# Planar Emulators Conjecture Is Nearly True for Cubic Graphs

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**Abstract.** We prove that a cubic nonprojective graph cannot have a finite planar emulator, unless one of two very special cases happen (in which the answer is open). This shows that Fellows' planar emulator conjecture, disproved for general graphs by Rieck and Yamashita in 2008, is nearly true on cubic graphs, and might very well be true there definitely.

## 1 Introduction

A graph $G$ has a finite *planar emulator* $H$ if $H$ is a planar graph and there is a graph homomorphism $\varphi : V(H) \to V(G)$ where $\varphi$ is locally surjective, i.e. for every vertex $v \in V(H)$, the neighbours of $v$ in $H$ are mapped surjectively onto the neighbours of $\varphi(v)$ in $G$. We also say that such a $G$ is planar-emulable. If we insist on $\varphi$ being locally bijective, we get a *planar cover.*

The concept of planar emulators was proposed in 1985 by M. Fellows [5], and it tightly relates (although of independent origin) to the better known *planar cover conjecture* of Negami [10]. Fellows also raised the main question: What is the class of graphs with finite planar emulators? Soon later he conjectured that the class of planar-emulable graphs coincides with the class of graphs with finite planar covers (conjectured to be the class of projective graphs by Negami [10]—still open nowadays). This was later restated as follows:

**Conjecture 1.1 (M. Fellows, falsified in 2008).** *A connected graph has a finite planar emulator if and only if it embeds in the projective plane.*

For two decades the research focus was exclusively on Negami's conjecture and no substantial new results on planar emulators had been presented until 2008, when emulators for two non-projective graphs were given by Rieck and Yamashita [12], effectively disproving Conjecture 1.1.

**Planar emulable nonprojective graphs.** Following Rieck and Yamashita, Chimani et al [2] constructed finite planar emulators of all the minor minimal obstructions for the projective plane except those which have been shown non-planar-emulable already by Fellows ($K_{3,5}$ and "two disjoint k-graphs" cases, Def. 2.2), and $K_{4,4} - e$. The graph $K_{4,4} - e$ is thus the only forbidden minor for the projective plane where the existence of a finite planar emulator remains open. Even though we do not have a definite replacement for falsified Conjecture 1.1 yet, the results obtained so far [4, 2] suggest that, vaguely speaking, up to some trivial operations ("planar expansions"), there is only a finite family of nonprojective planar-emulable graphs. A result like that would nicely correspond with the current state-of-art [9] of Negami's conjecture.

While characterization of planar-emulable graphs has proven itself to be difficult in general, significant progress can be made in a special case. Negami's conjecture has been confirmed in the case of cubic graphs in [11], and the same readily follows from [9]. Here we prove:

**Theorem 1.2.** *If a cubic nonprojective graph $H$ has a finite planar emulator, then $H$ is a planar expansion (Def. 2.1) of one of two minimal cubic nonprojective graphs shown in Fig. 1.*

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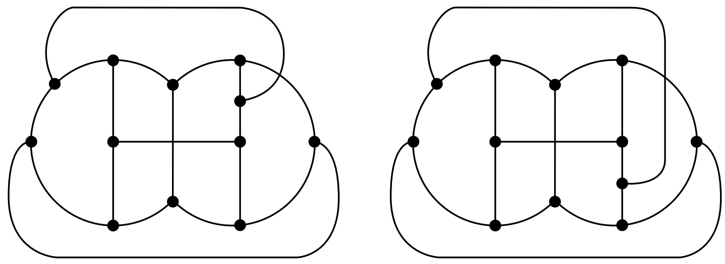


**Fig. 1.** Two (out of six in total) cubic irreducible obstructions for the projective plane [6]. Although these graphs result by splitting nonprojective graphs for which we have finite planar emulators [2] (namely $K_7 - C_4$ and its "relatives"), it is still open whether they are planar-emulable.

## 2 Cubic planar-emulable graphs

The purpose of this section is to prove Theorem 1.2. In order to do so, we need to define two important concepts as follows.

**Definition 2.1.** *A* planar expansion *of a graph $G$ is a graph which results from $G$ by repeatedly adding a planar graph sharing one vertex with $G$, or by replacing an edge or a cubic vertex with a connected planar graph with its attachments (two or three, resp.) on the outer face.*

**Definition 2.2.** *Graph $G$ is said to* contain two disjoint k-graphs *if there exist two vertex-disjoint subgraphs $J_1, J_2 \subseteq G$ such that, for $i = 1, 2$, the graph $J_i$ is isomorphic to a subdivision of $K_4$ or $K_{2,3}$, the subgraph $G - V(J_i)$ is connected and adjacent to $J_i$, and contracting in $G$ all the vertices of $V(G) \setminus V(J_i)$ into one results in a nonplanar graph.*

The next claim describes some folklore known facts about planar-emulable graphs.

**Proposition 2.3.** *Let $G$ be a connected graph.*

1. *The class of planar-emulable graphs is closed under taking minors.*
2. *If $G$ is projective, then $G$ has a finite planar emulator in form of its finite planar cover.*
3. *If $G$ contains two disjoint k-graphs or a $K_{3,5}$ minor, then $G$ is not planar-emulable.*
4. *$G$ is planar-emulable if, and only if, so is any planar expansion of $G$.*

A computerized search for all possible counterexamples to Conjecture 1.1, carried out so far [4], shows that a nonprojective planar-emulable graph $G$ cannot be cubic, unless $G$ contains a minor isomorphic to $\mathcal{E}_2$, $K_{4,5} - 4K_2$, or a member of the so called "$K_7 - C_4$ family". Our new approach, Theorem 1.2, actually dismisses the former two possibilities completely and strongly restricts the latter one.

**Proof of Theorem 1.2.** Glover and Huneke [6] characterized the cubic graphs with projective embedding using six minimal forbidden cubic topological minors (see Fig. 1 for two of them).

**Theorem 2.4 (Glover–Huneke [6]).** *There is a set $\mathcal{I}$ of six cubic graphs such that; if $H$ is a cubic graph that does not embed in the projective plane, then $H$ contains a graph $G \in \mathcal{I}$ as a topological minor.*

Let us point out that four out of the six graphs in $\mathcal{I}$ contain two disjoint k-graphs, and so only the remaining two— $G_1 \in \mathcal{I}$ and $G_2 \in \mathcal{I}$ of Fig. 1, can potentially be planar-emulable.

Hence the cubic graph $H$ in Theorem 1.2 contains one of $G_1, G_2$ as a topological minor. In other words, there is a subgraph $G' \subseteq H$ being a subdivision of a cubic $G \in \{G_1, G_2\}$.

We call a *bridge of* $G'$ in $H$ any connected component $B$ of $H - V(G')$ together will all the incident edges. In a degenerate case, $B$ might consist just of one edge from $E(H) \setminus E(G')$ with both ends in $G'$. We would like, for simplicity, to speak about positions of bridges with respect to the underlying cubic graph $G$: Such a bridge $B$ connects to vertices $u$ of $G'$ which subdivide edges $f$ of $G$—this is due to the cubic degree bound, and we (with neglectable abuse of terminology) say that $B$ *attaches to* this edge $f$ in $G$ itself.

A bridge $B$ is *nontrivial* if $B$ attaches to some two nonadjacent edges of $G$, and $B$ is *trivial* otherwise. For a trivial bridge $B$; either $B$ attaches to only one edge in $G$, and we say *exclusively*, or all the edges to which $B$ attaches in $G$ have a vertex $w$ in common (since $G$ contains no triangle), and we say that $B$ *attaches to* this $w$.

We divide the rest of the proof into two main cases; that either some bridge of $G'$ in $H$ is nontrivial or all such bridges are trivial. We moreover assume that $G' \subseteq H$ being a subdivision of $G$ is chosen such that it has a nontrivial bridge if possible. In the "all-trivial" case one more technical condition has to be observed: Suppose $B_1, B_2$ are bridges such that $B_1$ attaches to $w$ and $B_2$ attaches to an edge $f$ incident to $w$ in $G$ (perhaps $B_2$ exclusively to $f$). On the path $P_f$ which replaces (subdivides) $f$ in $G'$, suppose that $B_2$ connects to some vertex which is closer to $w$ on $P_f$ than some other vertex to which $B_1$ connects to. Then we *declare that* $B_2$ *attaches to* $w$, too. The transitive closure of declared attachment is well defined because of the following:

**Lemma 2.5.** *Let $G' \subseteq H$ be a subdivision of $G$ where $G, H$ are cubic graphs. Suppose that all bridges of $G'$ in $H$ are trivial, and that a bridge $B_0$ attaches (or, is declared to) both to $w_1$ and $w_2$, where $w_1w_2 \in E(G)$. Then there is $G'' \subseteq H$ which is a subdivision of $G$, too, and a nontrivial bridge of $G''$ in $H$ exists.*

*Proof (sketch).* Let $P_f$ be the path representing $f = w_1w_2$ in $H$. In the described situation, we call $B_0$ a *conflicting* bridge, and assume that $H - B_0$ has no conflicting bridge of $G'$. By the definition of declared attachment there exist vertices $u_1, u_2 \in V(P_f)$ such that the following holds for $i = 1, 2$: Either $u_i = w_i$ and $B_0$ attaches to at least two edges incident to $w_i$, or there is a bridge $B_i$ connecting to $u_i$ such that $B_i$ attaches (or, is declared to) to $w_i$ in $G$ and $B_0$ connects the two components of $P_f - u_i$ together. Notice that $B_1 \neq B_2$ and $u_1$ is closer to $w_1$ on $P_f$ than $u_2$ (since $H - B_0$ has no conflicting bridge).

One can now easily check that there exist two internally disjoint paths from $u_i$ to the two neighbours of $w_i$ not on $P_f$, for each $i = 1, 2$ (Fig. 2). Hence there exists new $G'' \subseteq H$ a subdivision of $G$ such that the vertices $w_1, w_2$ now correspond to $u_1, u_2$, respectively, and the bridge of $G''$ arising from $B_0$ is nontrivial. ◻

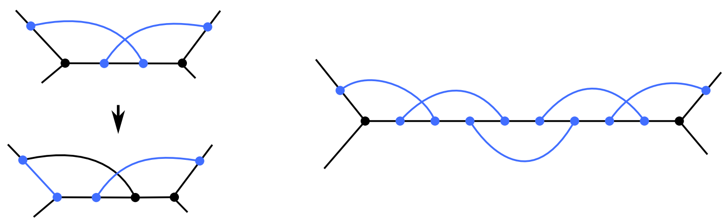

**Fig. 2.** Illustration for sketch proof of Lemma 2.5. The trivial bridge on the left takes over the role of a branch vertex of $G$ in the graph $G'$, resulting in existence of a nontrivial bridge. The other case shows when the transitive closure of declared attachment becomes important.

**Lemma 2.6.** *Let $G' \subseteq H$ be a subdivision of $G$ where $G, H$ are cubic nonprojective graphs and $G$ does not contain two disjoint k-graphs. Suppose that all bridges of $G'$ in $H$ are trivial, and no one is conflicting (cf. Lemma 2.5). Then $H$ does not contain two disjoint k-graphs if, and only if, $H$ is a planar expansion of $G$.*

*Proof (sketch).* If $H$ is a planar expansion of $G$, then two disjoint k-graphs in $H$ would imply containment of those in $G$ itself, which is not possible. In the converse direction, we assume that $H$ is not a planar expansion of $G$. Let $B_v$ be the union of all trivial bridges of $G'$ in $H$ that attach or are declared to attach to a vertex $v \in V(G)$. Let $B_f$ be the union of all trivial bridges of $G'$ in $H$ that attach exclusively to an edge $f \in E(G)$. Since $H$ is not a planar expansion of $G$, for at least one $x \in V(G) \cup E(G)$ the subgraph $H_x = G' \cup B_x$ is not a planar expansion of $G$, too. For simplicity, we consider only the more interesting case $x = u \in V(G)$. See an illustration in Fig. 3.

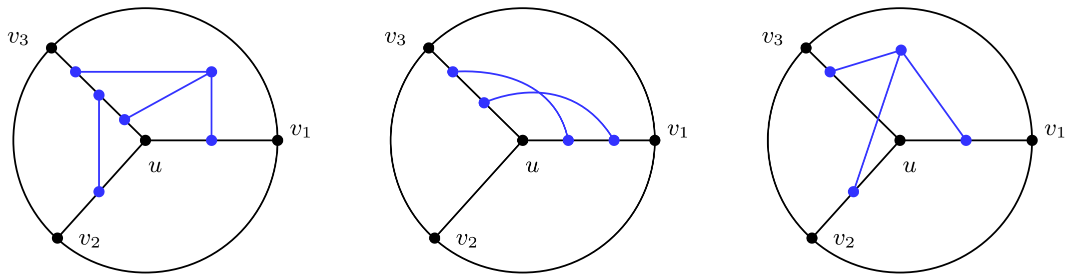


**Fig. 3.** Illustration of three collections of trivial bridges that attach to a cubic vertex $u$. The first collection gives a planar expansion, while the other two are “minimal” non-planar-expansion cases.

Let $G'_u \subseteq G'$ denote the corresponding subdivision of $G-u$. Let $C = \{e_1, e_2, e_3\}$ be a minimal edge-cut in $H_u$ which separates $G'_u$ on one side and $B'_u \supset B_u \cup \{u\}$ on the other side. Then our graph $H_u$ is not a planar expansion of $G'$ iff $B'_u$ is not planar with all the three connections to $C$ on the outer face. The latter can be characterized by containment of a $K_{2,3}$ subdivision in $B'_u$ with the size-three part incident to $C$. Then it is easy to show that $G' \cup B_u$ confirms to Def. 2.2 of two disjoint k-graphs, since $G - u$ is connected and particularly $G$ is nonplanar. □

**Lemma 2.7.** *Let $G' \subseteq H$ be a subdivision of $G \in \{G_1, G_2\}$ (Fig. 1) in a cubic graph $H$. If there exists a nontrivial bridge of $G'$ in $H$, then $H$ does not have a finite planar emulator.*

*Proof (sketch).* We have exhaustively verified that for $G \in \{G_1, G_2\}$, all the graphs $G' + e$ where $e$ is a nontrivial bridge of $G$ do not admit existence of finite planar emulator. Up to one case, all such graphs contain two disjoint k-graphs. In the one special case, the graph $G'_2 + e$ does not contain two disjoint k-graphs, but it contains a $K_{3,5}$ minor. We would like to point out that due to the necessity of $K_{3,5}$ in that one case, there is likely no simple argument summarizing the cases similarly as done in Lemma 2.6. □

Theorem 1.2 is then an immediate corollary of Lemmas 2.7 and 2.6.

## 3 Conclusions

While our main effort (started in [4, 2]) is to provide a new finite characterization of nonprojective graphs with finite planar emulators, this paper shows that the problem becomes significantly easier when only a restricted class of graphs is considered. We identified two graphs (Fig. 1), for which existence of finitie planar emulator now becomes extremely interesting. We would like to point out that similarity of these two graphs suggest that if one has a finite planar emulator, so does the other one. If we however elaborate on this idea and attempt to "unify" the graphs as depicted in Fig. 4, we have to use a nontrivial bridge. Perhaps, this provides a clue that these two graphs should not be planar-emulable. Thus, providing an answer for any of these two graphs would bring a better insight to the problem of planar emulations not only for the cubic case, but also in general.

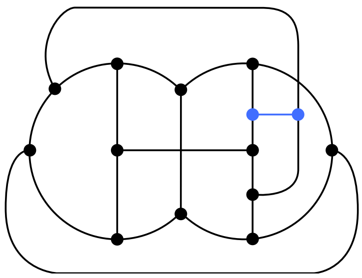

**Fig. 4.** "Unification" of pictures of $G_1$ and $G_2$ using a nontrivial bridge.

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