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New almost-planar crossing-critical graph families

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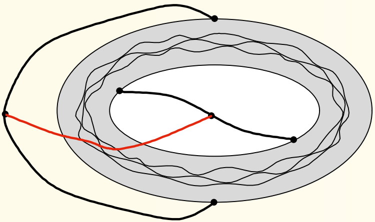

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– $\mathrm{cr}(H) \geq k$ and $\mathrm{cr}(H - e) < k$ for all edges $e \in E(H)$.

We study crossing-critical graphs to understand what structural properties force the crossing number of a graph to be large.

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Remarks:

– 1-crossing-critical graphs are K_5 and $K_{3,3}$ (up to vertices of degree 2).

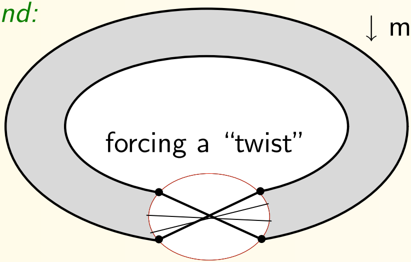
– Infinite classes of $3, 2$-crossing-critical graphs, [Širáň 84, Kochol 87].

– Many infinite classes of crossing-critical graphs are known today, and they tend to have similar "global" structure.

Constructing crossing-critical graphs

Twisted Möbius band:
(a classical idea)

↓ many homogeneous planar "tiles"



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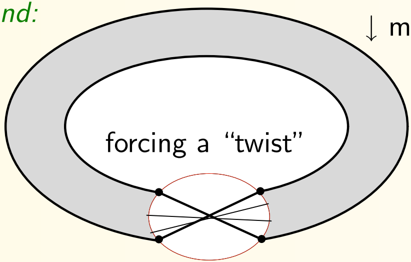

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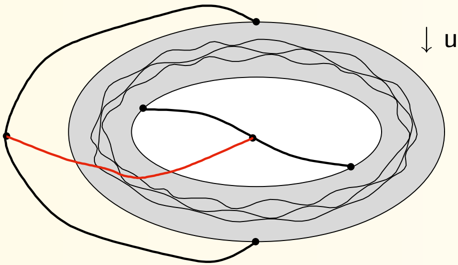
Crossed planar belt:
[PH, 2001]



↓ union of k edge-disjoint cycles

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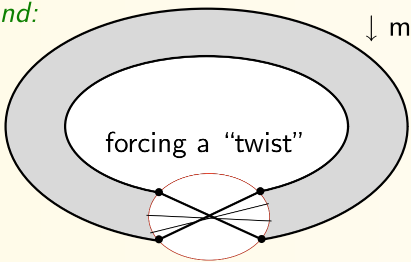

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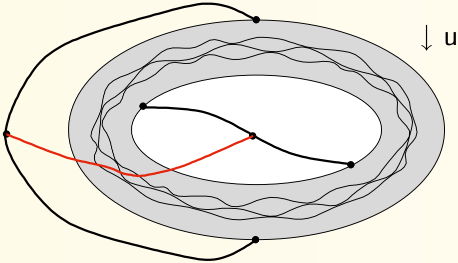

↓ union of k edge-disjoint cycles

Zip-product: [Bokal, 2005] Composing crossing-critical graphs...

Looking at vertex degrees

- [folklore] Infinite families of simple 3-connected crossing-critical graphs can have average degree in $(3, 6]$.
  (Lower bound by connectivity and graph minors, upper via Euler.)

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- The case of odd degrees > 3 remains open...

2 Our "Belt" Construction

- Constructing simple 3-connected *almost-planar* crossing-critical graphs (such that deleting one edge leaves them planar).
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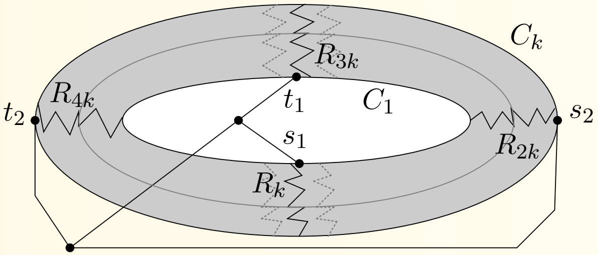
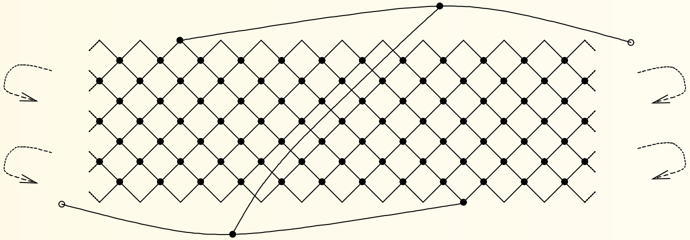
Definition. **Crossed k-belt graphs**:



- Edge-disj. planar union $C_1 \cup \ldots \cup C_k$, with a 4-terminal "bridge".
- Forming many disjoint "radial" paths, separating the bridge terminals.
- No vertex of degree > 4 on C_k, that is, $C_k \cap C_{k-2} = \emptyset$.

Getting high-degree vertices

- We start with a "crossed fence" from [PH 2001],



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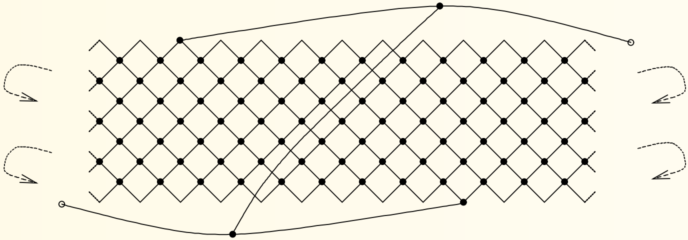
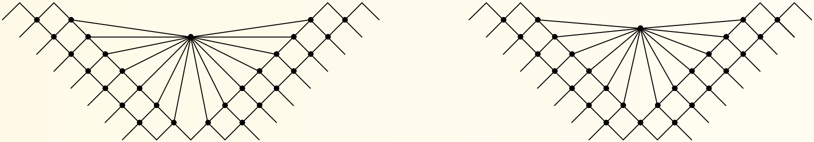


- The following modif. produce vertices of degrees $2k-2$, $2k-4$, ...



— the resulting graphs are all *crossed k-belt graphs*.

Crossing-criticality

Proposition 1. *Let k be fixed. For every integer m there is a crossed k-belt graph which is simple 3-connected and which contains more than m vertices of each of even degrees $\ell = 4, 6, 8, \ldots, 2k-2$.*

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Theorem 2. *For $k \geq 3$, every crossed k-belt graph is k-crossing-critical.*

Proof. By induction on k:

- $k = 1 \rightarrow$ a subdivision of nonplanar $K_{3,3}$ ($k = 2$ – a false statement),
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- If C_1 is crossed, then remove it, forming a crossed $(k-1)$-belt graph, and continue.
- If C_1 is not crossed, then it forms a face in the optimal drawing. Then the radial paths witness $2k-2$ C_1-ears separating the bridge terminals on C_1, forcing too many crossings.

■

3 Average Degrees in $[4, 6)$

Theorem 3. *For every odd $k > 3$ there are infinitely many simple 3-connected crossed k-belt graphs with the average degree equal to any rational value in the interval* $[\mathbf{4},\ \mathbf{6} - \frac{\mathbf{8}}{\boldsymbol{k}+\mathbf{1}})$.

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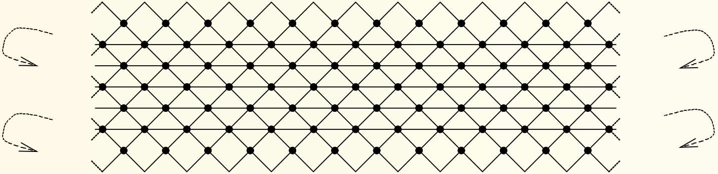
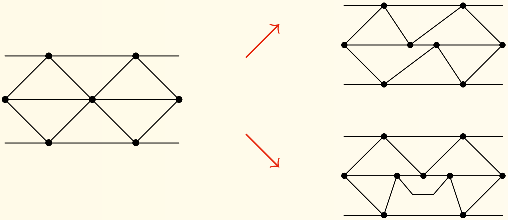
Proof. We start with the following belt, and apply suitably local splittings...





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Any solutions or counterexamples?