Another Two Graphs With No Planar Covers

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Abstract. A graph $\boldsymbol{H}$ is a cover of a graph $\boldsymbol{G}$ if there exists a mapping φ from $V(\boldsymbol{H})$ onto $V(\boldsymbol{G})$ such that φ maps the neighbors of every vertex v in $\boldsymbol{H}$ bijectively to the neighbors of $\varphi(v)$ in $\boldsymbol{G}$. Negami conjectured in 1986 that a connected graph has a finite planar cover if and only if it embeds in the projective plane. It follows from the results of Archdeacon, Fellows, Negami, and the author that the conjecture holds as long as $\boldsymbol{K}_{1,2,2,2}$ has no finite planar cover. However, this is still an open question, and $\boldsymbol{K}_{1,2,2,2}$ is not the only minor-minimal graph in doubt. Let $\boldsymbol{\mathcal{C}}_4$ ($\boldsymbol{\mathcal{E}}_2$) denote the graph obtained from $\boldsymbol{K}_{1,2,2,2}$ by replacing two vertex-disjoint triangles (four edge-disjoint triangles) not incident with the vertex of degree 6 with cubic vertices. We prove that the graphs $\boldsymbol{\mathcal{C}}_4$ and $\boldsymbol{\mathcal{E}}_2$ have no planar covers. This fact is used in [P. Hliněný, R. Thomas, *On Possible Counterexamples to Negami's Planar Cover Conjecture*, manuscript 1999] to show that that there are, up to obvious constructions, at most 16 possible counterexamples to Negami's conjecture.

1 Introduction

All *graphs* in this paper are finite, and have no loops or multiple edges. The vertex set of a graph $\boldsymbol{G}$ is denoted by $V(\boldsymbol{G})$, the edge set by $E(\boldsymbol{G})$. A *plane graph* $\boldsymbol{G}$ is a planar graph embedded in the plane, and $F(\boldsymbol{G})$ denotes its set of faces. A graph $\boldsymbol{H}$ is a *cover* of a graph $\boldsymbol{G}$ if there exist an onto mapping $\varphi : V(\boldsymbol{H}) \to V(\boldsymbol{G})$, called a (cover) *projection*, such that φ maps the neighbors of any vertex v in $\boldsymbol{H}$ bijectively onto the neighbors of $\varphi(v)$ in $\boldsymbol{G}$. A cover is called *planar* if it is a finite planar graph. (Notice that every graph can be covered by an infinite tree, but that is not what we are looking for.)

If a graph $\boldsymbol{G}$ has an embedding in the projective plane, then the lifting of the embedding of $\boldsymbol{G}$ into the universal covering surface of the projective plane (the sphere) is a double planar cover of $\boldsymbol{G}$. Thus every projective-planar graph has a planar cover. The converse is false in general, because for instance the graph consisting of two disjoint copies of $\boldsymbol{K}_5$ has a planar cover, and yet has no embedding in the projective plane. On the other hand, Negami made the following interesting conjecture.

⋆ Partially supported by NSF under Grant No. DMS-9623031.

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Conjecture 1. (S. Negami, 1986) *A connected graph has a finite planar cover if and only if it has an embedding in the projective plane.*

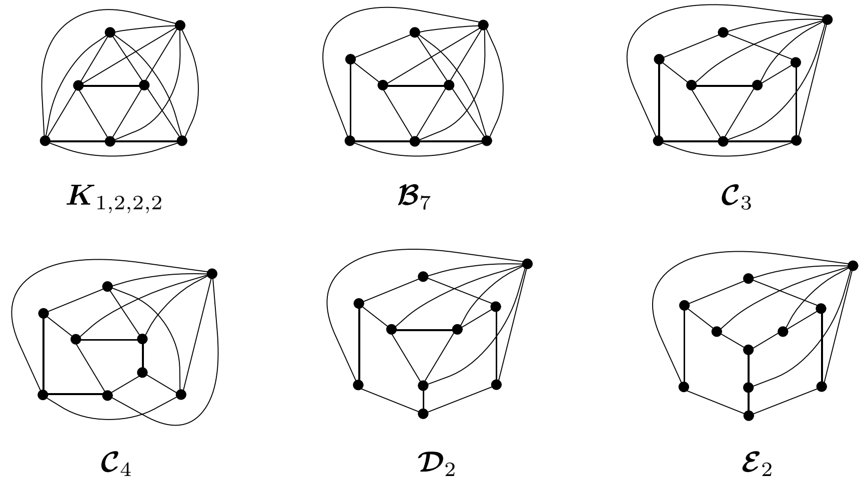

Fig. 1.

Note that the property of having a planar cover is hereditary under the minor ordering. Thus, in order to prove Conjecture 1, it is sufficient to prove that no graph from the family Λ of all 32 connected minor-minimal nonprojective graphs [1, 3] has a planar cover. Let $\boldsymbol{K}_{1,2,2,2}, \boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{C}}_4, \boldsymbol{\mathcal{D}}_2, \boldsymbol{\mathcal{E}}_2 \in \Lambda$ denote the graphs depicted in Fig. 1. (Our notation of these graphs mostly follows [3].) Archdeacon [2], Fellows, Negami [7], and the author [4] have shown the following.

Theorem 2. (D. Archdeacon, M. Fellows, S. Negami, P. Hliněný, 1987–1996) *No member of the family* $\Lambda - \{\boldsymbol{K}_{1,2,2,2}, \boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{C}}_4, \boldsymbol{\mathcal{D}}_2, \boldsymbol{\mathcal{E}}_2\}$ *has a planar cover.*

A vertex of degree 3 with three distinct neighbors is called *cubic*. A *$Y\Delta$-transformation* is the operation replacing a cubic vertex v in a graph by a triangle on the three neighbors of v. Note that every $Y\Delta$-transformation clearly preserves the property of having a planar cover, as observed by Archdeacon. Since each of the graphs $\boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{C}}_4, \boldsymbol{\mathcal{D}}_2, \boldsymbol{\mathcal{E}}_2$ can be $Y\Delta$-transformed to $\boldsymbol{K}_{1,2,2,2}$, Theorem 2 implies that Conjecture 1 is equivalent to the statement that $\boldsymbol{K}_{1,2,2,2}$ has no finite planar cover. However, this is still an open question, and the arguments outlined above seem to say little about possible counterexamples.

We prove the following result, that is used in [6] to show that there are, up to obvious constructions, at most 16 possible counterexamples to Negami's conjecture. The result is also contained in the author's Ph.D. dissertation [5].

Theorem 3. *The graphs (a)* $\boldsymbol{\mathcal{C}}_4$*, and (b)* $\boldsymbol{\mathcal{E}}_2$ *have no planar covers.*

2 The graph $\mathcal{C}_4$

Let $\boldsymbol{H}$ be a plane graph, and let f be the outer face of $\boldsymbol{H}$. The graph $\boldsymbol{H}$ is called a *semi-cover* of a graph $\boldsymbol{G}$ if there exists an onto mapping $\varphi : V(\boldsymbol{H}) \to V(\boldsymbol{G})$, called a *semi-projection*, such that for each vertex v of $\boldsymbol{H}$ not incident with f, the neighbors of v are mapped bijectively onto the neighbors of $\varphi(v)$, and for each vertex w of $\boldsymbol{H}$ incident with f, the neighbors of w in $\boldsymbol{H}$ are mapped injectively to the neighbors of $\varphi(w)$. (Informally, the vertices of the outer face are "allowed to miss some neighbors" in a semi-cover.) Clearly, each cover is a semi-cover, but the converse is false.

Assume that a connected plane graph $\boldsymbol{H}$ is a semi-cover of a connected graph $\boldsymbol{G}$, and $\varphi : V(\boldsymbol{H}) \to V(\boldsymbol{G})$ is a semi-projection. Let $\psi : E(\boldsymbol{H}) \to E(\boldsymbol{G})$ denote the corresponding edge projection, defined by $\psi(uv) = \varphi(u)\varphi(v)$. If $\boldsymbol{G}'$ is a subgraph of $\boldsymbol{G}$, then the graph $\boldsymbol{H}'$ with vertex set $\varphi^{-1}(V(\boldsymbol{G}'))$ and edge set $\psi^{-1}(E(\boldsymbol{G}'))$ is called a *lifting* of $\boldsymbol{G}'$ into $\boldsymbol{H}$. Assuming C is a cycle in $\boldsymbol{G}$, the semi-cover $\boldsymbol{H}$ is said to be C*-fixed* if the lifting of C into $\boldsymbol{H}$ consists of finite facial cycles of the same length as C.

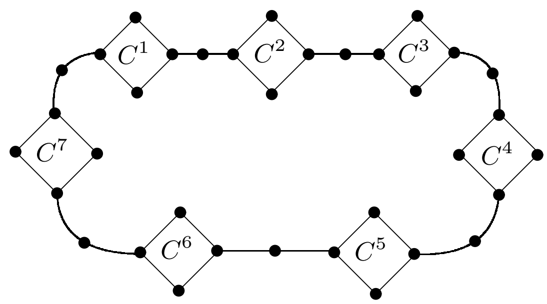

Fig. 2. An illustration of a necklace.

The idea of a necklace was introduced by Archdeacon in [2]. For our purpose it is formally defined as follows. Suppose that C is an induced 4-cycle in $\boldsymbol{G}$, w is a vertex of $\boldsymbol{G} - V(C)$, and $\boldsymbol{F}$ is a subgraph isomorphic to $\boldsymbol{K}_{2,3}$ such that $C \subset \boldsymbol{F} \subset \boldsymbol{G}$, $V(\boldsymbol{F}) = V(C) \cup \{w\}$. Furthermore, suppose that a plane graph $\boldsymbol{H}$ is a semi-cover of $\boldsymbol{G}$, and that φ is the corresponding semi-projection. A connected component $\boldsymbol{N}$ of the lifting of $\boldsymbol{F}$ into $\boldsymbol{H}$ is called a (C, w)*-necklace*, if $\boldsymbol{N}$ is C-fixed, and the restriction of φ to $V(\boldsymbol{N})$ is a projection onto $\boldsymbol{F}$ (and hence $\boldsymbol{N}$ is actually a cover of $\boldsymbol{F}$). Let the lifting of C into $\boldsymbol{N}$ consist of l facial 4-cycles $C^1, C^2, \ldots, C^l$; then these cycles are called the *beads* of the necklace, and l is the *length* of the necklace. (See Fig. 2 for an illustration.) The finite face of $\boldsymbol{N}$ not bounded by any of the beads is called the *interior* of $\boldsymbol{N}$.

Let C_1, C_2 be two induced 4-cycles in a graph $\boldsymbol{G}$. The graph $\boldsymbol{G}$ is said to have the (C_1, C_2, w)*-necklace property* if

- the sets $V(C_1), V(C_2), \{w\}$ are pairwise disjoint, and $V(C_1) \cup V(C_2) \cup \{w\} = V(\boldsymbol{G})$, i.e. $\boldsymbol{G}$ has 9 vertices;

– for $i = 1, 2$, the vertices of C_i can be denoted by a, b, c, d in this cyclic order so that aw, cw are edges of $\boldsymbol{G}$, and that each of b, d is adjacent by an edge to exactly one vertex of the other cycle C_{3-i} in $\boldsymbol{G}$.

Examples of two graphs having the necklace property are shown in Fig. 3. (At the first look, the necklace property may seem to be similar to the property of "having two disjoint k-graphs", as defined in [3]. However, unlike the latter one, the necklace property may hold also for some projective-planar graphs, such as for the right-hand side graph in Fig. 3.)

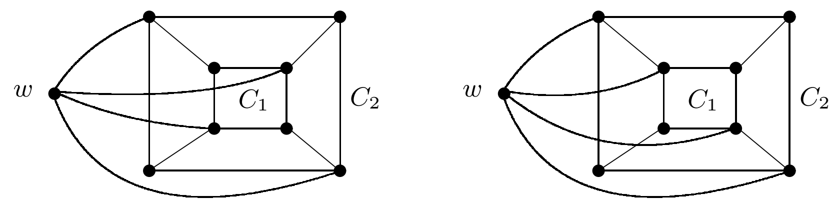

Fig. 3. Examples of two graphs having the necklace property.

The arguments given in this section generalize the proofs used by Archdeacon [2] to show that the graphs $\boldsymbol{K}_7 - \boldsymbol{C}_4$ and $\boldsymbol{K}_{4,5} - \boldsymbol{M}_4$ (Fig. 3 left) have no planar covers. From now on, it is assumed that $\boldsymbol{G}$ is a graph having a (C_1, C_2, w)-necklace property. Let a connected plane graph $\boldsymbol{H}$ be a semi-cover of $\boldsymbol{G}$. Suppose that $V(\boldsymbol{H})$ can be partitioned into the vertex set of a (C_1, w)-necklace $\boldsymbol{N}$, and the vertex set of a C_2-fixed lifting of C_2 into $\boldsymbol{H}$. Furthermore, suppose that the only vertices incident with the outer face of $\boldsymbol{H}$ are those of $\boldsymbol{N}$, i.e. $\boldsymbol{H} - V(\boldsymbol{N})$ is embedded in the interior of the necklace $\boldsymbol{N}$. Then $\boldsymbol{H}$ is called a *reduced semi-cover* of $\boldsymbol{G}$ bounded by the necklace $\boldsymbol{N}$.

Lemma 2.1. *Suppose that a graph $\boldsymbol{G}$ has a (C_1, C_2, w)-necklace property. If $\boldsymbol{G}$ has a planar cover, then, for some $i \in \{1, 2\}$, $\boldsymbol{G}$ has a reduced semi-cover bounded by a (C_i, w)-necklace.*

Proof. Let a connected plane graph $\boldsymbol{H}$ be a cover of $\boldsymbol{G}$, and let $\boldsymbol{H}^o$ denote the lifting of the graph $C_1 \cup C_2$ into $\boldsymbol{H}$. Clearly, $\boldsymbol{H}^o$ is a collection of disjoint cycles of $\boldsymbol{H}$. Notice that if C' is a cycle in the lifting of C_i, and C' is longer than C_i, then the cover projection "winds" C' several times around C_i. So if C' is a face of $\boldsymbol{H}$, it can be easily broken down into facial 4-cycles covering C_i. Hence it may be assumed that the cycles of $\boldsymbol{H}^o$ that are faces in $\boldsymbol{H}$ have length 4, and that they bound finite faces. If it happens that all cycles of $\boldsymbol{H}^o$ are faces in $\boldsymbol{H}$, then the arguments in the next paragraph may be skipped since the lifting of $\boldsymbol{G}'$ into $\boldsymbol{H}$ (see below) consists only of (C_1, w)-necklaces.

Let C^o denote some cycle of $\boldsymbol{H}^o$ that bounds an innermost open disc containing at least one vertex of $\boldsymbol{H}$. By the previous assumption, the subgraph embedded inside C^o is C_1- and C_2-fixed. Without loss of generality, we may assume that C^o belongs to the lifting of C_2. Let $C_1 = abcd$, and let $\boldsymbol{G}'$ be the subgraph of $\boldsymbol{G}$ with vertex-set $V(C_1) \cup \{w\}$ and edge-set $E(C_1) \cup \{aw, cw\}$. (See the definition of the (C_1, C_2, w)-necklace property.) Now, since $\boldsymbol{G} - V(C_2) \supseteq \boldsymbol{G}'$ is connected, and since C^o is not a face of $\boldsymbol{H}$, some component $\boldsymbol{N}$ of the lifting of $\boldsymbol{G}'$ into $\boldsymbol{H}$ must be embedded inside C^o. Hence $\boldsymbol{N}$ is a (C_1, w)-necklace.

Let $\boldsymbol{N}^o$ be a (C_1, w)-necklace inside C^o with minimal interior with respect to inclusion. Then all vertices in the interior of $\boldsymbol{N}^o$ belong to the lifting of C_2 (which is C_2-fixed); otherwise there would be a (C_1, w)-necklace with its interior properly contained in the interior of $\boldsymbol{N}^o$, by repeating the previous argument. Thus $\boldsymbol{N}^o$ bounds a reduced semi-cover of $\boldsymbol{G}$. ∎

Lemma 2.2. *Suppose that a graph $\boldsymbol{G}$ has a (C_1, C_2, w)-necklace property. If there exists, for some $i \in \{1, 2\}$, a reduced semi-cover of $\boldsymbol{G}$ bounded by a (C_i, w)-necklace of length $l > 2$, then there exists a reduced semi-cover of $\boldsymbol{G}$ bounded by a (C_i, w)-necklace of length smaller than l.*

Proof. The proof of this lemma is the heart of our argument. Without loss of generality, we may assume that $\boldsymbol{H}$ is a reduced semi-cover of $\boldsymbol{G}$ bounded by a (C_1, w)-necklace $\boldsymbol{N}$, and $\varphi : V(\boldsymbol{H}) \to V(\boldsymbol{G})$ is the semi-projection. Let the vertices of C_1 be a, b, c, d in this cyclic order so that $\boldsymbol{N}$ is the lifting of $(V(C_1) \cup \{w\}, E(C_1) \cup \{aw, cw\})$. Notice that if C^1, C^2 are two beads of the necklace $\boldsymbol{N}$, and v_1, v_2 are the vertices of C^1, C^2, respectively, encountered first when traversing the necklace in the clockwise orientation, then $\varphi(v_1) = \varphi(v_2) \in \{a, c\}$.

Since $\boldsymbol{N}$ is not outerplanar, and $\boldsymbol{G} - \{a, c, w\} \supset C_2$ is connected, there exists some cycle C' in $\boldsymbol{H}$ with $\varphi(C') = C_2$. From the necklace property of $\boldsymbol{G}$ it follows that there exists exactly one vertex $b_1 \in V(\boldsymbol{N})$ such that $\varphi(b_1) = b$, and that b_1 is adjacent to some vertex of C'. Similarly, there exists exactly one vertex $d_2 \in V(\boldsymbol{N})$, $\varphi(d_2) = d$ adjacent to some vertex of C'. Let C^1 and C^2 be the beads of $\boldsymbol{N}$ such that $b_1 \in V(C^1)$ and $d_2 \in V(C^2)$. Clearly, $C^1 \neq C^2$ since $\boldsymbol{H}$ is a plane graph. The subgraph of $\boldsymbol{H}$ induced on $V(C') \cup \{b_1, d_2\}$ is denoted by $\boldsymbol{B}$ (for "bridge").

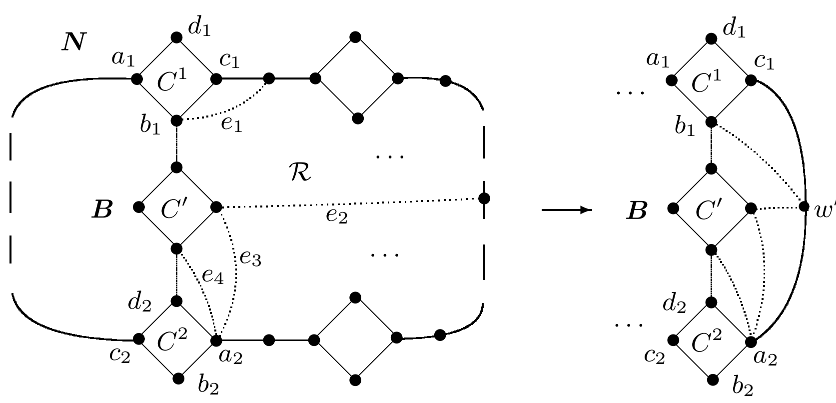

Fig. 4.

Since the length of the necklace $\boldsymbol{N}$ is greater than two, one of the two regions that $\boldsymbol{B}$ separates the interior of $\boldsymbol{N}$ into, say $\mathcal{R}$, has at least one bead other than C^1, C^2 on its boundary. The left-hand side of Fig. 4 illustrates the situation. Notice, however, that the other ends of edges joining b_1 and d_2 with the cycle C' (according to the necklace property of $\boldsymbol{G}$) need not be diagonally opposite on C'. Let the vertices of C^1 be a_1, d_1, c_1, b_1, the vertices of C^2 be a_2, b_2, c_2, d_2 (both in clockwise orientation), and $\varphi(c_1) = c$. Then it follows that $\varphi(a_2) = a$.

Let $e_1, e_2, \ldots, e_k$ be the edges that have exactly one endpoint in $V(\boldsymbol{B})$ and that belong to the interior of $\mathcal{R}$, ordered by their appearance on the boundary of $\mathcal{R}$ from b_1 to d_2. Let $u_1, \ldots, u_k$ be the ends of $e_1, \ldots, e_k$, respectively, not in $V(\boldsymbol{B})$. Since the subgraph $\boldsymbol{B}$ is actually isomorphic to $\boldsymbol{G} - \{a, c, w\}$, it follows that $\varphi(u_i) \in \{a, c, w\}$ for $i = 1, \ldots, k$, and hence $u_1, \ldots, u_k$ are incident with the outer face of $\boldsymbol{H}$. Now, suppose that there exist $0 \le i \le j \le k$ such that $\varphi(u_1) = \ldots = \varphi(u_i) = c$, $\varphi(u_{i+1}) = \ldots = \varphi(u_j) = w$, and $\varphi(u_{j+1}) = \ldots = \varphi(u_k) = a$. In such a case, the part of $\boldsymbol{H}$ embedded in $\mathcal{R}$ is deleted, and the section of the bounding necklace between c_1 and a_2 is replaced by a new path $c_1 w' a_2$. Instead of the edges $e_1, \ldots, e_k$, corresponding new edges $e'_1, \ldots, e'_k$ between vertices of $\boldsymbol{B}$ and $\{c_1, w', a_2\}$ are drawn, as needed. Clearly, no multiple edges are created, and the new graph $\boldsymbol{H}'$ is a reduced semi-cover of $\boldsymbol{G}$ bounded by a necklace of shorter length.

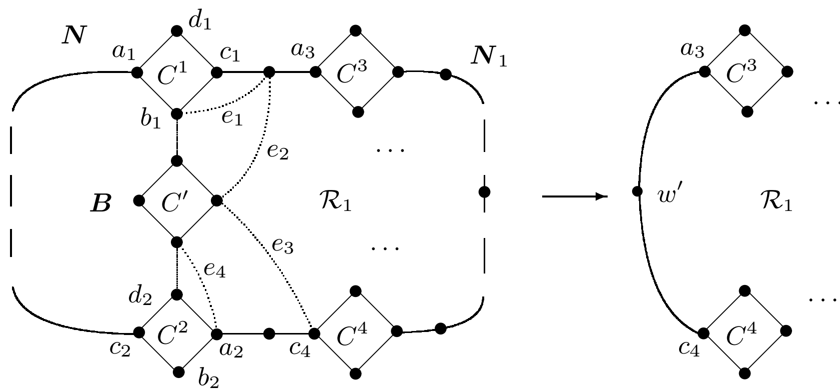

Fig. 5.

Otherwise, that is, if the above case does not happen, then there exists $1 \le i < k$ such that $\varphi(u_i) \in \{w, a\}$, $\varphi(u_{i+1}) \in \{c, w\}$, and $\varphi(u_i) \ne \varphi(u_{i+1})$. (See Fig. 5, where $\varphi(u_2) = w$, $\varphi(u_3) = c$.) Each of the edges e_i, e_{i+1} separates $\mathcal{R}$ into two regions, and e_i, e_{i+1} are disjoint up to possible common endvertex in $\boldsymbol{B}$. Let $\mathcal{R}_1$ be the connected component of the set $\mathcal{R} - e_i - e_{i+1}$ incident with both e_i, e_{i+1}, and let $\boldsymbol{N}_1$ be the section of $\boldsymbol{N}$ incident with the boundary of $\mathcal{R}_1$. In this case, by the choice of u_i, u_{i+1}, $\boldsymbol{N}_1$ must contain at least one bead. So let C^3, C^4 be the ending beads of $\boldsymbol{N}_1$, and let a_3, c_4 be the vertices of C^3, C^4 closest to u_i, u_{i+1}, respectively. (Notice that C^3, C^4 are not necessarily next to C^1, C^2, and they may be equal.) Then at most one of the vertices u_i, u_{i+1}, say u_i, is distinct from both a_3, c_4, and if this happens, then $\varphi(u_i) = w$. A new graph $\boldsymbol{H}'$ is formed as the part of $\boldsymbol{H}$ embedded in the region $\mathcal{R}_1$, bounded by the section $\boldsymbol{N}_1$ and a new path $c_4 w' a_3$. Possible edges between some vertices of $\boldsymbol{H}'$ and u_i if $u_i \neq a_3, c_4$ are rerouted to the endvertex w'. Again, $\boldsymbol{H}'$ is clearly a reduced semi-cover of $\boldsymbol{G}$ bounded by a necklace of shorter length than the length of $\boldsymbol{N}$. ∎

Lemma 2.3. *Suppose that a graph $\boldsymbol{G}$ has a (C_1, C_2, w)-necklace property. If $\boldsymbol{G}$ has a planar cover, then $\boldsymbol{G}$ has an embedding in the projective plane.*

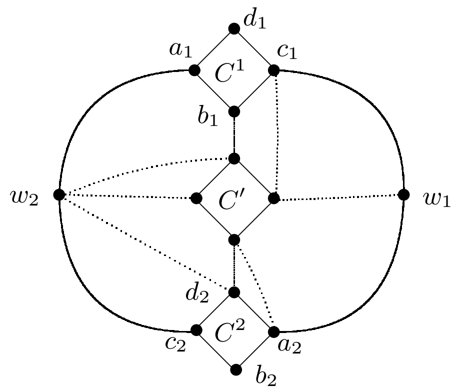

Fig. 6.

Proof. By Lemma 2.1, there exists a reduced semi-cover of $\boldsymbol{G}$ bounded by a (C_i, w)-necklace for some $i \in \{1, 2\}$, so assume that $i = 1$. By repeatedly applying Lemma 2.2, it can be deduced that there exists a reduced semi-cover $\boldsymbol{H}_2$ of $\boldsymbol{G}$, $\varphi : V(\boldsymbol{H}_2) \to V(\boldsymbol{G})$, bounded by a (C_1, w)-necklace of length at most two. As it was implicitly found in the proof of Lemma 2.2, the lifting of C_1 into $\boldsymbol{H}_2$ consists of two cycles C^1, C^2, and the lifting of C_2 is a single cycle C' in the interior of the necklace. Let the vertices of C^1, C^2 be $C^1 = a_1 b_1 c_1 d_1$ and $C^2 = a_2 d_2 c_2 b_2$ in the counterclockwise orientation so that b_1, d_2, $\varphi(b_1) = b, \varphi(d_2) = d$ are the vertices not incident with the outer face of $\boldsymbol{H}_2$, and hence adjacent to the cycle C'. Let w_1 be the common neighbor of c_1, a_2, and w_2 be the common neighbor of c_2, a_1, in the bounding necklace. (See Fig. 6 for an example.)

An embedded projective-planar graph $\boldsymbol{H}_p$ is formed from $\boldsymbol{H}_2$ by deleting the vertices d_1, b_2, and identifying the opposite pairs $w_1 = w_2$, $a_1 = a_2$, $c_1 = c_2$. Let φ' be the restriction of φ onto $V(C') \cup \{a_1, b_1, c_1, w_1, d_2\}$. It is claimed that $\varphi' : V(\boldsymbol{H}_p) \to V(\boldsymbol{G})$ is an isomorphism. Indeed, the vertices of $V(C') \cup \{b_1, d_2\}$ are not incident with the outer face of $\boldsymbol{H}_2$, and hence they are incident with all the required edges in the isomorphism relation by the definition of φ and φ'. In particular, all the required edges between the sets $V(C') \cup \{b_1, d_2\}$ and $\{a_1, c_1, w_1\}$ are present also in $\boldsymbol{H}_p$, and the edges $a_1 w_1, c_2 w_1$ are in $\boldsymbol{H}_p$ as well. (Recall that there is no edge between a, c in $\boldsymbol{G}$.) ∎

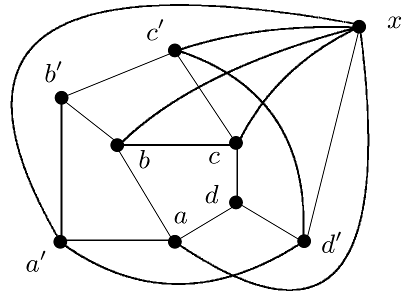

Fig. 7. The graph $\mathcal{C}_4$.

Proof of Theorem 3(a). Let the vertices of the graph $\mathcal{C}_4$ be denoted by $a, b, c, d, a', b', c', d', x$ as depicted in Fig. 7. It is easy to verify that $\mathcal{C}_4$ has the (C_1, C_2, x)-necklace property for $C_1 = abcd$ and $C_2 = a'b'c'd'$. By [3], $\mathcal{C}_4$ has no projective embedding, and hence it has no planar cover by Lemma 2.3. ∎

Remark. It is possible to generalize the definition of a necklace, allowing it to be a lifting of a subgraph isomorphic to $\boldsymbol{K}_4$, with pairs of edge-sharing facial triangles as the beads. (A variant that was used by Archdeacon in [2].) Then the above arguments work as well, and they also include the graph $\boldsymbol{K}_7 - \boldsymbol{C}_4$.

Remark. It is likely that Lemma 2.3 can be proved for much wider definition of a necklace, assuming more cycles of different sizes to be potential beads of a necklace, and allowing more additional vertices and interconnecting edges. Unfortunately, this does not seem to be useful for any one of the 16 graphs that might be (up to obvious constructions) possible counterexamples to Negami's conjecture [6].

3 The graph $\mathcal{E}_2$

Let the vertices of $\mathcal{E}_2$ be denoted by $a_1, a_2, a_3, a_4,\ b_1, b_2, b_3, b_4, b_5, b_6,\ x$, as in Fig. 8. Suppose, for a contradiction, that there exists a connected planar cover $\boldsymbol{H}$ of $\mathcal{E}_2$, determined by a projection $\varphi : V(\boldsymbol{H}) \to V(\mathcal{E}_2)$. The graph $\boldsymbol{H}$ is treated as a plane graph here. Clearly, $|\varphi^{-1}(v)|$ is the same number for every vertex $v \in V(\mathcal{E}_2)$, so let us denote this number by n.

If $\boldsymbol{G}$ is a plane graph and u is a vertex of degree 2 with neighbors v, v' in $\boldsymbol{G}$, then *suppressing* u in $\boldsymbol{G}$ means deleting the vertex u and adding a new edge vv' drawn along the original path vuv'. A plane graph $\boldsymbol{H}'$ on the vertex set $V(\boldsymbol{H}') = \varphi^{-1}(a_1) \cup \varphi^{-1}(a_2) \cup \varphi^{-1}(a_3) \cup \varphi^{-1}(a_4)$ is constructed from $\boldsymbol{H}$ by deleting all vertices u of $\boldsymbol{H}$ for which $\varphi(u) = x$, and by suppressing all vertices w of $\boldsymbol{H}$ such that $\varphi(w) \in \{b_1, b_2, b_3, b_4, b_5, b_6\}$. Our proofs need to work with a connected graph, while $\boldsymbol{H}'$ may not be connected. So we define a graph $\boldsymbol{H}_4$ as an arbitrary connected component of $\boldsymbol{H}'$ which bounds an innermost region of the plane (i.e. $\boldsymbol{H}' - V(\boldsymbol{H}_4)$ is embedded in the outer face of $\boldsymbol{H}_4$).

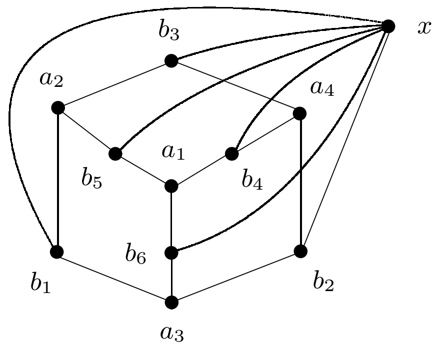

Fig. 8. The graph $\boldsymbol{\mathcal{E}}_2$.

Let the mappings $\psi : V(\boldsymbol{H}_4) \to \{1,2,3,4\}$, $\eta : E(\boldsymbol{H}_4) \to \{1,2,\dots,n\}$, and $\lambda : \{1,2,\dots,n\} \to F(\boldsymbol{H}_4)$ be defined by the following rules: For a vertex v of $\boldsymbol{H}_4$, let $\psi(v) = i$ if $\varphi(v) = a_i$. Assuming $\varphi^{-1}(x) = \{x_1, x_2, \dots, x_n\}$, define $\eta(e) = j$ if e is an edge of $\boldsymbol{H}_4$ that was formed by suppressing a vertex $w \in V(\boldsymbol{H})$ with $wx_j \in E(\boldsymbol{H})$. For $1 \le j \le n$, define $\lambda(j) = f$ if f is a face of $\boldsymbol{H}_4$ containing x_j. (Notice that x_j is a vertex of $\boldsymbol{H}$, but not of $\boldsymbol{H}_4$, so x_j lies inside some face of $\boldsymbol{H}_4$.) Let $m = |\psi^{-1}(1)|$. It follows easily from the fact that $\boldsymbol{H}$ covers $\boldsymbol{\mathcal{E}}_2$ that these mappings are well-defined. After all, the mapping ψ may be viewed as an m-fold covering projection of $\boldsymbol{H}_4$ onto a complete graph on vertices $1,2,3,4$, and the mappings η and λ describe "relations" of edges and faces of $\boldsymbol{H}_4$ to the vertices $x_1, \dots, x_n$ of $\boldsymbol{H}$.

Lemma 3.1. *The plane graph $\boldsymbol{H}_4$, and the mappings ψ, η, λ, satisfy the following properties.*

(a) $\boldsymbol{H}_4$ is a simple 2-connected 3-regular graph on $4m$ vertices, and ψ is a cover projection of $\boldsymbol{H}_4$ onto $\boldsymbol{K}_4$, the complete graph on the vertex set $\{1,2,3,4\}$. In particular, any two vertices $v \ne w$ of $\boldsymbol{H}_4$ satisfying $\psi(v) = \psi(w)$ must be at distance of at least 3 apart.

(b) If e is an edge of $\boldsymbol{H}_4$, then $\lambda(\eta(e))$ is a face incident with e. Consequently, for a face f and $j \in \lambda^{-1}(f)$, the edges of $\eta^{-1}(j)$ lie on the boundary of f.

(c) If $j \in \{1,2,\dots,n\}$ such that $\lambda(j)$ is a finite face of $\boldsymbol{H}_4$, then $\eta^{-1}(j)$ has six elements, and $\{\{\psi(u),\psi(v)\} : uv \in \eta^{-1}(j)\}$ is the collection of all six two-element subsets of $\{1,2,3,4\}$.

(d) Let f be a face of $\boldsymbol{H}_4$, and $j_1, j_2 \in \lambda^{-1}(f)$. If e_1, e_2, e_3, e_4 are four edges of f in this cyclic order (not necessarily consecutive), and $\eta(e_1) = \eta(e_3) = j_1$, $\eta(e_2) = \eta(e_4) = j_2$, then $j_1 = j_2$.

Proof. (a) All these properties, except the first one, follow immediately from the definition of $\boldsymbol{H}_4$. Since $\boldsymbol{H}_4$ is connected and 3-regular, it is enough to show that it is 2-edge-connected. Indeed, for each edge uv of $\boldsymbol{H}_4$ there is a triangle C in $\boldsymbol{K}_4$ containing the vertices $\psi(u), \psi(v)$. Hence uv belongs to a cycle that is a component of the lifting of C into $\boldsymbol{H}_4$. Thus $\boldsymbol{H}_4$ is 2-edge-connected.

(b) Let $w \in V(\boldsymbol{H})$ be the vertex that was suppressed when forming the edge e, and let f, f' be the two faces of $\boldsymbol{H}_4$ incident with e. By definition, $\eta(e) = j$ if and only if wx_j is an edge in $\boldsymbol{H}$ ($\varphi(x_j) = x$). Since $\boldsymbol{H}$ is planar, the vertex x_j is embedded in one of the faces f, f' of $\boldsymbol{H}_4$, thus $\lambda(\eta(e)) \in \{f, f'\}$.

(c) The vertex x_j in $\boldsymbol{H}$ has six neighbors $w_1, \dots, w_6$, where $\varphi(w_i) = b_i$. Let $e_1, \dots, e_6$ denote the edges of $\boldsymbol{H}'$ formed by suppressing the vertices $w_1, \dots, w_6$, respectively. Since x_j lies in a finite face of $\boldsymbol{H}_4$, all edges $e_1, \dots, e_6$ belong to $\boldsymbol{H}_4$ from planarity, and hence $\eta^{-1}(j) = \{e_1, \dots, e_6\}$. Moreover, each of the six vertices $b_1, \dots, b_6$ of $\boldsymbol{\mathcal{E}}_2$ has a different pair of vertices a_1, a_2, a_3, a_4 as neighbors. Therefore $\{\psi(u_i), \psi(v_i)\}$ for $e_i = u_i v_i$, $i = 1, \dots, 6$ are six different pairs of numbers from $1, 2, 3, 4$.

(d) Let w_1, w_2, w_3, w_4 be the vertices that were suppressed when forming the edges e_1, e_2, e_3, e_4, respectively, and let C be the cycle in $\boldsymbol{H}$ corresponding to the boundary of f. If $j_1 \neq j_2$, then $\{w_1, x_{j_1}, w_3\}$ and $\{w_2, x_{j_2}, w_4\}$ are the vertex sets of two disjoint paths embedded in the same face of C. However, this contradicts planarity of $\boldsymbol{H}$ since w_1, w_2, w_3, w_4 lie in this cyclic order on the boundary of C. ∎

A discharging argument is used to show that the graph $\boldsymbol{H}_4$ and the mappings ψ, η, λ with the properties described by Lemma 3.1 cannot exist. Generally, a *discharging argument* first assigns certain *charge* to vertices, edges, and/or faces of a graph, then it redistributes the charge according to specified *discharging rules*, and finally it shows that the total sum of charge has changed, which leads to a contradiction. In this particular case, the starting charges and the discharging rules are defined as follows.

Initial charges. Each face f of $\boldsymbol{H}_4$ starts with a charge of $3k$, where k is the length of f. All edges of $\boldsymbol{H}_4$ start with no charge.

Discharging rules. For any face f of $\boldsymbol{H}_4$, and for any four consecutive vertices u_1, u_2, u_3, u_4 on the boundary of f such that $\psi(u_1) = \psi(u_4)$ (possibly $u_1 = u_4$), the following rule applies: If $\lambda(\eta(u_2u_3)) = f$, then the edge u_2u_3 receives a charge of 1 from f, otherwise u_2u_3 sends a charge of 1 to f. (See also Fig. 9.)

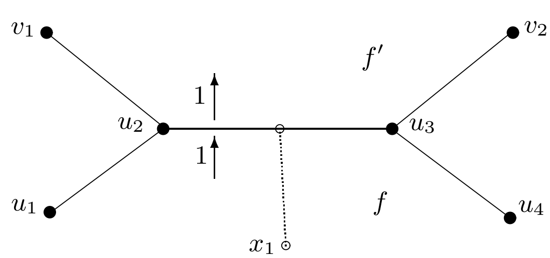

Fig. 9. An illustration of the discharging rule, $\psi(u_1) = \psi(u_4)$, $\lambda(\eta(u_2u_3)) = f$.

Lemma 3.2. *Each edge of $\boldsymbol{H}_4$ ends up with a charge of* 0.

Proof. Let e be an edge of $\boldsymbol{H}_4$. By Lemma 3.1(b), $\lambda(\eta(e)) = f$ is a face incident with e. Let u_1, u_2, u_3, u_4 denote four consecutive vertices of f such that $e = u_2u_3$. Let f' denote the other face incident with e, and let v_1, u_2, u_3, v_2 be four consecutive vertices of f'. (See Fig. 9.) By Lemma 3.1(a), $\psi(u_1), \psi(u_2), \psi(u_3), \psi(v_1)$ form a permutation of $1, 2, 3, 4$. Similarly, $\psi(u_4), \psi(u_2), \psi(u_3), \psi(v_2)$ form a permutation of $1, 2, 3, 4$. Thus $\psi(u_1) = \psi(u_4)$ if and only if $\psi(v_1) = \psi(v_2)$. So if $\psi(u_1) \neq \psi(u_4)$, then no discharging rule applies to e. If $\psi(u_1) = \psi(u_4)$, then $\psi(v_1) = \psi(v_2)$. Therefore the edge e receives a charge of 1 from the face f and sends a charge of 1 to f', and hence it ends up with no charge. ∎

Since $\boldsymbol{H}_4$ is a 2-connected graph, each face is bounded by a cycle of length of at least 3. In order to use induction in the proof of the next lemma, the assumptions about the graph $\boldsymbol{H}_4$ need to be restricted to each face of $\boldsymbol{H}_4$ alone. The following claim is an immediate corollary of Lemma 3.1.

Claim 1. Suppose that $\boldsymbol{C}$ is the cycle bounding a finite face f of $\boldsymbol{H}_4$. Let $Y \subseteq E(\boldsymbol{C})$ be the set defined by $Y = \eta^{-1}(\lambda^{-1}(f))$. Let ψ' be the restriction of ψ to $V(\boldsymbol{C})$, and let η' be the restriction of η to Y.

(a) If $v \neq w$ are two vertices of $\boldsymbol{C}$, and $\psi'(v) = \psi'(w)$, then the distance between v, w is at least 3.

(b) For $j \in \eta'(Y)$, $\eta'^{-1}(j)$ is a set of six edges of $\boldsymbol{C}$, and $\{\{\psi'(u), \psi'(v)\} : uv \in \eta'^{-1}(j)\}$ is the collection of all six two-element subsets of $\{1, 2, 3, 4\}$. (The symbol $f(A)$ stands for the image of A under f.)

(c) If $e_1, e_2, e_3, e_4 \in Y$ are four edges of the cycle $\boldsymbol{C}$ in this cyclic order (not necessarily consecutive), and $\eta'(e_1) = \eta'(e_3) = j_1$, $\eta'(e_2) = \eta'(e_4) = j_2$, then $j_1 = j_2$.

The discharging rules are reformulated for the cycle $\boldsymbol{C}$ (which stands for the cycle bounding f now), the set Y, and the mapping ψ' as follows:

Claim 2. The cycle $\boldsymbol{C}$ starts with a charge of $3|V(\boldsymbol{C})|$. Whenever u_1, u_2, u_3, u_4 are four consecutive vertices of $\boldsymbol{C}$ (possibly $u_1 = u_4$) such that $\psi'(u_1) = \psi'(u_4)$, the edge u_2u_3 receives a charge of 1 from $\boldsymbol{C}$ if $u_2u_3 \in Y$, and u_2u_3 sends a charge of 1 to $\boldsymbol{C}$ else.

Lemma 3.3. *Suppose that a cycle $\boldsymbol{C}$ of the length at least 3, a set $Y \subseteq E(\boldsymbol{C})$, and mappings $\psi' : V(\boldsymbol{C}) \to \{1, 2, 3, 4\}$, $\eta' : Y \to \{1, \ldots, n\}$ satisfy the conditions described by Claim 1. If the discharging rules from Claim 2 are applied to $\boldsymbol{C}$, then $\boldsymbol{C}$ ends up with a charge of at least $12 \cdot |\,\eta'(Y)\,| + 12$.*

Proof. Let $k = |V(\boldsymbol{C})|$, and $p = |\,\eta'(Y)\,|$. Notice that $|Y| = 6p$ by Claim 1(b). So the charge of $\boldsymbol{C}$ may decrease by at most $6p$ in the discharging process. If $k \geq 6p + 4$, then $3k - 6p \geq 12p + 12$, and hence the lemma holds. Thus it is necessary to consider only cycles with $k \leq 6p + 3$. If $p = 0$, then $|Y| = 0$ and $k = 3$, so $\boldsymbol{C}$ is a triangle. In such a case, $u_1 = u_4$ holds for any four consecutive vertices u_1, u_2, u_3, u_4 of $\boldsymbol{C}$, so $\boldsymbol{C}$ receives a charge of 1 from each of its edges. Therefore it ends up with a charge of $9 + 3 = 12$, as desired.

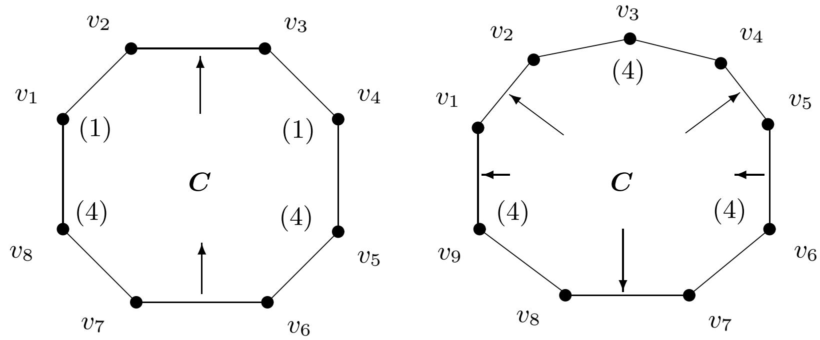

Fig. 10.

The rest of the statement is proved by induction on $p \geq 1$. The base case $p = 1$ needs to be considered for cycles of length $k \leq 9$. Let the vertices of $\boldsymbol{C}$ be denoted by $v_1, v_2, \ldots, v_k$ in order, see Fig. 10. By Claim 1(b), $\{\psi'(v_1), \psi'(v_2)\}$, $\{\psi'(v_2), \psi'(v_3)\}$, $\ldots, \{\psi'(v_k), \psi'(v_1)\}$ include all two-element subsets of $\{1, 2, 3, 4\}$. In other words, $\psi'(v_1)\psi'(v_2)\ldots\psi'(v_k)\psi'(v_1)$ is a closed walk in $\boldsymbol{K}_4$ visiting all edges. Hence, in particular, each of the four ψ'-values $1, 2, 3, 4$ occurs at least twice among the vertices of $\boldsymbol{C}$, so the length of $\boldsymbol{C}$ is 8 or 9.

Consider first $k = 8$ (Fig. 10 left). Assume, without loss of generality, that the edge v_2v_3 receives a charge of 1 from $\boldsymbol{C}$, so $v_2v_3 \in Y$, $\psi'(v_1) = \psi'(v_4) = 1$, and $\{\psi'(v_2), \psi'(v_3)\} = \{2, 3\}$. Since the ψ'-values of two of the vertices v_5, v_6, v_7, v_8 should be 4, Claim 1(a) implies that necessarily $\psi'(v_5) = \psi'(v_8) = 4$, and $\{\psi'(v_6), \psi'(v_7)\} = \{2, 3\}$. Now, since $\{\psi'(v_2), \psi'(v_3)\} = \{\psi'(v_6), \psi'(v_7)\}$, at most one of the edges v_2v_3, v_6v_7 may be in Y by Claim 1(b), so $v_6v_7 \notin Y$, and hence v_6v_7 sends a charge of 1 to $\boldsymbol{C}$. That means, whenever an edge e of $\boldsymbol{C}$ receives a charge from $\boldsymbol{C}$, then the edge opposite to e sends a charge to $\boldsymbol{C}$. Therefore $\boldsymbol{C}$ always ends up with at least the initial charge of $3 \cdot 8 = 24 = 12p + 12$, as required.

Consider $k = 9$ now (Fig. 10 right). In this case, one of the ψ'-values $1, 2, 3, 4$ occurs three times at distance three on the boundary of $\boldsymbol{C}$, so let it be $\psi'(v_3) = \psi'(v_6) = \psi'(v_9) = 4$. Then, whatever the other values of ψ' are, the edges v_1v_2, v_4v_5, v_7v_8 receive a charge of 1 each from $\boldsymbol{C}$. (Actually, all three edges v_1v_2, v_4v_5, v_7v_8 belong to Y by Claim 1(b).) If no other edge receives a charge from $\boldsymbol{C}$, then $\boldsymbol{C}$ ends up with a charge of $3 \cdot 9 - 3 = 24 = 12p + 12$, as desired. Otherwise, assume, without loss of generality, that the edge v_1v_9 receives a charge of 1 from $\boldsymbol{C}$, so $\psi'(v_2) = \psi'(v_8) = 1$. By Claim 1(a), there is only one possibility for the remaining values $\psi'(v_4) = \psi'(v_7) = 3$, $\psi'(v_1) = \psi'(v_5) = 2$, up to symmetry. Since $\{\psi'(v_1), \psi'(v_9)\} = \{\psi'(v_5), \psi'(v_6)\}$, the same argument as in the previous paragraph implies that the edge v_5v_6 sends a charge of 1 to $\boldsymbol{C}$. No discharging rule applies elsewhere, so $\boldsymbol{C}$ ends up with a charge of $3 \cdot 9 - 4 + 1 = 24 = 12p + 12$.

Assume the induction hypothesis. (Every cycle $\boldsymbol{C}$ satisfying our assumptions for some $p \geq 1$ ends up with a charge of at least $12p + 12$.) Let $\boldsymbol{C}$ be a cycle for which $|\,\eta'(Y)\,| = p+1 \geq 2$, and let $X = \eta'(Y)$. First, it is shown that there exist distinct $j_1, j_2 \in X$, and two disjoint paths $P_1 = s_1 s_2 \dots s_q \supseteq \eta'^{-1}(j_1)$, $P_2 = t_1 t_2 \dots t_{q'} \supseteq \eta'^{-1}(j_2)$ on the boundary of $\boldsymbol{C}$ such that, for $i = 1,2$, $P_i \cap \eta'^{-1}(k) = \emptyset$ whenever $k \in X - \{j_1, j_2\}$. Let $j_1 \in X$ be chosen such that the path P_1 in $\boldsymbol{C}$ has the smallest possible length. If there is some $j_1' \in X$ such that $\eta'^{-1}(j_1') \cap P_1 \neq \emptyset$, then $\eta'^{-1}(j_1')$ is strictly contained in P_1 by Claim 1(c), which is a contradiction to the choice of j_1. The other path P_2 is found in a similar way in $\boldsymbol{C} - V(P_1)$ (which is connected).

Similarly as in the base induction case, it follows from Claim 1(b) that $\psi'(s_1)\psi'(s_2)\dots\psi'(s_q)$ and $\psi'(t_1)\psi'(t_2)\dots\psi'(t_{q'})$ are walks (not necessarily closed) in $\boldsymbol{K}_4$, both visiting all of its edges. Hence, in particular, each of the ψ'-values $1,2,3,4$ occurs at least twice among the vertices of P_1 and of P_2, so $q, q' \geq 8$. And since $|\eta'^{-1}(j)| = 6$ for each $j \in X - \{j_1, j_2\}$, the length of $\boldsymbol{C}$ is at least $k \geq 6(p+1-2) + q - 1 + q' - 1 \geq 6(p+1) + 2$. (Recall that $k \leq 6(p+1)+3$ can be assumed.)

If $k = 6(p+1)+3$, then, without loss of generality, $q = 8$ and $q' \leq 9$. In the case when the *net charge* edges of P_1 receive from $\boldsymbol{C}$ (considering also a charge that some edges of P_1 might send to $\boldsymbol{C}$) is at most 3, the cycle $\boldsymbol{C}$ ends up with a charge of at least $3k - 6p - 3 = 12p + 24 = 12(p+1) + 12$, as desired. Similarly, if $k = 6(p+1)+2$, then $q = q' = 8$. In the case when the net charge edges of each of P_1, P_2 receive from $\boldsymbol{C}$ is at most 3, the cycle $\boldsymbol{C}$ ends up with a charge of at least $3k - 6(p-1) - 3 - 3 = 12p + 24 = 12(p+1) + 12$ again. Thus, up to symmetry, it remains to consider the case when the path P_1 of length 7 receives the net charge of at least 4 from the cycle $\boldsymbol{C}$ (regardless of k).

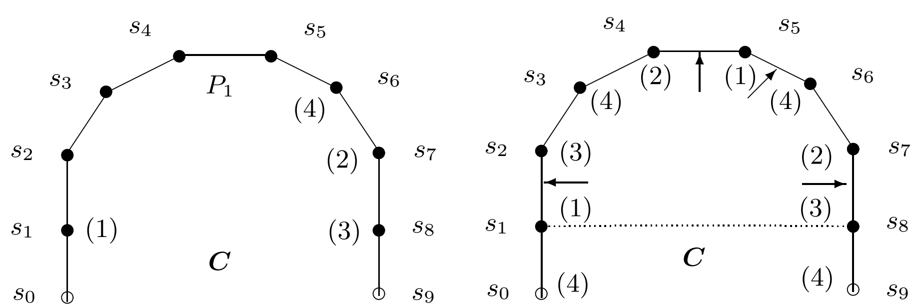

Fig. 11.

Recall that each of the ψ'-values $1,2,3,4$ occurs twice among the vertices of P_1. If $\psi'(s_1) = \psi'(s_8)$, then P_1 would correspond to a closed walk in $\boldsymbol{K}_4$ of length 7 visiting all edges, which is impossible. So assume, without loss of generality, that $\psi'(s_1) = 1$, $\psi'(s_8) = 3$, see Fig. 11. Since $\psi'(s_2) = 3$, $\psi'(s_7) = 1$ is not possible due to Claim 1(a), it can also be assume that $\psi'(s_7) = 2$. Now, if $\psi'(s_6) = 1$, then necessarily $\psi'(s_2) = \psi'(s_5) = 4$, and no edge of P_1 has values $\{1,3\}$, which is a contradiction to Claim 1(b). So $\psi'(s_6) = 4$. Further, the three possible values of $\psi'(s_2)$ are considered.

If $\psi'(s_2) = 4$, then $\psi'(s_3) \neq 1 \neq \psi'(s_5)$ by Claim 1(a) and (b), hence $\psi'(s_4) = 1$, $\psi'(s_3) = 2$ and $\psi'(s_5) = 3$ by Claim 1(a). In such situation, only the edges s_1s_2, s_7s_8, and one of s_2s_3, s_6s_7 may receive charges from $\boldsymbol{C}$, so edges of P_1 receive the net charge of at most 3 from $\boldsymbol{C}$, which is an already covered case. If $\psi'(s_2) = 2$, then $\psi'(s_3) = 4$, $\{\psi'(s_4), \psi'(s_5)\} = \{1, 3\}$, and the edges of P_1 again receive the net charge of at most 3 from $\boldsymbol{C}$: For $\psi'(s_4) = 3$ and $\psi'(s_5) = 1$, the discharging rule may apply only to s_1s_2, s_7s_8, and s_4s_5. For $\psi'(s_4) = 1$ and $\psi'(s_5) = 3$, the three edges s_1s_2, s_7s_8, s_4s_5, and one of s_2s_3, s_6s_7 may receive a charge of 1 from $\boldsymbol{C}$, but the other one of the edges s_2s_3, s_6s_7 sends a charge of 1 to $\boldsymbol{C}$ since it is not in Y by Claim 1(b). (So the net charge of at most $4 - 1 = 3$ is sent from $\boldsymbol{C}$.)

The remaining possibility is $\psi'(s_2) = 3$, hence $\psi'(s_3) = 4$, $\psi'(s_5) = 1$, and $\psi'(s_4) = 2$. In such a situation, $\boldsymbol{C}$ may send charge of up to 4 to the edges s_1s_2, s_4s_5, s_5s_6, s_7s_8 of P_1, provided that $\psi'(s_0) = 4$ and $\psi'(s_9) = 4$, see Fig. 11 right.

If the latter case happens, a new cycle $\boldsymbol{C}'$ is formed by replacing the path P_1 with the edge s_1s_8, a new set $Y' = Y - E(P_1)$ is defined, and the mappings ψ', η' are restricted to $E(\boldsymbol{C}')$, Y', respectively. It is easy to check that the conditions in Claim 1(b,c) are still satisfied for $\boldsymbol{C}'$ by the choice of P_1. Also, validity of Claim 1(a) is preserved in this special case. (See the picture.) Since $|\,\eta'(Y')\,| = |\,\eta'(Y)\,| - 1 = p$, the new cycle $\boldsymbol{C}'$ ends up with a charge of at least $12p + 12$ by the induction assumption. Now, the cycle $\boldsymbol{C}$ is longer by 6 than $\boldsymbol{C}'$, hence $\boldsymbol{C}$ starts with a charge larger by 18 than $\boldsymbol{C}'$ does. The same discharging rules (cf. Claim 2) apply in $\boldsymbol{C}$ as in $\boldsymbol{C}'$ to all edges of $\boldsymbol{C}'$ except for two, namely s_1s_8 and s_8s_9. (The edge s_1s_8 does not exist in $\boldsymbol{C}$, and s_8s_9 has a neighbor of a different ψ'-value in $\boldsymbol{C}$ than in $\boldsymbol{C}'$.) Additionally, exactly four edges of P_1 receive a charge of 1 from $\boldsymbol{C}$. Therefore the cycle $\boldsymbol{C}$ ends up with a charge of at least $(12p + 12) + 18 - 2 - 4 = 12(p + 1) + 12$, as desired. ∎

Corollary 3.4. *Each finite face f of $\boldsymbol{H}_4$ ends up with a charge of at least $12|\lambda^{-1}(f)| + 12$.*

Proof. Lemma 3.3 is applied to the cycle $\boldsymbol{C}$ bounding f, and to the set Y and the mappings ψ', η' defined as in Claim 1. Notice that $\lambda^{-1}(f) = \eta'(Y)$ by definition. ∎

Proof of Theorem 3(b). Since $\boldsymbol{H}_4$ is a 3-regular graph on $4m$ vertices, the total charge at the beginning is

$$3 \sum_{f \in F(H_4)} |f| = 3 \cdot 12m = 36m.$$

The number of faces of $\boldsymbol{H}_4$ is $2m+2$ by Euler's formula. It is a trivial observation that each face of $\boldsymbol{H}_4$, and hence also the outer one, ends up with a charge of at least twice bigger than its length. Let us denote the outer face of $\boldsymbol{H}_4$ by f_0, and let $\mathcal{F} = F(\boldsymbol{H}_4) - \{f_0\}$. If $L = \bigcup_{f \in \mathcal{F}} \lambda^{-1}(f)$, then all edges of $E(\boldsymbol{H}_4) - \eta^{-1}(L)$ belong to f_0 from definition, and so $|f_0| \geq 6m - 6|L|$ using Lemma 3.1(c).

Furthermore, by Lemma 3.2 and Corollary 3.4, the total sum of charges at the end of the discharging process is at least

$$0 + \sum_{f\in\mathcal{F}} \big(12|\lambda^{-1}(f)| + 12\big) + 2|f_0| = 12\cdot|\mathcal{F}| + 12\cdot\sum_{f\in\mathcal{F}} |\lambda^{-1}(f)| + 2|f_0| \geq$$

$$\geq 12\cdot|\mathcal{F}| + 12|L| + 2(6m - 6|L|) = 12(2m+2-1) + 12m = 36m + 12 > 36m.$$

However, the discharging process just redistributes existing charges, and no new charge is introduced during the process. This contradiction shows that the graph $\boldsymbol{H}_4$, and hence also a planar cover of $\boldsymbol{\mathcal{E}}_2$, cannot exist. ∎

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