Where Myhill–Nerode Theorem Meets Parameterized Algorithmics

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- Explicit comb. extensions of this concept appeared e.g. in the works [Abrahamson and Fellows, 93], [PH, 03], or [Ganian and PH, 08].

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- Informally, the classes of $\approx_{\mathcal{P},k}$ capture all information about the property $\mathcal{P}$ that can "cross" our boundary of size k
(regardless of actual meaning of "boundary" and "join").

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- For simplicity, solution fragments φ can be "embedded" in $\mathcal{U}_k$ and $\otimes$.
- Can, e.g., count the solutions in each class of $\approx_{\mathcal{P},k}$, or keep an opt. one.

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  – yes, still works quite nicely, cf. [Ganian, PH, Obdržálek, 09].

  – brings new application issues such as "quantification inside $\otimes$" (cf. sol. fragments), or a "second-level" congruence on top of $\approx_{\mathcal{P},k}$.

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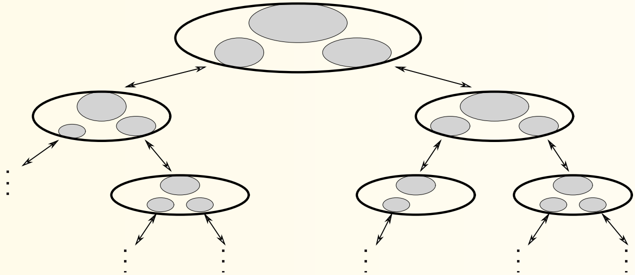
- This can be (visually) seen as. . .



3 Measuring Graphs: Clique-width and Rank-width

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$\longrightarrow$ giving the *expression tree* (parse tree) for clique-width.

Rank-decomposition

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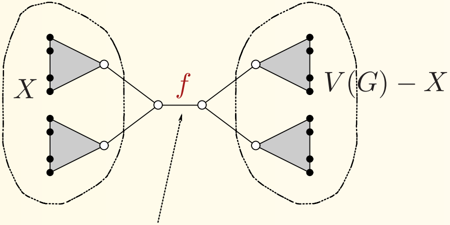

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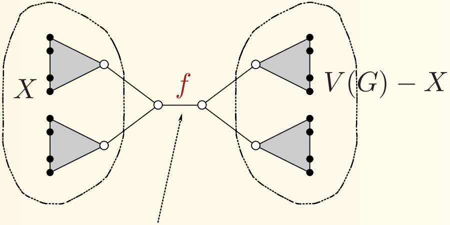

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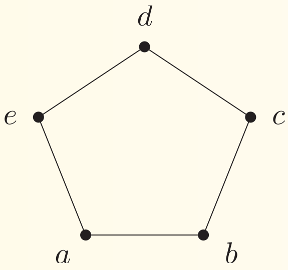
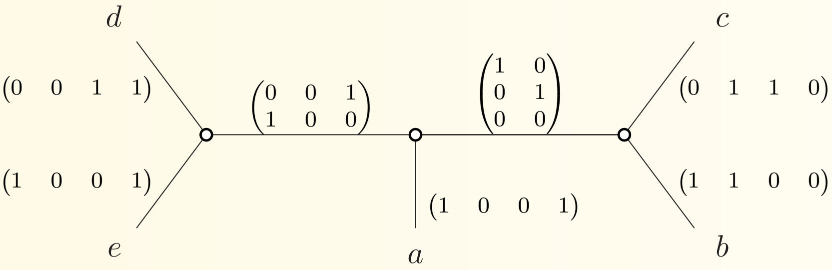
- **Rank-width** $= \min_{\text{rank-decs. of } G} \max \{\text{width}(f) : f \text{ tree edge}\}$

An example. Cycle C_5 and its *rank-decomposition* of width 2:





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- [Oum and PH, 07] There is an *FPT algorithm* for computing an optimal width-t rank-decomposition of a graph in time $O(f(t) \cdot n^3)$.
- And new results show that certain algorithms designed on rank-decompositions run faster than their analogues designed on clique-width expressions... (subst. $poly(t)$ in place of cw, instead of 2^t)

Parse trees for rank-decompositions

Unlike for tree- or clique-decompositions with obvious parse trees, what is the "boundary" and "join" operation for rank-width?

Our "boundary" includes all vertices, and "join" is just an implicit matrix rank.

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    edge $uv \;\leftrightarrow\; lab(u) \cdot \mathbf{g} \cdot lab(v) = 1.$

- Join $\;\to\;$ a *composition* operator with relabelings f_1, f_2;

  $$(G_1, lab^1) \;\otimes[\mathbf{g} \mid f_1, f_2]\; (G_2, lab^2) \;=\; (H, lab)$$

  $\Longrightarrow$ the rank-width **parse tree** [Ganian and PH, 08]:

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  $\Longrightarrow$ the rank-width **parse tree** [Ganian and PH, 08]:

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Parse trees for rank-decompositions

Unlike for tree- or clique- decompositions with obvious parse trees, what is the "boundary" and "join" operation for rank-width?

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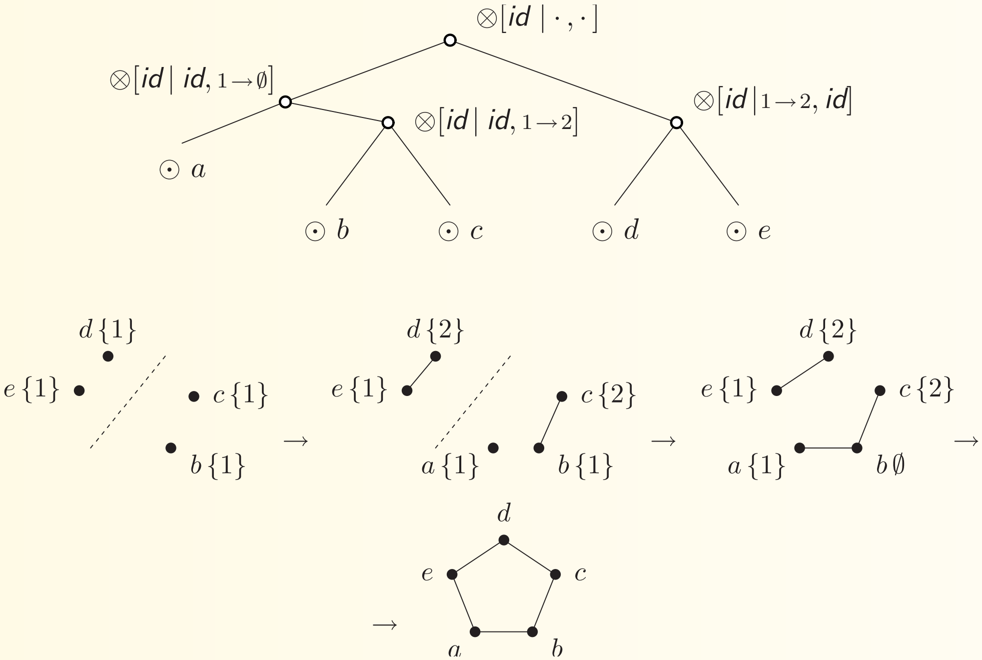
- Independently considered related notion of R_t-*join* decompositions by [Bui-Xuan, Telle, and Vatshelle, 08].

A parse tree. An example generating the cycle C_5 (of rank-width 2):



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Where is the problem?

A resulting double-exponential worst-case dependency on a width estimate!

The problem, again

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- A clear and rigorous algorithm employing many of the above tricks.

Theorem. [Ganian, PH, Obdržálek, 10] #SAT solved in FPT time

$$\mathcal{O}(t^3 \cdot 2^{3t(t+1)/2} \cdot |\phi|)$$

where t is the *signed rank-width* of the input instance (CNF formula) ϕ.

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  Then

  $$G_1 \oplus G_2 \;=\; \left(G_1^+ \oplus G_2^+\right) \cup \left(G_1^- \oplus G_2^-\right)$$

  and the same decomposition is used.

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Easy to prove. . . , but does it help?

Subsets of labels from $2^{\{1,2,\ldots,t\}}$ $\longrightarrow$ $\Omega\left(2^{2^t}\right)$ classes!

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Conclusion. Breaking the satisfying assignments of ϕ into $S(t)^4$ classes, and processing a node of the parse tree in $O^*\big(S(t)^6\big)$. □

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THANK YOU FOR YOUR ATTENTION