Faster than Courcelle's thm. . .

. . . (really ???)

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- Yet, more on "optimality": cannot get much *above bd. tree-width*, for MSO_2 by [Kreutzer–Tazari], and col.-MSO_1 by [Ganian et al].

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MSO$_2$ on graphs: additionally using edges (and edge-set variables), and an $inc(x,e)$ predicate,

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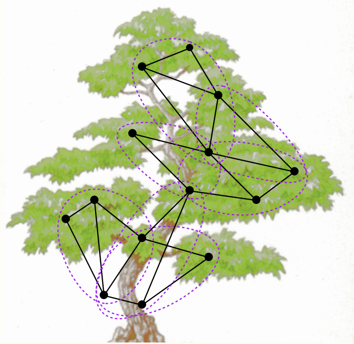
- can do Hamiltonian, spanning tree (MSO_2, but not MSO_1),
- and extensions can enumerate / optimize over solutions...

Courcelle's MSO_2 Theorem, once again

Tree-width $tw(G) \le k$ if whole G can be covered by bags of size $\le k+1$, arranged in a "tree-like fashion".



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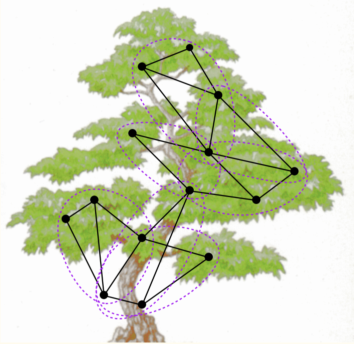

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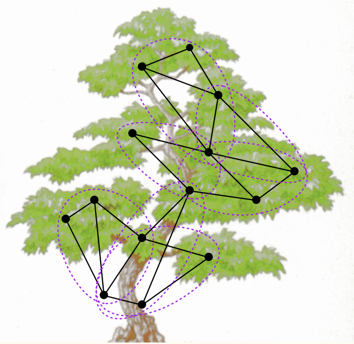

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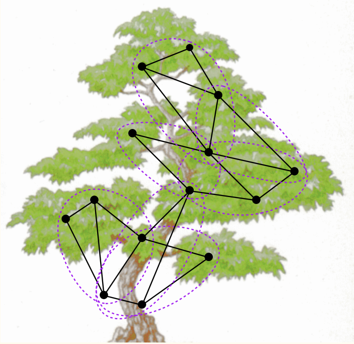

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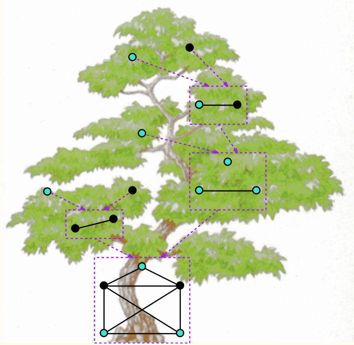
Theorem. (Courcelle)

Assume ϕ is an MSO_2 sentence, and G is of tree-width k, given along with a tree-decomposition. Then $G \models \phi$ can be decided by an FPT algorithm, in time $\mathcal{O}\big(g(k,\phi) \cdot |V(G)|\big)$ for some g.

Courcelle–Makowsky–Rotics MSO_1 Theorem

Clique-width $cwd(G) \leq k$ if G given by a k-*expression* (over k-labelled gr.),

k-expression $\sim$ disjoint unions, relabelling, edge-add. between labels.



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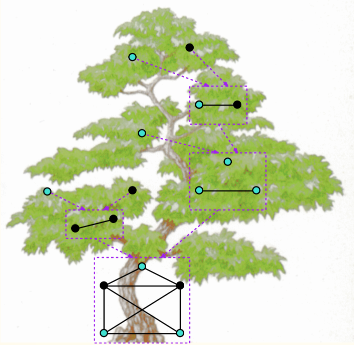

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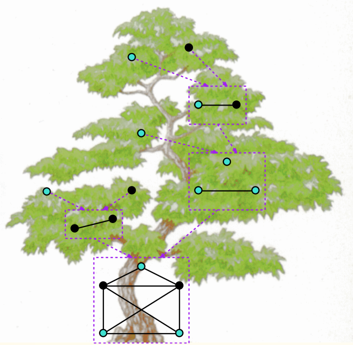

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Theorem. (Courcelle–Makowsky–Rotics)
Assume ψ is an MSO_1 sentence, and G is of clique-width k, given along with a k-expression. Then $G \models \psi$ can be decided by an FPT algorithm, in time $\mathcal{O}\big(g(k,\psi) \cdot |V(G)|\big)$ for some g.

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- tree-decomposition $\rightarrow$ bag intersections $\rightarrow$ tree edges, and
- MSO_2 sentence $\rightarrow$ MSO over the coloured tree.
- (Similarly for clique-width and $MSO_1 \dots$)

The conclusion. Enough to study MSO properties of coloured trees!

4 The Ground: Trees vs. Shrubs

Coloured MSO model checking in time. . .

$$|T| \cdot \left. 2^{2^{\cdot^{\cdot^{*}}}} \right\} \text{quant-alt}(\phi) \qquad \text{vs.} \qquad |T| + \left. 2^{2^{\cdot^{\cdot^{*}}}} \right\} \text{shrub height}$$

About the Shrub Case – FO

Claim. (almost folklore) A given FO sentence ϱ cannot distinguish too many copies of an arb. relational structure R.

$$R^+ = \quad \boxed{R} \; \boxed{R} \; \boxed{R} \; \dots \; \boxed{R} \; \boxed{R}$$

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$\leadsto$ a direct kernelization routine (sim. to tree isomorphism)

So, trim your Shrub

Corollary. For a given tree T of bounded height, there is (efficiently—the *kernel*) a subtree $T' \subseteq T$ such that $T \models \varrho \iff T' \models \varrho$, and T' is of bounded size, all for any FO ϱ.

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Hence there is a kernelization FPT algorithm with runtime

$$\mathcal{O}\big(|T| + |T'|^q\big) \quad \text{where } |T'| \sim 2^{2^{\cdot^{\cdot^{\cdot}}}}\Big\}{height}.$$

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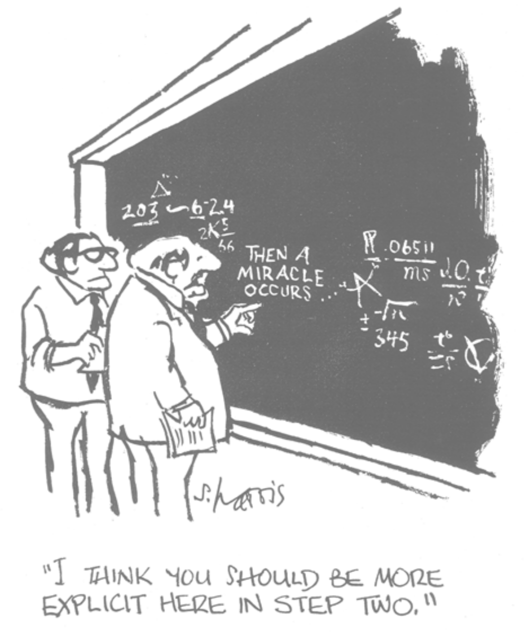

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Cons., the repetition threshold depends on ϕ and on the size of R.

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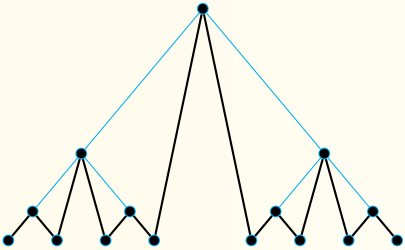

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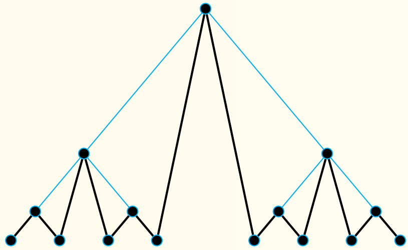

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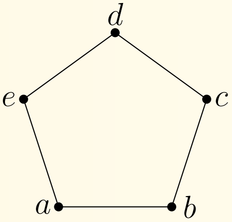

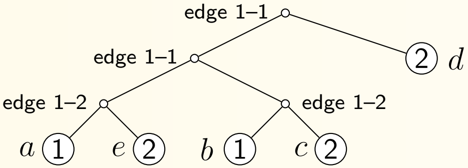

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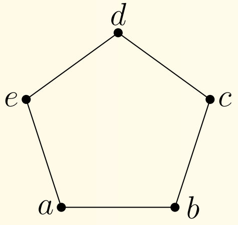

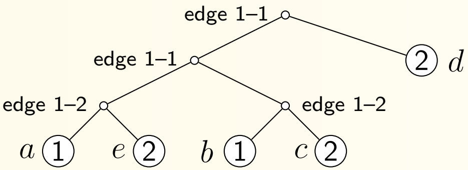

- *Shrub-depth* (of a graph class) = the smallest depth for which all the graphs are m-partite cographs (for some m).

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- Say, the *ECML logic* by [Michal Pilipczuk]. . .