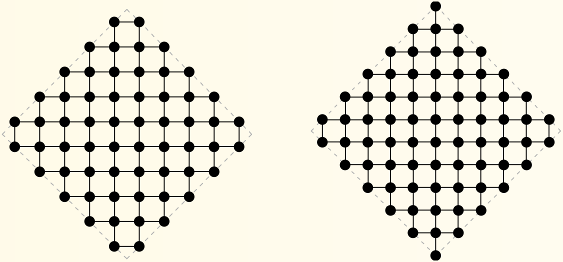
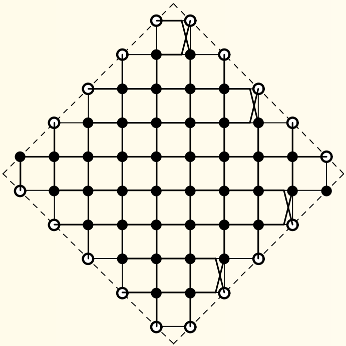


Proof. Again, cut the projective embedding of G at r points (and open it to the plane, to $2r$ points).

Find two "orthogonal" collections of r paths each between those points, by Menger's theorem.

By planarity, these two collections form P_r... ∎

Definition. *I-collection* – each two cycles have connected intersection, and no vertex is in more than two cycles.



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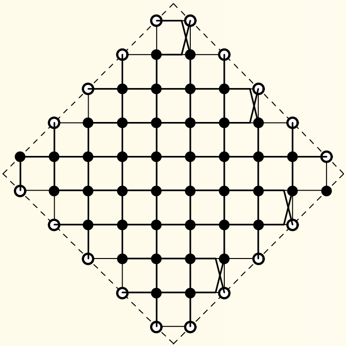

Proposition 6. *The projective diamond grid P_r of size r contains an I-collection of $r-1$ cycles.*

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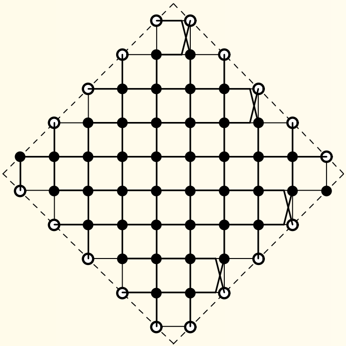

Proposition 6. *The projective diamond grid P_r of size r contains an I-collection of $r-1$ cycles.*

Proposition 7. *If an I-collection $\mathcal{C}$ is embedded in the plane, then $|\mathcal{C}| \leq 4$.*

Theorem 8. *If G contains an I-collection of size $k > 4$, then the crossing number of G is at least $k(k-1)/20$.*

Proof. Any 5-tuple of cycles in the I-collection must induce a crossing by Proposition 7. Each such crossing is counted at most $\binom{k-2}{3}$ times. Hence we have at least this many crossings in G:

$$\binom{k}{5} \Big/ \binom{k-2}{3} \;=\; k(k-1) \,/\, 5 \cdot 4$$

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Regarding **Theorem 3**, we continue:

- We have $k = r - 1$ by Proposition 6.
- So the number of crossings is by Theorem 8, for $r \geq 6$,

$$(r-1)(r-2) \,/\, 20 \;\geq\; r^2/36 \,.$$

∎

3 Approximation algorithm

Theorem 9. *For every fixed Δ there is a polynomial time approximation algorithm that computes the crossing number of a projective graph with maximum degree Δ within a constant factor.*

- We test whether the input graph G is planar in $O(n)$ time.
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- Since $\binom{|F|}{2} < |F|^2/2 \leq r^2\Delta^2/8$, we have an approximation of $\mathrm{cr}(G)$ within factor $4.5\Delta^2$.

∎

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Consider the crossing number on a fixed *orientable surface* Σ_g ...

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- Hence an extension of Theorem 3 gives a lower bound of $r^2/(M_g+2)^2$ crossings.
- An extension of the approximation algorithm is also straightforward.

Conclusions

- Only very few graphs classes have efficient constant-factor approximaions for the crossing number; e.g. *almost-planar* graphs of bounded degrees.

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- What further generalization are possible?
- Thank you for attention!