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Part G.

Stable Model Theory

G1. Stable theories

1. Definable types

A key property of stable theories is the definability of types. The relation $\overset{\mathrm{df}}{\surd}$ will thus play a major role in this chapter. In the next section, we will study its properties in the context of stable theories. But first, we consider the relation $\overset{\mathrm{df}}{\surd}$ in an arbitrary theory, where it is usually much less well-behaved: if U and B are small enough it might happen that $\mathrm{tp}(\bar{a}/UB)$ is definable over U just because some formula 'accidentally' is a φ-definition, although it ceases to be a definition for every extension $\mathrm{tp}(\bar{a}/UB')$ with $B' \supseteq B$. Therefore, when investigating a statement of the form $A \overset{\mathrm{df}}{\surd}_U B$ we usually assume that one of the sets B and U is large. In particular, there is hope that the derived relation $^*(\overset{\mathrm{df}}{\surd})$ is much better behaved. We start with relating $\overset{\mathrm{df}}{\surd}$ to the relation $\overset{\mathrm{u}}{\surd}$.

Lemma 1.1. *Let $A, A', B, U \subseteq \mathbb{M}$.*

(a) $A \overset{\mathrm{df}}{\surd}_U B$ *and* $B \overset{\mathrm{u}}{\surd}_U U$ *implies* $B \overset{\mathrm{u}}{\surd}_U A$.

(b) $A \overset{\mathrm{df}}{\surd}_U B$, $B \overset{\mathrm{u}}{\surd}_U A'$, *and* $A \equiv_U A'$ *implies* $A \equiv_{UB} A'$.

Proof. (a) Let $A \overset{\mathrm{df}}{\surd}_U B$ and $B \overset{\mathrm{u}}{\surd}_U U$. To show that $B \overset{\mathrm{u}}{\surd}_U A$, suppose that $\mathbb{M} \vDash \varphi(\bar{a}; \bar{b})$, where $\bar{a} \subseteq A$, $\bar{b} \subseteq B$, and $\varphi(\bar{x}; \bar{y})$ is a formula over U. We have to find a tuple $\bar{c} \subseteq U$ such that $\mathbb{M} \vDash \varphi(\bar{a}; \bar{c})$. Since $\bar{a} \overset{\mathrm{df}}{\surd}_U \bar{b}$, the type $\mathrm{tp}(\bar{a}/U\bar{b})$ has a φ-definition δ over U. Hence,

$$\mathbb{M} \vDash \varphi(\bar{a}; \bar{b}) \quad \text{implies} \quad \mathbb{M} \vDash \delta(\bar{b})\,.$$

Since $\bar{b} \overset{\mathrm{u}}{\surd}_U U$, there is some $\bar{c} \subseteq U$ with $\mathbb{M} \vDash \delta(\bar{c})$. Consequently, $\mathbb{M} \vDash \varphi(\bar{a}; \bar{c})$.

(b) Let $\bar{a}$ be an enumeration of A and $\bar{a}'$ the corresponding enumeration of A'. For every formula $\varphi(\bar{x};\bar{y})$ over U, we fix a φ-definition $\delta_\varphi(\bar{y})$ of $\operatorname{tp}(\bar{a}/UB)$ over U. It is sufficient to prove that δ_φ is also a φ-definition of $\operatorname{tp}(\bar{a}'/UB)$ since this implies that

$$\mathbb{M} \vDash \varphi(\bar{a};\bar{b}) \quad \text{iff} \quad \mathbb{M} \vDash \delta(\bar{b}) \quad \text{iff} \quad \mathbb{M} \vDash \varphi(\bar{a}';\bar{b}),$$

for all $\bar{b} \subseteq U \cup B$.

For a contradiction, suppose that the formula δ_φ is not a φ-definition of $\operatorname{tp}(\bar{a}'/UB)$. Then there exists a tuple $\bar{b} \subseteq U \cup B$ such that

$$\mathbb{M} \vDash \neg(\varphi(\bar{a}';\bar{b}) \leftrightarrow \delta_\varphi(\bar{b})).$$

Since $B \mathrel{\overset{\mathrm{u}}{\surd}}_U \bar{a}'$, there is a tuple $\bar{c} \subseteq U$ such that

$$\mathbb{M} \vDash \neg(\varphi(\bar{a}';\bar{c}) \leftrightarrow \delta_\varphi(\bar{c})).$$

As $\bar{a} \equiv_U \bar{a}'$, this implies that

$$\mathbb{M} \vDash \neg(\varphi(\bar{a};\bar{c}) \leftrightarrow \delta_\varphi(\bar{c})).$$

Consequently, δ_φ is no φ-definition of $\operatorname{tp}(\bar{a}/UB)$. A contradiction. $\square$

The relation $\overset{\mathrm{df}}{\surd}$ is particularly well-behaved if the base set is a model.

Lemma 1.2. *Let T be a complete first-order theory, $\mathfrak{M}$ a model of T, and $A \subseteq \mathbb{M}$ a set such that $A \mathrel{\overset{\mathrm{df}}{\surd}}_M M$.*

(a) $A \;{}^*(\overset{\mathrm{df}}{\surd})_M\; M$.

(b) *For every set $B \subseteq \mathbb{M}$, there exists a set $A' \equiv_M A$ such that $B \mathrel{\overset{\mathrm{u}}{\surd}}_M A'$.*

(c) *If $A', B \subseteq \mathbb{M}$ are sets such that*

$$B \mathrel{\overset{\mathrm{u}}{\surd}}_M A, \quad B \mathrel{\overset{\mathrm{u}}{\surd}}_M A' \quad \text{and} \quad A \equiv_M A', \quad \text{then} \quad A \equiv_{MB} A'.$$

Proof. (a) Suppose that $\bar{a} \overset{\mathrm{df}}{\smile}_M M$. To show that $\bar{a} \ {}^*(\overset{\mathrm{df}}{\smile})_M M$, consider a set $C \supseteq M$. For every formula $\varphi(\bar{x};\bar{y})$ fix a φ-definition $\delta_\varphi(\bar{y})$ of $\mathrm{tp}(\bar{a}/M)$ over M and set

$$\Phi(\bar{x}) := \big\{ \varphi(\bar{x};\bar{c}) \bigm| \bar{c} \subseteq C,\ \mathbb{M} \vDash \delta_\varphi(\bar{c}) \big\}.$$

Note that $\mathrm{tp}(\bar{a}/M) \subseteq \Phi$. Hence, if $\bar{a}'$ is a tuple satisfying Φ, then $\bar{a}' \equiv_M \bar{a}$ and $\bar{a}' \overset{\mathrm{df}}{\smile}_M C$ since each formula δ_φ is a φ-definition of $\mathrm{tp}(\bar{a}'/C)$.

Thus, it remains to prove that Φ is satisfiable. Consider a finite subset $\Phi_0 = \{\varphi_0(\bar{x};\bar{c}_0),\dots,\varphi_n(\bar{x};\bar{c}_n)\} \subseteq \Phi$. By definition of Φ we have

$$\mathbb{M} \vDash \delta_{\varphi_0}(\bar{c}_0) \wedge \dots \wedge \delta_{\varphi_n}(\bar{c}_n).$$

We have seen in Lemma F2.3.15 that $C \overset{\mathrm{u}}{\smile}_M M$. Hence, we can find tuples $\bar{b}_0,\dots,\bar{b}_n \subseteq M$ such that

$$\mathbb{M} \vDash \delta_{\varphi_0}(\bar{b}_0) \wedge \dots \wedge \delta_{\varphi_n}(\bar{b}_n).$$

This implies that

$$\mathbb{M} \vDash \varphi_0(\bar{a};\bar{b}_0) \wedge \dots \wedge \varphi_n(\bar{a};\bar{b}_n).$$

Consequently, the model $\mathbb{M}$ satisfies Φ_0 if we interpret the variables $\bar{x}$ by $\bar{a}$ and the constants $\bar{c}_i$ by $\bar{b}_i$.

(b) Given $B \subseteq \mathbb{M}$, we can use (a) to find a set $A' \equiv_M A$ such that $A' \overset{\mathrm{df}}{\smile}_M B$. Furthermore, we have $B \overset{\mathrm{u}}{\smile}_M M$, by Lemma F2.3.15. Hence, Lemma 1.1 (a) implies that $B \overset{\mathrm{u}}{\smile}_M A'$.

(c) Let $\bar{a}$ be an enumeration of A and let $\bar{a}'$ be the corresponding enumeration of A'. By (a), we can find a tuple $\bar{a}'' \equiv_M \bar{a}$ such that $\bar{a}'' \overset{\mathrm{df}}{\smile}_M B$. By Lemma 1.1 (b), $B \overset{\mathrm{u}}{\smile}_M \bar{a}$ implies that $\bar{a}'' \equiv_{MB} \bar{a}$. Since $\bar{a}'' \equiv_M \bar{a}'$ and $B \overset{\mathrm{u}}{\smile}_M \bar{a}'$, it follows in the same way that $\bar{a}'' \equiv_{MB} \bar{a}'$. Hence, $\bar{a} \equiv_{MB} \bar{a}'' \equiv_{MB} \bar{a}'$. □

Lemma 1.3 (Harrington). *Let $\mathfrak{M}$ be an $\aleph_0$-saturated model, $\mathfrak{p}(\bar{x}), \mathfrak{q}(\bar{y}) \in S^{<\omega}(M)$, and let $\varphi(\bar{x};\bar{y})$ be a formula over M that does not have the order property. If $\delta(\bar{y})$ is a φ-definition of $\mathfrak{p}$ and $\varepsilon(\bar{x})$ a φ-definition of $\mathfrak{q}$, then*

$$\varepsilon(\bar{x}) \in \mathfrak{p}(\bar{x}) \quad \text{iff} \quad \delta(\bar{y}) \in \mathfrak{q}(\bar{y}).$$

Proof. Let $\bar{c} \subseteq M$ be the parameters occurring in φ. By induction on n, we construct two sequences $(\bar{a}_n)_{n<\omega}$ and $(\bar{b}_n)_{n<\omega}$ in M as follows. Suppose that we have already defined $\bar{a}_0, \dots, \bar{a}_{n-1}$ and $\bar{b}_0, \dots, \bar{b}_{n-1}$. As $\mathfrak{M}$ is $\aleph_0$-saturated, we can choose a tuple $\bar{b}_n \subseteq M$ realising $\mathfrak{q} \upharpoonright \bar{c}\bar{a}_0 \dots \bar{a}_{n-1}$ and a tuple $\bar{a}_n \subseteq M$ realising $\mathfrak{p} \upharpoonright \bar{c}\bar{b}_0 \dots \bar{b}_n$.

Having constructed $(\bar{a}_n)_{n<\omega}$ and $(\bar{b}_n)_{n<\omega}$ it follows, for $i < k$, that

$$\begin{aligned} \mathfrak{M} \vDash \varphi(\bar{a}_i; \bar{b}_k) \quad &\text{iff} \quad \varphi(\bar{a}_i; \bar{y}) \in \mathfrak{q} \upharpoonright \bar{c}\bar{a}_0 \dots \bar{a}_{k-1} \\ &\text{iff} \quad \mathfrak{M} \vDash \varepsilon(\bar{a}_i) \\ &\text{iff} \quad \varepsilon(\bar{x}) \in \mathfrak{p}, \end{aligned}$$

and, for $i \geq k$,

$$\begin{aligned} \mathfrak{M} \vDash \varphi(\bar{a}_i; \bar{b}_k) \quad &\text{iff} \quad \varphi(\bar{x}; \bar{b}_k) \in \mathfrak{p} \upharpoonright \bar{c}\bar{b}_0 \dots \bar{b}_i \\ &\text{iff} \quad \mathfrak{M} \vDash \delta(\bar{b}_k) \\ &\text{iff} \quad \delta(\bar{x}) \in \mathfrak{q}. \end{aligned}$$

Hence, if $\delta \notin \mathfrak{q}$ and $\varepsilon \in \mathfrak{p}$, then

$$\mathfrak{M} \vDash \varphi(\bar{a}_i; \bar{b}_k) \quad \text{iff} \quad i < k,$$

and φ has the order property. A contradiction. In the same way we obtain a contradiction if we assume that $\delta \in \mathfrak{q}$ and $\varepsilon \notin \mathfrak{p}$. □

We have already seen in Lemma F2.3.3 (a) that $\overset{\text{df}}{\smile\!\!\!\!/} \subseteq \overset{\text{s}}{\smile\!\!\!\!/}$. The converse holds only in special circumstances.

Lemma 1.4. *Suppose that $\mathfrak{M}$ is a κ-saturated and strongly κ-homogeneous model and let $U \subseteq M$ be a set of size $|U| < \kappa$. If $A \subseteq \mathbb{M}$ is a set such that $A \overset{\text{df}}{\smile\!\!\!\!/}_M M$, then*

$$A \overset{\text{df}}{\smile\!\!\!\!/}_U M \quad \text{iff} \quad A \overset{\text{s}}{\smile\!\!\!\!/}_U M.$$

Proof. ($\Rightarrow$) We have seen in Lemma F2.3.3 (a) that $\overset{\text{df}}{\smile\!\!\!\!/} \subseteq \overset{\text{s}}{\smile\!\!\!\!/}$.

($\Leftarrow$) Let $\bar{a} \subseteq A$, let $\varphi(\bar{x};\bar{y})$ be a formula, and let $\delta(\bar{y})$ be a φ-definition of $\mathrm{tp}(\bar{a}/M)$ over M. It is sufficient to show that the relation $\delta^{\mathfrak{M}}$ is definable over U. By definition of $\sqrt[\mathrm{s}]{}$,

$$\bar{b} \equiv_U \bar{b}' \quad \text{implies} \quad \bar{b} \equiv_{U\bar{a}} \bar{b}' , \quad \text{for all } \bar{b}, \bar{b}' \subseteq M .$$

Hence, if $\bar{b}, \bar{b}' \subseteq M$ are tuples such that $\bar{b} \equiv_U \bar{b}'$, then

$$\begin{aligned} \mathbb{M} \vDash \delta(\bar{b}) \quad &\text{iff} \quad \mathbb{M} \vDash \varphi(\bar{a};\bar{b}) \\ &\text{iff} \quad \mathbb{M} \vDash \varphi(\bar{a};\bar{b}') \quad \text{iff} \quad \mathbb{M} \vDash \delta(\bar{b}') . \end{aligned}$$

Consequently, we have

$$\mathfrak{M} \vDash \delta(\bar{b}) \quad \text{iff} \quad \mathfrak{M} \vDash \delta(\pi(\bar{b})) , \quad \text{for all } \pi \in \mathrm{Aut}\,\mathfrak{M}_U ,$$

and it follows by Lemma E2.1.10 that $\delta^{\mathfrak{M}}$ is definable over U. □

Another immediate consequence of the inclusion $\sqrt[\mathrm{df}]{} \subseteq \sqrt[\mathrm{s}]{}$ is the corresponding inclusion between the starred relations.

Proposition 1.5. $^*(\sqrt[\mathrm{df}]{}) \subseteq \sqrt[\mathrm{i}]{}$

Proof. We have seen in Lemma F2.3.3 (a) that $\sqrt[\mathrm{df}]{} \subseteq \sqrt[\mathrm{s}]{}$. This implies that $^*(\sqrt[\mathrm{df}]{}) \subseteq {}^*(\sqrt[\mathrm{s}]{}) = \sqrt[\mathrm{i}]{}$. □

We conclude this section with a comparison of $\sqrt[\mathrm{df}]{}$ with $\sqrt[\mathrm{d}]{}$.

Lemma 1.6. *Let $\mathfrak{M}$ be a κ-saturated model and let $U \subseteq M$ be a set of size $|U| < \kappa$. Then*

$$A \sqrt[\mathrm{df}]{}_U\, M \quad \textit{implies} \quad A \sqrt[\mathrm{d}]{}_U\, M , \quad \textit{for all } A \subseteq \mathbb{M} .$$

Proof. Suppose that $\bar{a} \sqrt[\mathrm{df}]{}_U\, M$. To show that $\bar{a} \sqrt[\mathrm{d}]{}_U\, M$ it is sufficient, by Lemma F3.1.3 and (DEF), to prove that, for every indiscernible sequence

$(\bar{b}_n)_{n<\omega}$ over U with $\bar{b}_0 \subseteq M$ and $|\bar{b}_0| < \aleph_0$, there exists a tuple $\bar{a}' \equiv_{U\bar{b}_0} \bar{a}$ such that

$$\bar{b}_m \equiv_{U\bar{a}'} \bar{b}_n \,, \quad \text{for all } m, n < \omega \,.$$

Hence, let $(\bar{b}_n)_{n<\omega}$ be such a sequence. As M is κ-saturated, we can find tuples $\bar{c}_n \subseteq M$, $n < \omega$, such that

$$(\bar{c}_n)_n \equiv_{U\bar{b}_0} (\bar{b}_n)_n \,.$$

Note that, by Lemma F2.3.3 (a),

$$\bar{a} \overset{\mathrm{df}}{\not\downarrow}_U M \quad \text{implies} \quad \bar{a} \overset{\mathrm{s}}{\not\downarrow}_U M \,.$$

Consequently,

$$\bar{c}_m \equiv_U \bar{c}_n \quad \text{implies} \quad \bar{c}_m \equiv_{U\bar{a}} \bar{c}_n \,, \quad \text{for } m, n < \omega \,.$$

Fixing a tuple $\bar{a}'$ such that

$$\bar{a}(\bar{c}_n)_n \equiv_{U\bar{b}_0} \bar{a}'(\bar{b}_n)_n \,,$$

it follows that

$$\bar{c}_m \equiv_{U\bar{a}} \bar{c}_n \quad \text{implies} \quad \bar{b}_m \equiv_{U\bar{a}'} \bar{b}_n \,, \quad \text{for } m, n < \omega \,. \qquad \square$$

2. *Forking in stable theories*

In this section we collect properties of preforking relations in stable theories. First, note that we have seen in Corollary F3.2.19 that every stable theory is simple. Hence, in stable theories there is no difference between dividing and forking and both relations are symmetric.

Furthermore, in stable theories the relation $\overset{\mathrm{df}}{\not\downarrow}$ is much better behaved than usual. For instance, we have already seen in Theorem C3.5.17 that $\overset{\mathrm{df}}{\not\downarrow}$ is right reflexive.

Theorem 2.1. *Let T be a stable theory. Then*

$$A \mathrel{\overset{\mathrm{df}}{\surd}}_U U\,, \quad \textit{for all } A, U \subseteq \mathbb{M}\,.$$

Let us consider the case where the base set is a model. In this case it will turn out that most preforking relations coincide.

Lemma 2.2. *Let T be a stable theory and $\mathfrak{M}$ a model of T. Then*

$$A \mathop{\downarrow}^{\mathrm{d}}_M B \quad \textit{implies} \quad A \mathrel{\overset{\mathrm{u}}{\surd}}_M B\,, \quad \textit{for all } A, B \subseteq \mathbb{M}\,.$$

Proof. Suppose that $\bar{a} \mathop{\downarrow}^{\mathrm{d}}_M \bar{b}$ and $\mathbb{M} \vDash \varphi(\bar{a}; \bar{b})$. We have to find a tuple $\bar{a}' \subseteq M$ such that $\mathbb{M} \vDash \varphi(\bar{a}'; \bar{b})$.

By Lemma F2.3.15, we have $\bar{a} \mathrel{\overset{\mathrm{u}}{\surd}}_M M$. Hence, we can use Proposition F2.4.10 to construct a $\overset{\mathrm{u}}{\surd}$-Morley sequence $(\bar{a}^n)_{n<\omega}$ for $\mathrm{tp}(\bar{a}/M)$ over M. Since $\bar{b} \mathop{\downarrow}^{\mathrm{d}}_M \bar{a}$, it follows by Lemma F3.1.3 that there exists a tuple $\bar{b}' \equiv_{M\bar{a}} \bar{b}$ such that $(\bar{a}^n)_{n<\omega}$ is indiscernible over $M \cup \bar{b}'$. Consequently, $\mathbb{M} \vDash \varphi(\bar{a}^n; \bar{b}')$, for all $n < \omega$. By Theorem 2.1, we have

$$\bar{b}' \mathrel{\overset{\mathrm{df}}{\surd}}_{M \cup \bigcup_{n<\omega} \bar{a}^n} M \cup \bigcup_{n<\omega} \bar{a}^n\,.$$

Let $\delta(\bar{y})$ be a φ-definition of $\mathrm{tp}(\bar{b}'/M \cup \bigcup_n \bar{a}^n)$ over $M \cup \bigcup_n \bar{a}^n$ and choose $n < \omega$ such that δ is a formula over $M \cup \bar{a}^0 \dots \bar{a}^{n-1}$. Since

$$\bar{a}^n \mathrel{\overset{\mathrm{u}}{\surd}}_M M\bar{a}^0 \dots \bar{a}^{n-1} \quad \text{and} \quad \mathbb{M} \vDash \delta(\bar{a}^n)\,,$$

there exists a tuple $\bar{a}' \subseteq M$ such that $\mathbb{M} \vDash \delta(\bar{a}')$. Hence, $\mathbb{M} \vDash \varphi(\bar{a}'; \bar{b})$, as desired. ☐

Corollary 2.3. *Let T be a stable theory and $\mathfrak{M}$ a model of T. Then*

$$A \mathop{\downarrow}^{\mathrm{d}}_M B \quad \textit{implies} \quad A \mathrel{\overset{\mathrm{df}}{\surd}}_M B\,, \quad \textit{for all } A, B \subseteq \mathbb{M}\,.$$

Proof. According to Theorem 2.1, we have $A \mathrel{\overset{\mathrm{df}}{\surd}}_M M$, which implies that $A \;{}^*(\overset{\mathrm{df}}{\surd})_M\; M$, by Lemma 1.2 (a). Hence, we can find a set $A' \equiv_M A$

with $A' \overset{\mathrm{df}}{\surd}_M B$. Furthermore, $B \downarrow^{\mathrm{d}}_M A$ implies $B \overset{\mathrm{u}}{\surd}_M A$, by Lemma 2.2. Consequently, it follows by Lemma 1.1 (b) that $A' \equiv_{MB} A$. Therefore, $A' \overset{\mathrm{df}}{\surd}_M B$ implies that $A \overset{\mathrm{df}}{\surd}_M B$. $\square$

The previous results imply that, in a stable theory, the relations $\overset{\mathrm{u}}{\surd}$, $\overset{\mathrm{i}}{\surd}$, $\overset{\mathrm{df}}{\surd}$, $\downarrow^{\mathrm{d}}$, and $\downarrow^{\mathrm{f}}$ are all equivalent, at least over models.

Theorem 2.4. *Let T be a stable theory and $\mathfrak{M}$ a model of T. Then*

$$\begin{aligned} A \overset{\mathrm{u}}{\surd}_M B \quad \text{iff} \quad & A \overset{\mathrm{i}}{\surd}_M B \quad \text{iff} \quad A \overset{\mathrm{df}}{\surd}_M B \\ \text{iff} \quad & A \downarrow^{\mathrm{f}}_M B \quad \text{iff} \quad A \downarrow^{\mathrm{d}}_M B\,. \end{aligned}$$

Proof. We have already seen in Proposition F3.1.12 that

$$A \overset{\mathrm{u}}{\surd}_M B \quad \Rightarrow \quad A \overset{\mathrm{i}}{\surd}_M B \quad \Rightarrow \quad A \downarrow^{\mathrm{f}}_M B \quad \Rightarrow \quad A \downarrow^{\mathrm{d}}_M B\,.$$

For stable theories, the implication $A \downarrow^{\mathrm{d}}_M B \Rightarrow A \overset{\mathrm{u}}{\surd}_M B$ is provided by Lemma 2.2. Furthermore, we can use Corollary 2.3 to show that

$$A \downarrow^{\mathrm{d}}_M B \quad \Rightarrow \quad A \overset{\mathrm{df}}{\surd}_M B\,,$$

while Lemma 1.1 (a) implies that

$$A \overset{\mathrm{df}}{\surd}_M B \quad \Rightarrow \quad B \overset{\mathrm{u}}{\surd}_M A\,.$$

Since we have already proved that, over models, $\overset{\mathrm{u}}{\surd}$ is equivalent to $\downarrow^{\mathrm{d}}$, it follows by symmetry of $\downarrow^{\mathrm{d}}$ that

$$A \overset{\mathrm{df}}{\surd}_M B \quad \Rightarrow \quad B \overset{\mathrm{u}}{\surd}_M A \quad \Leftrightarrow \quad A \overset{\mathrm{u}}{\surd}_M B\,. \qquad \square$$

There is an even closer connection between $\overset{\mathrm{i}}{\surd}$, $^*(\overset{\mathrm{df}}{\surd})$, and $\downarrow^{\mathrm{f}}$.

Proposition 2.5. *Let T be a stable theory.*

(a) $\overset{\mathrm{i}}{\surd} = {}^*(\overset{\mathrm{df}}{\surd})$.

(b) $A \mathop{\downarrow}^{\mathrm{f}}_{U} B \quad$ iff $\quad A \sqrt[\mathrm{i}]{\,}\!/_{\mathrm{acl}^{\mathrm{eq}}(U)} B$.

Proof. (a) ($\supseteq$) was already proved in Proposition 1.5. For ($\subseteq$), suppose that $\bar{a} \sqrt[\mathrm{i}]{\,}\!/_{U} B$. To show that $\bar{a}\ {}^{*}(\sqrt[\mathrm{df}]{\,}\!/)_{U} B$, we consider a set $C \supseteq B$. We have to find a tuple $\bar{a}' \equiv_{UB} \bar{a}$ with $\bar{a}' \sqrt[\mathrm{df}]{\,}\!/_{U} C$. Let $\mathfrak{M}$ be a $|U|^{+}$-saturated and strongly $|U|^{+}$-homogeneous model of T that contains $U \cup C$. Since $\bar{a} \sqrt[\mathrm{i}]{\,}\!/_{U} B$, there exists a tuple $\bar{a}' \equiv_{UB} \bar{a}$ with $\bar{a}' \sqrt[\mathrm{s}]{\,}\!/_{U} M$. Furthermore, we have seen in Theorem 2.1 that $\bar{a}' \sqrt[\mathrm{df}]{\,}\!/_{M} M$. Consequently, it follows by Lemma 1.4 that $\bar{a}' \sqrt[\mathrm{df}]{\,}\!/_{U} M$. In particular, $\bar{a}' \sqrt[\mathrm{df}]{\,}\!/_{U} C$.

(b) ($\Leftarrow$) According to Proposition F3.1.12, $A \sqrt[\mathrm{i}]{\,}\!/_{\mathrm{acl}^{\mathrm{eq}}(U)} B$ implies that $A \mathop{\downarrow}^{\mathrm{f}}_{\mathrm{acl}^{\mathrm{eq}}(U)} B$. Moreover, we have $\mathrm{acl}^{\mathrm{eq}}(U) \mathop{\downarrow}^{\mathrm{f}}_{U} A$, by Corollary F2.2.12 and Lemma F3.1.8. By symmetry and transitivity, it therefore follows that $A \mathop{\downarrow}^{\mathrm{f}}_{U} B$.

($\Rightarrow$) Suppose that $\bar{a} \mathop{\downarrow}^{\mathrm{f}}_{U} B$. We will prove that $\bar{a}\ {}^{*}(\sqrt[\mathrm{df}]{\,}\!/)_{\mathrm{acl}^{\mathrm{eq}}(U)} B$. Then the claim follows by (a). Hence, consider a set $C \supseteq \mathrm{acl}^{\mathrm{eq}}(U) \cup B$. We fix a $(|T| \oplus |U|)^{+}$-saturated and strongly $(|T| \oplus |U|)^{+}$-homogeneous model $\mathfrak{N}$ containing C and a tuple $\bar{a}' \equiv_{UB} \bar{a}$ with $\bar{a}' \mathop{\downarrow}^{\mathrm{f}}_{U} N$. We will prove that

$$\mathrm{Gb}(\mathrm{tp}(\bar{a}'/N)) \subseteq M^{\mathrm{eq}}, \quad \text{for every model } \mathfrak{M} \text{ with } U \subseteq M \subseteq N.$$

By Lemma E2.1.9, this implies that $\mathrm{Gb}(\mathrm{tp}(\bar{a}'/N)) \subseteq \mathrm{acl}^{\mathrm{eq}}(U)$. Consequently, we can use Lemma E2.3.10 to show that $\mathrm{tp}(\bar{a}'/N)$ is definable over $\mathrm{acl}^{\mathrm{eq}}(U)$. In particular, $\bar{a}' \sqrt[\mathrm{df}]{\,}\!/_{\mathrm{acl}^{\mathrm{eq}}(U)} C$.

It remains to prove the above claim. Consider a model $\mathfrak{M}$ with $U \subseteq M \subseteq N$. Then $\bar{a}' \mathop{\downarrow}^{\mathrm{f}}_{M} N$ implies that $\bar{a}' \sqrt[\mathrm{df}]{\,}\!/_{M} N$, by Theorem 2.4. Therefore, it follows by Lemma E2.3.8 that

$$\mathrm{Gb}(\mathrm{tp}(\bar{a}'/N)) \subseteq \mathrm{dcl}^{\mathrm{eq}}(M) = M^{\mathrm{eq}}. \qquad \square$$

Corollary 2.6. *In a stable theory,*

$$A \mathop{\downarrow}^{\mathrm{f}}_{U} B \quad \textit{implies} \quad A \sqrt[\mathrm{df}]{\,}\!/_{\mathrm{acl}^{\mathrm{eq}}(U)} B\,.$$

3. Stationary types

In this section we study types with a unique free extension over every set of parameters. Such types are called *stationary.*

Definition 3.1. A type $\mathfrak{p}$ over U is *stationary* if, for every set $C \subseteq \mathbb{M}$, $\mathfrak{p}$ has a unique free extension to a complete type over $U \cup C$.

We start by proving that stationary types exist.

Proposition 3.2. *In a stable theory, every type over a set of the form* $\mathrm{acl}^{\mathrm{eq}}(U)$ *is stationary.*

Proof. Note that a type $\mathfrak{p}(\bar{x})$ is stationary if, and only if, for every finite tuple $\bar{x}' \subseteq \bar{x}$ of variables, the restriction $\mathfrak{p} \upharpoonright \bar{x}'$ is stationary. Hence, it is sufficient to consider types $\mathfrak{p} \in S^{<\omega}(\mathrm{acl}^{\mathrm{eq}}(U))$. Let $C \supseteq \mathrm{acl}^{\mathrm{eq}}(U)$ be a set and suppose that $\bar{a}$ and $\bar{a}'$ are two realisations of $\mathfrak{p}$ with

$$\bar{a} \mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}_{\mathrm{acl}^{\mathrm{eq}}(U)} C \quad \text{and} \quad \bar{a}' \mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}_{\mathrm{acl}^{\mathrm{eq}}(U)} C\,.$$

We have to show that $\bar{a} \equiv_C \bar{a}'$. Hence, consider a formula $\varphi(\bar{x};\bar{c})$ with $\bar{c} \subseteq C$. We choose an $\aleph_0$-saturated model $\mathfrak{M}$ containing $C \cup \bar{a}\bar{a}'$. There are tuples $\bar{a}_* \equiv_C \bar{a}$ and $\bar{a}'_* \equiv_C \bar{a}'$ with

$$\bar{a}_* \mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}_{\mathrm{acl}^{\mathrm{eq}}(U)} M \quad \text{and} \quad \bar{a}'_* \mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}_{\mathrm{acl}^{\mathrm{eq}}(U)} M\,.$$

Since $\bar{c} \mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}_{\mathrm{acl}^{\mathrm{eq}}(U)} \bar{a}$, there is a tuple $\bar{c}_* \equiv_{\mathrm{acl}^{\mathrm{eq}}(U) \cup \bar{a}} \bar{c}$ with

$$\bar{c}_* \mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}_{\mathrm{acl}^{\mathrm{eq}}(U)} M\,.$$

By Corollary 2.6, the types

$$\mathfrak{q} := \mathrm{tp}(\bar{a}_*/M)\,, \quad \mathfrak{q}' := \mathrm{tp}(\bar{a}'_*/M)\,, \quad \text{and} \quad \mathfrak{r} := \mathrm{tp}(\bar{c}_*/M)$$

are definable over $\mathrm{acl}^{\mathrm{eq}}(U)$. Let $\delta(\bar{y})$, $\delta'(\bar{y})$, and $\varepsilon(\bar{x})$ be the corresponding φ-definitions. By Lemma 1.3, it follows that

$$\begin{aligned}
\mathbb{M} \vDash \varphi(\bar{a};\bar{c}) \quad &\text{iff} \quad \mathbb{M} \vDash \varphi(\bar{a}_*;\bar{c}) \\
&\text{iff} \quad \mathbb{M} \vDash \delta(\bar{c}) \\
&\text{iff} \quad \mathbb{M} \vDash \delta(\bar{c}_*) \\
&\text{iff} \quad \delta(\bar{y}) \in \mathfrak{r} \\
&\text{iff} \quad \varepsilon(\bar{y}) \in \mathfrak{q} \\
&\text{iff} \quad \varepsilon(\bar{y}) \in \mathfrak{q}' \\
&\text{iff} \quad \delta'(\bar{y}) \in \mathfrak{r} \\
&\text{iff} \quad \mathbb{M} \vDash \delta'(\bar{c}_*) \\
&\text{iff} \quad \mathbb{M} \vDash \delta'(\bar{c}) \\
&\text{iff} \quad \mathbb{M} \vDash \varphi(\bar{a}'_*;\bar{c}) \quad \text{iff} \quad \mathbb{M} \vDash \varphi(\bar{a}';\bar{c})\,. \qquad \square
\end{aligned}$$

Corollary 3.3. *In a stable theory types over models are stationary.*

Proof. Note that every type over a model M has a unique extension to a type over $M^{\mathrm{eq}} = \mathrm{dcl}^{\mathrm{eq}}(M)$, which is an algebraically closed set. Hence, the claim follows by Proposition 3.2. □

In Proposition 3.7 below we will present a characterisation of stationary types in terms of the relation $\sqrt[\mathrm{i}]{}$. We start with two technical lemmas. In the first one, we prove that all free extensions of a given type are conjugate.

Lemma 3.4. *Let T be a stable theory, $\kappa > |T|$ a cardinal, $\mathfrak{M}$ a strongly κ-homogeneous model of T, and $U \subseteq M$ a set of size $|U| < \kappa$. If*

$$\bar{a} \equiv_U \bar{a}'\,, \quad \bar{a} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_U M\,, \quad \text{and} \quad \bar{a}' \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_U M\,,$$

then there exists an automorphism $\pi \in \mathrm{Aut}\,\mathbb{M}_U$ such that $\pi(\bar{a}') = \bar{a}$ and $\pi[M] = M$.

Proof. Since $\bar{a} \equiv_U \bar{a}'$, there exists an automorphism $\pi \in \operatorname{Aut} \mathbb{M}_U$ with $\pi(\bar{a}') = \bar{a}$. By Corollary E2.1.7, we have

$$\pi[\operatorname{acl}^{\mathrm{eq}}(U)] = \operatorname{acl}^{\mathrm{eq}}(U) .$$

As $\mathfrak{M}$ is strongly $|\operatorname{acl}^{\mathrm{eq}}(U)|^+$-homogeneous, we can find an automorphism $\sigma_0 \in \operatorname{Aut} \mathfrak{M}$ with

$$\sigma_0 \upharpoonright \operatorname{acl}^{\mathrm{eq}}(U) = \pi \upharpoonright \operatorname{acl}^{\mathrm{eq}}(U) .$$

Let $\sigma \in \operatorname{Aut} \mathbb{M}$ be an extension of σ_0. For every formula $\varphi(\bar{x}; \bar{c})$ with parameters $\bar{c} \subseteq \operatorname{acl}^{\mathrm{eq}}(U)$, it follows that

$$\begin{aligned} \mathbb{M} \vDash \varphi(\sigma(\bar{a}'); \bar{c}) \quad & \text{iff} \quad \mathbb{M} \vDash \varphi(\bar{a}'; \sigma^{-1}(\bar{c})) \\ & \text{iff} \quad \mathbb{M} \vDash \varphi(\pi(\bar{a}'); \pi(\sigma^{-1}(\bar{c}))) \\ & \text{iff} \quad \mathbb{M} \vDash \varphi(\bar{a}; \bar{c}) . \end{aligned}$$

Hence, $\sigma(\bar{a}') \equiv_{\operatorname{acl}^{\mathrm{eq}}(U)} \bar{a}$. By invariance, we have

$$\sigma(\bar{a}') \mathop{\downarrow}^{\mathrm{f}}_{\operatorname{acl}^{\mathrm{eq}}(U)} M \quad \text{and} \quad \bar{a} \mathop{\downarrow}^{\mathrm{f}}_{\operatorname{acl}^{\mathrm{eq}}(U)} M .$$

Moreover, $\operatorname{tp}(\bar{a}/\operatorname{acl}^{\mathrm{eq}}(U))$ is stationary according to Proposition 3.2. Therefore, $\sigma(\bar{a}') \equiv_M \bar{a}$ and there exists an automorphism $\rho \in \operatorname{Aut} \mathbb{M}_M$ mapping $\sigma(\bar{a}')$ to $\bar{a}$. Since

$$\rho[\sigma[M]] = \rho[\sigma_0[M]] = \rho[M] = M ,$$

the composition $\rho \circ \sigma \in \operatorname{Aut} \mathbb{M}_U$ is the desired automorphism mapping $\bar{a}'$ to $\bar{a}$. □

The second lemma characterises those free extensions that are unique.

Definition 3.5. We write

$$\bar{a} \mathop{\downarrow}^{!}_{U} B \quad :\text{iff} \quad \begin{array}{l} \operatorname{tp}(\bar{a}/UB) \text{ is the unique free extension of} \\ \operatorname{tp}(\bar{a}/U) \text{ over } U \cup B . \end{array}$$

Lemma 3.6. *Let T be a stable theory, $\kappa > |T|$ a cardinal, $\mathfrak{M}$ a strongly κ-homogeneous model of T, and $U \subseteq M$ a set of size $|U| < \kappa$. Then*

$$\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{!}_{U} M \quad \text{iff} \quad \bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_{U} M \quad \textit{and} \quad \bar{a} \sqrt[\mathrm{s}]{}_{U} M\,.$$

Proof. ($\Leftarrow$) As $\mathrm{tp}(\bar{a}/M)$ is a free extension of $\mathrm{tp}(\bar{a}/U)$, we only need to prove uniqueness. If $\mathrm{tp}(\bar{a}/M)$ and $\mathrm{tp}(\bar{a}'/M)$ are two free extensions of $\mathrm{tp}(\bar{a}/U)$, we can use Lemma 3.4 to find an automorphism $\pi \in \mathrm{Aut}\,\mathbb{M}_U$ with $\pi(\bar{a}') = \bar{a}$ and $\pi[M] = M$. Hence, for every formula $\varphi(\bar{x};\bar{y})$ over U and every $\bar{b} \subseteq M$, $\bar{a} \sqrt[\mathrm{s}]{}_{U} M$ implies that

$$\mathbb{M} \vDash \varphi(\bar{a}';\bar{b}) \quad \text{iff} \quad \mathbb{M} \vDash \varphi(\bar{a};\pi(\bar{b})) \quad \text{iff} \quad \mathbb{M} \vDash \varphi(\bar{a};\bar{b})\,.$$

Consequently, $\bar{a} \equiv_M \bar{a}'$.

($\Rightarrow$) If $\bar{a} \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_{U} M$, the type $\mathrm{tp}(\bar{a}/M)$ is not a free extension of $\mathrm{tp}(\bar{a}/U)$ and we are done. Hence, suppose that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_{U} M$ and $\bar{a} \not\sqrt[\mathrm{s}]{}_{U} M$. We claim that $\mathrm{tp}(\bar{a}/U)$ has at least two free extensions over M. By assumption, we can find finite tuples $\bar{b}, \bar{b}' \subseteq M$ with $\bar{b} \equiv_U \bar{b}'$ and $\bar{b} \not\equiv_{U\bar{a}} \bar{b}'$. Let $\varphi(\bar{x};\bar{y})$ be a formula over U such that

$$\mathbb{M} \vDash \varphi(\bar{a};\bar{b}) \wedge \neg\varphi(\bar{a};\bar{b}')\,.$$

Since $\bar{b} \equiv_U \bar{b}'$ and $\mathfrak{M}$ is strongly κ-homogeneous, there exists an automorphism $\pi_0 \in \mathrm{Aut}\,\mathfrak{M}_U$ mapping $\bar{b}$ to $\bar{b}'$. Let π be an automorphism of $\mathbb{M}$ extending π_0. Then

$$\mathbb{M} \vDash \varphi(\bar{a};\bar{b}) \quad \text{implies} \quad \mathbb{M} \vDash \varphi(\pi(\bar{a});\bar{b}')\,.$$

Hence, $\pi(\bar{a}) \not\equiv_M \bar{a}$. Furthermore,

$$\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_{U} M \quad \text{implies} \quad \pi(\bar{a}) \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_{U} \pi[M]\,.$$

Since $\pi[M] = \pi_0[M] = M$, it follows that $\mathrm{tp}(\bar{a}/M)$ and $\mathrm{tp}(\pi(\bar{a})/M)$ are two different free extensions of $\mathrm{tp}(\bar{a}/U)$. □

Proposition 3.7. *Let T be stable. Then*

$$\mathrm{tp}(\bar{a}/U) \textit{ is stationary} \quad \text{iff} \quad \bar{a} \sqrt[\mathrm{i}]{}_{U} U\,.$$

Proof. (⇒) Suppose that $\mathfrak{p} := \mathrm{tp}(\bar a/U)$ is stationary. Since $\sqrt[\mathrm{i}]{} = {}^*(\sqrt[\mathrm{s}]{})$, it is sufficient to show that $\bar a \; {}^*(\sqrt[\mathrm{s}]{})_U \; U$. Hence, consider a set $B \supseteq U$. We fix a strongly $(|T|^+ \oplus |U|^+)$-homogeneous model $\mathfrak{M}$ containing B. Let $\mathfrak{q}$ be the unique free extension of $\mathfrak{p}$ to M and let $\bar a'$ be a realisation of $\mathfrak{q}$. Then $\bar a' \equiv_U \bar a$ and it follows by Lemma 3.6 that $\bar a' \sqrt[\mathrm{s}]{}_U M$. In particular, we have $\bar a' \sqrt[\mathrm{s}]{}_U B$.

(⇐) Suppose that $\bar a \sqrt[\mathrm{i}]{}_U U$. To show that $\mathfrak{p} := \mathrm{tp}(\bar a/U)$ is stationary, consider a set $B \supseteq U$. We fix a strongly $(|T|^+ \oplus |U|^+)$-homogeneous model $\mathfrak{M}$ containing B. Let $\bar a' \equiv_U \bar a$ be a tuple with $\bar a' \sqrt[\mathrm{i}]{}_U M$. By Proposition F3.1.12, it follows that $\bar a' \mathop{\downarrow}^{\mathrm{f}}_U M$ and $\bar a' \sqrt[\mathrm{s}]{}_U M$. Therefore, Lemma 3.6 implies that $\bar a' \mathop{\downarrow}^{!}_U M$. In particular, $\mathrm{tp}(\bar a'/B)$ is the unique free extension of $\mathfrak{p}$ over B. □

As an application of stationary types, we present the following topological characterisation of the set of free extensions of a type.

Theorem 3.8 (Open Mapping Theorem). *Let T be a stable theory, $U \subseteq A$ sets, and let $\mathfrak{F}^{\bar s}(A/U)$ denote the subspace of $\mathfrak{S}^{\bar s}(A)$ consisting of all types that do not fork over U.*

(a) *$F^{\bar s}(A/U)$ is a closed subset of $\mathfrak{S}^{\bar s}(A)$.*

(b) *The restriction map $\rho : \mathfrak{F}^{\bar s}(A/U) \to \mathfrak{S}^{\bar s}(U) : \mathfrak{p} \mapsto \mathfrak{p}|_U$ is continuous, closed, open, and surjective.*

Proof. (a) We use (DEF) to fix, for every type $\mathfrak{p} \in S^{\bar s}(A) \setminus F^{\bar s}(A/U)$, a formula $\varphi_{\mathfrak{p}}(\bar x) \in \mathfrak{p}$ such that

$$\mathbb{M} \vDash \varphi_{\mathfrak{p}}(\bar c) \quad \text{implies} \quad \bar c \not\mathop{\downarrow}^{\mathrm{f}}_U A\,.$$

Setting

$$\Phi := \{\, \neg\varphi_{\mathfrak{p}} \mid \mathfrak{p} \in S^{\bar s}(A) \setminus F^{\bar s}(A/U) \,\}$$

it follows that $\Phi \subseteq \mathfrak{p}$, for every $\mathfrak{p} \in F^{\bar s}(A/U)$, while $\neg\varphi_{\mathfrak{p}} \in \Phi \setminus \mathfrak{p}$, for every $\mathfrak{p} \notin F^{\bar s}(A/U)$. Hence,

$$F^{\bar s}(A/U) = \langle \Phi \rangle_{\mathfrak{S}^{\bar s}(A)}\,,$$

which is a closed set.

(b) We have seen in Corollary c3.2.22 that the restriction map

$$\rho^+ : \mathfrak{S}^{\bar{s}}(A) \to \mathfrak{S}^{\bar{s}}(U) : \mathfrak{p} \mapsto \mathfrak{p}|_U$$

is continuous, closed, and surjective. By (a) it follows that the restriction

$$\rho = \rho^+ \upharpoonright F^{\bar{s}}(A/U) : \mathfrak{F}^{\bar{s}}(A/U) \to \mathfrak{S}^{\bar{s}}(U)$$

is also continuous and closed. Furthermore, it follows by (EXT) that every type over U has a free extension to a type over A. Hence, ρ is surjective and it remains to prove that it is open.

First, we consider the case where A is a strongly $|U|^+$-homogeneous model. Every open set $O \subseteq F^{\bar{s}}(A/U)$ is a union of basic open sets of the form

$$\langle \varphi \rangle_{\mathfrak{F}^{\bar{s}}(A/U)} := \{ \mathfrak{p} \in F^{\bar{s}}(A/U) \mid \varphi \in \mathfrak{p} \} .$$

Therefore it is sufficient to prove that the image of a basic open set is open. Let $\varphi(\bar{x};\bar{y})$ be a formula over U, $\bar{c} \subseteq A$ parameters, and let

$$\overline{\langle \varphi(\bar{x};\bar{c}) \rangle}_{\mathfrak{F}^{\bar{s}}(A/U)} := \bigcup \{ \langle \varphi(\bar{x};\pi(\bar{c})) \rangle_{\mathfrak{F}^{\bar{s}}(A/U)} \mid \pi \in \operatorname{Aut} \mathfrak{M}_U \}$$

be the closure of $\langle \varphi(\bar{x};\bar{c}) \rangle_{\mathfrak{F}^{\bar{s}}(A/U)}$ under conjugates. Being a union of open sets, this set is also open. Furthermore,

$$\rho^{-1}\big[\rho\big[\langle \varphi(\bar{x};\bar{c}) \rangle_{\mathfrak{F}^{\bar{s}}(A/U)}\big]\big] = \overline{\langle \varphi(\bar{x};\bar{c}) \rangle}_{\mathfrak{F}^{\bar{s}}(A/U)} ,$$

since, for types $\mathfrak{p}, \mathfrak{q} \in F^{\bar{s}}(A/U)$ with $\mathfrak{p}|_U = \mathfrak{q}|_U$, we can use Lemma 3.4 to find an automorphism $\pi \in \operatorname{Aut} \mathfrak{M}_U$ with $\pi(\mathfrak{p}) = \mathfrak{q}$.

For a type $\mathfrak{p}_0 \in S^{\bar{s}}(U)$, it follows that

$$\begin{aligned}
& \mathfrak{p}_0 \notin \rho[\langle \varphi(\bar{x};\bar{c}) \rangle_{\mathfrak{F}^{\bar{s}}(A/U)}] \\
\text{iff}\quad & \rho^{-1}(\mathfrak{p}_0) \cap \langle \varphi(\bar{x};\bar{c}) \rangle_{\mathfrak{F}^{\bar{s}}(A/U)} = \varnothing \\
\text{iff}\quad & \rho^{-1}(\mathfrak{p}_0) \cap \overline{\langle \varphi(\bar{x};\bar{c}) \rangle}_{\mathfrak{F}^{\bar{s}}(A/U)} = \varnothing
\end{aligned}$$

$$\begin{aligned}
&\text{iff} \quad \rho^{-1}(\mathfrak{p}_0) \setminus \overline{\langle \varphi(\bar{x};\bar{c})\rangle_{\mathfrak{F}^{\bar{s}}(A/U)}} \neq \emptyset \\
&\text{iff} \quad \mathfrak{p}_0 \in \rho\big[F^{\bar{s}}(A/U) \setminus \overline{\langle \varphi(\bar{x};\bar{c})\rangle_{\mathfrak{F}^{\bar{s}}(A/U)}}\big].
\end{aligned}$$

Hence, the complement of $\rho[\langle \varphi(\bar{x};\bar{c})\rangle_{\mathfrak{F}^{\bar{s}}(A/U)}]$ is the image of a closed set. As we have shown above that the map ρ is closed, it follows that the complement is closed and the image $\rho[\langle \varphi(\bar{x};\bar{c})\rangle_{\mathfrak{F}^{\bar{s}}(A/U)}]$ is open.

It remains to prove the general case. We fix a strongly $|U|^+$-homogeneous model $\mathfrak{M}$ containing A and let

$$\rho' : \mathfrak{F}^{\bar{s}}(M/U) \to \mathfrak{F}^{\bar{s}}(A/U)$$

be the corresponding restriction map. Consider a basic open set

$$\langle \varphi(\bar{x};\bar{c})\rangle_{\mathfrak{F}^{\bar{s}}(A/U)}$$

in $\mathfrak{F}^{\bar{s}}(A/U)$. Then $\langle \varphi(\bar{x};\bar{c})\rangle_{\mathfrak{F}^{\bar{s}}(M/U)}$ is basic open in $\mathfrak{F}^{\bar{s}}(M/U)$ and

$$\langle \varphi(\bar{x};\bar{c})\rangle_{\mathfrak{F}^{\bar{s}}(M/U)} = (\rho')^{-1}\big[\langle \varphi(\bar{x};\bar{c})\rangle_{\mathfrak{F}^{\bar{s}}(A/U)}\big].$$

As ρ' is surjective, we have

$$\begin{aligned}
\rho\big[\langle \varphi(\bar{x};\bar{c})\rangle_{\mathfrak{F}^{\bar{s}}(A/U)}\big] &= \rho\big[\rho'\big[(\rho')^{-1}\big[\langle \varphi(\bar{x};\bar{c})\rangle_{\mathfrak{F}^{\bar{s}}(A/U)}\big]\big]\big] \\
&= (\rho \circ \rho')\big[\langle \varphi(\bar{x};\bar{c})\rangle_{\mathfrak{F}^{\bar{s}}(M/U)}\big]
\end{aligned}$$

This set is open, since we have shown above that the composition $\rho \circ \rho'$ is an open map. □

4. *The multiplicity of a type*

Most types have several free extensions. In this section we study their number. We will prove in Theorem 4.6 below that a theory is stable if, and only if, the number of such extensions is bounded.

Definition 4.1. Let T be a theory and $\surd$ a forking relation.

(a) The $\sqrt{}$-*multiplicity* $\operatorname{mult}_{\sqrt{}}(\mathfrak{p})$ of a type $\mathfrak{p}$ over A is the minimal cardinal κ such that, for every set $B \supseteq A$, there are at most κ complete types over B that are $\sqrt{}$-free extensions of $\mathfrak{p}$. If such a cardinal κ does not exist, we set $\operatorname{mult}_{\sqrt{}}(\mathfrak{p}) := \infty$. For $\sqrt{} = \sqrt[f]{}$, we drop the subscript and simply write $\operatorname{mult}(\mathfrak{p})$.

(b) The *multiplicity* $\operatorname{mult}(\sqrt{})$ of $\sqrt{}$ is the maximal $\sqrt{}$-multiplicity of some complete type (with finitely many variables). If there is no maximum, we set $\operatorname{mult}(\sqrt{}) := \infty$.

We start by proving that, in a stable theory, every type has bounded multiplicity.

Lemma 4.2. *Let T be a stable theory.*

(a) *For every type $\mathfrak{p} \in S^{\bar{s}}(A)$ with $|\bar{s}| < \omega$, there exists some model $\mathfrak{M}$ of T of size $|M| \leq |T|$ such that*

$$\operatorname{mult}_{\downarrow^{f}}(\mathfrak{p}) \leq |S^{\bar{s}}(M)|\,.$$

(b) $\operatorname{mult}(\downarrow^{f}) \leq \sup\big\{\, |S^{<\omega}(U)| \;\big|\; |U| \leq |T| \,\big\} \leq 2^{|T|}$

Proof. (a) For every type $\mathfrak{p} \in S^{\bar{s}}(A)$, there exists a set $U \subseteq A$ of size

$$|U| < \operatorname{loc}_0(\downarrow^{f}) \leq \operatorname{fc}(\downarrow^{f}) \leq |T|^{+}$$

such that $\mathfrak{p}$ does not fork over U. Since $|U| \leq |T|$, we can find a model $\mathfrak{M}$ of size $|M| = |T|$ containing U. Since every type has at least one free extension over any given set, it is sufficient to bound the number of free extensions of $\mathfrak{p}$ over sets B containing M. Hence, let $\mathfrak{q} \in S^{\bar{s}}(B)$ be an extension of $\mathfrak{p}$ with $B \supseteq M$. We have seen in Corollary 3.3 that types over models are stationary. Hence, $\mathfrak{q}$ is the unique free extension of $\mathfrak{q}|_M$. Consequently, if $\mathfrak{q}, \mathfrak{q}' \in S^{\bar{s}}(B)$ are distinct free extensions of $\mathfrak{p}$, then $\mathfrak{q}|_M \neq \mathfrak{q}'|_M$. Therefore, $\mathfrak{p}$ has at most $|S^{\bar{s}}(M)|$ free extensions.

(b) The first inequality follows immediately from (a). The second one follows from the fact that there are at most $2^{|T|}$ types over a set of size $|T|$. ☐

A characterisation of stable theories

Recall that we write $\mathfrak{p} \leq_{\surd} \mathfrak{q}$ if $\mathfrak{q}$ is a $\surd$-free extension of $\mathfrak{p}$. This is a definition of $\leq_{\surd}$ in terms of a given preforking relation $\surd$. Conversely, given an extension relation $\leq$ we can recover a corresponding preforking relation $\surd$. In the remainder of this section we present two characterisations of stable theories one in terms of the extension relation $\leq$ and one in terms of the parameter $\mathrm{mult}(\downarrow^{\mathrm{f}})$.

Proposition 4.3. *If* $\downarrow$ *is a symmetric forking relation with* $\mathrm{mult}(\downarrow) < \infty$, *then* $\leq_{\downarrow}$ *satisfies the following conditions:*

(INV) *Invariance.* $\mathfrak{p} \leq \mathfrak{q}$ *implies* $\pi(\mathfrak{p}) \leq \pi(\mathfrak{q})$, *for every automorphism* $\pi \in \mathrm{Aut}(\mathbb{M})$.

(LC) *Local Character. There exists a cardinal* κ *such that, for every set* U *and every type* $\mathfrak{p} \in S^{<\omega}(U)$, *there exists a subset* $U_0 \subseteq U$ *of size* $|U_0| < \kappa$ *such that* $\mathfrak{p} \restriction U_0 \leq \mathfrak{p}$.

(BND) *Boundedness. For every type* $\mathfrak{p} \in S^{<\omega}(U)$, *there exists a cardinal* μ *such that, for every set* $C \subseteq \mathbb{M}$, $\mathfrak{p}$ *has at most* μ *extensions* $\mathfrak{q} \in S^{<\omega}(U \cup C)$ *with* $\mathfrak{p} \leq \mathfrak{q}$.

(EXT) *Extension. For every* $\mathfrak{p} \in S^{<\omega}(U)$ *and every set* $C \subseteq \mathbb{M}$, *there exists some type* $\mathfrak{q} \in S^{<\omega}(U \cup C)$ *with* $\mathfrak{p} \leq \mathfrak{q}$.

(TR) *Transitivity.* $\mathfrak{p} \leq \mathfrak{q} \leq \mathfrak{r}$ *implies* $\mathfrak{p} \leq \mathfrak{r}$.

(MON) *Monotonicity.* $\mathfrak{p} \leq \mathfrak{r}$ *implies* $\mathfrak{p} \leq \mathfrak{q}$, *for all* $\mathfrak{p} \subseteq \mathfrak{q} \subseteq \mathfrak{r}$.

Proof. (BND) holds since $\mathrm{mult}(\downarrow) < \infty$. The other axioms follow from the fact that $\downarrow$ is a symmetric forking relation: (INV) follows by invariance; (LC) follows by right locality; (EXT) follows by the extension axiom; (TR) follows by left transitivity and symmetry; and (MON) follows by monotonicity. □

For the converse statement, we need a technical lemma.

Lemma 4.4. *Let $\mathfrak{M}$ be a κ^+-saturated, strongly κ^+-homogeneous model, $U \subseteq M$ a set of size $|U| < \kappa$, and $\bar{a} \in \mathbb{M}^{<\omega}$. If $\bar{a} \not\mathrel{\smile}^{\mathrm{d}}_U M$, then* $\mathrm{tp}(\bar{a}/M)$ *has at least κ conjugates over U.*

Proof. If $\bar{a} \not\mathrel{\smile}^{\mathrm{d}}_U M$, there exists a finite tuple $\bar{b} \subseteq M$ such that $\bar{a} \not\mathrel{\smile}^{\mathrm{d}}_U \bar{b}$. By Lemma F3.1.3, we can find an indiscernible sequence $(\bar{b}_n)_{n<\omega}$ over U with $\bar{b} = \bar{b}_0$ such that, for each tuple $\bar{a}' \equiv_{U\bar{b}} \bar{a}$, there are indices $m, n < \omega$ with

$$\bar{b}_m \not\equiv_{U\bar{a}'} \bar{b}_n \,.$$

Fix a formula $\varphi(\bar{x}; \bar{y})$ over U such that

$$\mathbb{M} \vDash \varphi(\bar{a}; \bar{b}_m) \wedge \neg\varphi(\bar{a}; \bar{b}_n)\,, \quad \text{for some } m, n < \omega\,.$$

Replacing φ be its negation, if necessary, we may assume that there are infinitely many indices n such that $\mathbb{M} \vDash \neg\varphi(\bar{a}; \bar{b}_n)$. By compactness, it follows that there exists a tuple $\bar{a}' \equiv_U \bar{a}$ and an indiscernible sequence $(\bar{b}'_\alpha)_{\alpha<\kappa}$ of length κ such that

$$\mathbb{M} \vDash \varphi(\bar{a}'; \bar{b}'_\alpha) \quad \text{iff} \quad \alpha = 0\,.$$

As $\mathfrak{M}$ is κ^+-saturated, we may choose the sequence $(\bar{b}'_\alpha)_{\alpha<\kappa}$ to be in M. By strong κ^+-homogeneity we can find, for every $\alpha < \kappa$, an automorphism $\sigma_\alpha \in \mathrm{Aut}\,\mathfrak{M}_U$ such that

$$\sigma_\alpha(\bar{b}'_\beta) = \bar{b}'_{\alpha+\beta}\,, \quad \text{for all } \beta < \kappa\,.$$

Let $\pi_\alpha \in \mathrm{Aut}\,\mathbb{M}$ be an extension of σ_α and set

$$\bar{a}_\alpha := \pi_\alpha(\bar{a}')\,, \quad \text{for } \alpha < \kappa\,.$$

Then

$$\begin{aligned} \mathbb{M} \vDash \varphi(\bar{a}_\alpha; \bar{b}'_{\alpha+\beta}) \quad &\text{iff} \quad \mathbb{M} \vDash \varphi\big(\pi_\alpha(\bar{a}'); \pi_\alpha(\bar{b}'_\beta)\big) \\ &\text{iff} \quad \mathbb{M} \vDash \varphi(\bar{a}'; \bar{b}'_\beta) \\ &\text{iff} \quad \beta = 0\,. \end{aligned}$$

Consequently, $\operatorname{tp}(\bar{a}_\alpha/M) \neq \operatorname{tp}(\bar{a}_\beta/M)$, for $\alpha \neq \beta$. Since these types are extensions of $\operatorname{tp}(\bar{a}/U)$ and they are conjugate over U, the claim follows. □

Theorem 4.5. *Let T be a complete first-order theory.*

(a) *T is stable if, and only if, there exists a relation $\leq$ on complete types satisfying* (INV), (LC), *and* (BND).

(b) *If $\leq$ is an extension relation satisfying* (INV), (LC), (BND), (EXT), (TR), *and* (MON), *then $\leq = \leq_{\text{⫝f}}$.*

Proof. (a) ($\Rightarrow$) If T is stable, the relation $\leq_{\text{⫝f}}$ has the desired properties by Proposition 4.3. For ($\Leftarrow$), suppose that $\leq$ satisfies (INV), (LC), and (BND). Let κ be the cardinal from (LC) and fix a κ-saturated model $\mathfrak{M}$. For $U \subseteq M$ and $\mathfrak{p} \in S^{<\omega}(U)$, we denote by $\mu(\mathfrak{p}; U)$ the cardinal from (BND). Set

$$\mu := \sup\left\{ \mu(\mathfrak{p}; U) \;\middle|\; U \subseteq M \text{ with } |U| < \kappa \text{ and } \mathfrak{p} \in S^{<\omega}(U) \right\}.$$

Since, for every subset $U \subseteq \mathbb{M}$ of size $|U| < \kappa$, there is some automorphism $\pi \in \operatorname{Aut}\mathbb{M}$ with $\pi[U] \subseteq M$, it follows by (INV) that, for all sets $U, C \subseteq \mathbb{M}$ with $|U| < \kappa$ and for every type $\mathfrak{p} \in S^{<\omega}(U)$, there are at most μ types $\mathfrak{q} \in S^{<\omega}(U \cup C)$ with $\mathfrak{p} \leq \mathfrak{q}$.

Fix a set $U \subseteq \mathbb{M}$ and a finite tuple $\bar{s}$ of sorts. For every type $\mathfrak{p} \in S^{\bar{s}}(U)$, we can fix, by (LC), a subset $C(\mathfrak{p}) \subseteq U$ of size $|C(\mathfrak{p})| < \kappa$ such that $\mathfrak{p}|_{C(\mathfrak{p})} \leq \mathfrak{p}$. Let $C \subseteq U$ be a set of size $|C| < \kappa$. Then

- $|S^{\bar{s}}(C)| \leq 2^{|T| \oplus \kappa}$ and,
- for every type $\mathfrak{q} \in S^{\bar{s}}(C)$, there are at most μ types $\mathfrak{p} \in S^{\bar{s}}(U)$ with $C(\mathfrak{p}) = C$ and $\mathfrak{p}|_C = \mathfrak{q}$.

Consequently, we have

$$\left|S^{\bar{s}}(U)\right| \leq |U|^{<\kappa} \otimes 2^{|T| \oplus \kappa} \otimes \mu\,.$$

Setting $\lambda_0 := \mu \oplus 2^{|T|}$ and $\lambda := \lambda_0^\kappa$, it follows that

$$\left|S^{\bar{s}}(U)\right| \leq \lambda^\kappa \otimes \lambda_0^\kappa \otimes \mu = \lambda\,, \quad \text{for every set } U \text{ of size } |U| \leq \lambda\,.$$

Hence, T is λ-stable.

(b) Let $\mathfrak{q} \in S^{\bar{s}}(B)$ be an extension of $\mathfrak{p} \in S^{\bar{s}}(A)$. We have to show that $\mathfrak{p} \leq \mathfrak{q}$ if, and only if, $\mathfrak{q}$ is a $\downarrow^{\mathrm{f}}$-free extension of $\mathfrak{p}$.

($\Rightarrow$) Suppose that $\mathfrak{p} \leq \mathfrak{q}$. By (BND), there is a cardinal μ such that, for every set C, $\mathfrak{p}$ has at most μ $\leq$-free extensions over $A \cup C$. Set $\kappa := \mu^+ \oplus |A|^+$ and let $\mathfrak{M}$ be a κ^+-saturated and strongly κ^+-homogeneous model containing B. We use (EXT) to find a type $\mathfrak{r} \geq \mathfrak{q}$ over M. By (TR), it follows that $\mathfrak{p} \leq \mathfrak{r}$. Hence, (INV) implies that $\mathfrak{r}$ has at most μ conjugates over A. By Lemma 4.4, it follows that $\mathfrak{r}$ does not fork over A. In particular, $\mathfrak{q}$ does not fork over A.

($\Leftarrow$) Suppose that $\mathfrak{q}$ is a free extension of $\mathfrak{p}$. Fix a strongly $(|T| \oplus |A|)^+$-homogeneous model $\mathfrak{M}$ containing B and let $\mathfrak{r}$ be a free extension of $\mathfrak{q}$ over M. By (EXT), there exists a type $\mathfrak{r}' \geq \mathfrak{p}$ over M. By the first part of the proof, $\mathfrak{p} \leq \mathfrak{r}'$ implies that $\mathfrak{r}'$ is a free extension of $\mathfrak{p}$. Let $\bar{a}$ and $\bar{a}'$ be realisations of, respectively, $\mathfrak{r}$ and $\mathfrak{r}'$. Then we can use Lemma 3.4 to find an automorphism $\pi \in \operatorname{Aut} \mathbb{M}_A$ such that $\pi(\bar{a}') = \bar{a}$ and $\pi[M] = M$. By (INV),

$$\mathfrak{p} \leq \mathfrak{q} \quad \text{implies} \quad \mathfrak{p} \leq \operatorname{tp}(\pi(\bar{a}')/\pi[M]) = \operatorname{tp}(\bar{a}/M)\,.$$

Hence, it follows by (MON) that $\mathfrak{p} \leq \operatorname{tp}(\bar{a}/B) = \mathfrak{q}$. □

Translating this theorem into the language of forking relations, we obtain the following characterisation of stable theories.

Theorem 4.6. *Let T be a complete first-order theory. The following statements are equivalent.*

(1) *T is stable.*

(2) *$\sqrt[\mathrm{f}]{}$ is symmetric and* $\operatorname{mult}(\sqrt[\mathrm{f}]{}) < \infty$.

(3) *There exists a symmetric forking relation $\downarrow$ with* $\operatorname{mult}(\downarrow) < \infty$.

Proof. (1) $\Rightarrow$ (2) follows by Lemma 4.2 and the implication (2) $\Rightarrow$ (3) is trivial. For (3) $\Rightarrow$ (1), suppose that $\downarrow$ is a symmetric forking relation with $\operatorname{mult}(\downarrow) < \infty$. By Proposition 4.3, the corresponding extension relation $\leq_{\downarrow}$ satisfies the three axioms from Theorem 4.5 (a). Hence, T is stable. □

Proposition 4.7. *If* $\downarrow$ *is a symmetric forking relation with* $\mathrm{mult}(\downarrow) < \infty$, *then* $\downarrow = \downarrow^{\mathrm{f}}$.

Proof. By Proposition 4.3, the extension relation $\leq_{\downarrow}$ satisfies the six axioms from Theorem 4.5 (b). Consequently, $\leq_{\downarrow} = \leq_{\downarrow^{\mathrm{f}}}$. For a finite tuple $\bar{a}$ and sets $U, B \subseteq \mathbb{M}$, it follows that

$$\begin{aligned} \bar{a} \downarrow_U B \quad &\text{iff} \quad \mathrm{tp}(\bar{a}/U) \leq_{\downarrow} \mathrm{tp}(\bar{a}/B) \\ &\text{iff} \quad \mathrm{tp}(\bar{a}/U) \leq_{\downarrow^{\mathrm{f}}} \mathrm{tp}(\bar{a}/B) \quad \text{iff} \quad \bar{a} \downarrow^{\mathrm{f}}_U B\,. \end{aligned}$$

Hence, finite character implies that $\downarrow = \downarrow^{\mathrm{f}}$. □

As a further application we derive a characterisation of forking in totally transcendental theories by showing that the relation of being a Morley-free extension (which was defined in Section F2.1) satisfies the conditions of the above theorem.

Corollary 4.8. *Let* T *be a totally transcendental theory.*

(a) $\bar{a} \downarrow^{\mathrm{f}}_U B \quad$ iff $\quad \mathrm{rk_M}(\bar{a}/UB) = \mathrm{rk_M}(\bar{a}/U)$, *for all finite* $\bar{a}$.

(b) $\mathrm{mult}_{\downarrow^{\mathrm{f}}}(\mathfrak{p}) < \aleph_0$.

Proof. (a) For types $\mathfrak{p} \in S^{\bar{s}}(U)$ and $\mathfrak{q} \in S^{\bar{s}}(V)$, we define

$$\mathfrak{p} \leq_{\mathrm{M}} \mathfrak{q} \quad :\text{iff} \quad \mathfrak{q} \text{ is a Morley-free extension of } \mathfrak{p}\,.$$

It is sufficient to show that $\leq_{\mathrm{M}}$ satisfies the conditions in Theorem 4.5 (b).

(INV) follows immediately from the definition. (BND) and (EXT) were already shown in Lemma F2.1.9 (a) and (b), respectively, while (LC) was proved in Lemma F2.1.6 (c).

For (TR), suppose that $\mathfrak{p} \leq_{\mathrm{M}} \mathfrak{q} \leq_{\mathrm{M}} \mathfrak{r}$. Then

$$\mathrm{rk_M}(\mathfrak{p}) = \mathrm{rk_M}(\mathfrak{q}) = \mathrm{rk_M}(\mathfrak{r})\,,$$

which implies that $\mathfrak{p} \leq_{\mathrm{M}} \mathfrak{r}$.

(MON) Suppose that $\mathfrak{p} \leq_{\mathrm{M}} \mathfrak{r}$ and $\mathfrak{p} \subseteq \mathfrak{q} \subseteq \mathfrak{r}$. By Lemma F2.1.6 (b), we have

$$\mathrm{rk}_{\mathrm{M}}(\mathfrak{p}) \geq \mathrm{rk}_{\mathrm{M}}(\mathfrak{q}) \geq \mathrm{rk}_{\mathrm{M}}(\mathfrak{r}) = \mathrm{rk}_{\mathrm{M}}(\mathfrak{p}) .$$

Hence, $\mathfrak{p} \leq_{\mathrm{M}} \mathfrak{q}$.

(b) We have seen in Lemma F2.1.9 that every type has only finitely many Morley-free extensions. Hence, the claim follows by (a). □

5. Morley sequences in stable theories

Let us collect several results on Morley sequences in stable theories. Many of the proofs rely on the notion of a stationary type. We start with a proof that we can drop the requirement of indiscernibility from the definition of a Morley sequence if the type in question is stationary.

Lemma 5.1. *Let T be a stable theory, $\mathfrak{p}$ a stationary type over U, and let $(\bar{a}_i)_{i \in I}$ be a sequence of realisations of $\mathfrak{p}$. If*

$$\bar{a}_i \mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}_U \bar{a}[<i] , \quad \textit{for all } i \in I ,$$

then $(\bar{a}_i)_{i \in I}$ is a $\mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}$-Morley sequence for $\mathfrak{p}$ over U.

Proof. We have to show that $(\bar{a}_i)_{i \in I}$ is indiscernible over U. By induction on $n < \omega$, we prove that

$$\bar{a}[\bar{\imath}] \equiv_{U\bar{a}[<l]\bar{a}[>m]} \bar{a}[\bar{k}] , \quad \text{for all } \bar{\imath}, \bar{k} \in [I]^n \text{ with } l < \bar{\imath}, \bar{k} < m .$$

Hence, let $\bar{\imath}, \bar{k} \in [I]^n$ and $l < \bar{\imath}, \bar{k} < m$. By symmetry, we may assume that $i_{n-1} < k_{n-1}$. According to Lemma F2.4.9, we have

$$\bar{a}_{i_{n-1}} \mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}_U \bar{a}[<i_{n-1}]\bar{a}[>m] \quad \text{and} \quad \bar{a}_{k_{n-1}} \mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}_U \bar{a}[<i_{n-1}]\bar{a}[>m] ,$$

Since $\mathfrak{p}$ is stationary, it follows that

$$\bar{a}_{i_{n-1}} \equiv_{U\bar{a}[<i_{n-1}]\bar{a}[>m]} \bar{a}_{k_{n-1}} .$$

By inductive hypothesis, we further have

$$\bar{a}_{i_{n-2}} \dots \bar{a}_{i_0} \equiv_{U\bar{a}[<l]\bar{a}[\geq k_{n-1}]} \bar{a}_{k_{n-2}} \dots \bar{a}_{k_0} .$$

Consequently,

$$\begin{aligned} \bar{a}_{i_{n-1}} \bar{a}_{i_{n-2}} \dots \bar{a}_{i_0} &\equiv_{U\bar{a}[<l]\bar{a}[>m]} \bar{a}_{k_{n-1}} \bar{a}_{i_{n-2}} \dots \bar{a}_{i_0} \\ &\equiv_{U\bar{a}[<l]\bar{a}[>m]} \bar{a}_{k_{n-1}} \bar{a}_{k_{n-2}} \dots \bar{a}_{k_0} . \end{aligned}$$

□

Lemma 5.2. *Let T be a stable theory, $\mathfrak{p}$ a stationary type over U and $\mathfrak{q}$ the unique free extensions of $\mathfrak{p}$ over $U \cup C$.*

(a) *Every $\mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}^{\mathrm{f}}$-Morley sequences $(\bar{a}_i)_{i\in I}$ for $\mathfrak{q}$ over $U \cup C$ is also a $\mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}^{\mathrm{f}}$-Morley sequence for $\mathfrak{p}$ over U.*

(b) *Let $(\bar{a}_i)_{i\in I}$ be a $\mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}^{\mathrm{f}}$-Morley sequence for $\mathfrak{p}$ over U. If $I_0 \subseteq I$ and $C \mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}^{\mathrm{f}}_{U\bar{a}[I_0]} \bar{a}[I]$, then $(\bar{a}_i)_{i\in I\setminus I_0}$ is a Morley sequence for $\mathfrak{q}$ over $U \cup C$.*

Proof. (a) As $(\bar{a}_i)_{i\in I}$ is indiscernible over $U \cup C$, it is trivially indiscernible over U. Furthermore,

$$\bar{a}_i \mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}^{\mathrm{f}}_U C \quad \text{and} \quad \bar{a}_i \mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}^{\mathrm{f}}_{UC} \bar{a}[<i] \quad \text{implies} \quad \bar{a}_i \mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}^{\mathrm{f}}_U \bar{a}[<i] .$$

(b) Set $A_0 := \bar{a}[I_0]$. We start by showing that

$$C\bar{a}[K] \mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}^{\mathrm{f}}_{UA_0} \bar{a}[I \setminus K] , \quad \text{for all finite } K \subseteq I \setminus I_0 .$$

The proof is by induction on $|K|$. If $K = \emptyset$, the claim holds by assumption. Hence, suppose that $K = K_0 \cup \{k\}$ and that we have already shown that

$$C\bar{a}[K_0] \mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}^{\mathrm{f}}_{UA_0} \bar{a}[I \setminus K_0] .$$

Then

$$C\bar{a}[K_0] \mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}^{\mathrm{f}}_{UA_0\bar{a}_k} \bar{a}[I \setminus K] .$$

Furthermore, we have seen in Lemma F2.4.9 that

$$\bar{a}_k \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_{U} \bar{a}[I \smallsetminus K]\,,$$

which implies that

$$\bar{a}_k \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_{UA_0} \bar{a}[I \smallsetminus K]\,.$$

Consequently, it follows by transitivity that

$$C\bar{a}[K_0]\bar{a}_k \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_{UA_0} \bar{a}[I \smallsetminus K]\,.$$

Having proved the claim, it follows by (FIN) that

$$C\bar{a}[<i] \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_{UA_0} \bar{a}_i\,, \quad \text{for all } i \in I\,,$$

which implies that

$$\bar{a}_i \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_{UCA_0} \bar{a}[<i]\,, \quad \text{for all } i \in I \smallsetminus I_0\,.$$

Hence, it follows by Lemma 5.1 that $(\bar{a}_i)_{i \in I \smallsetminus I_0}$ is a $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$-Morley sequences over $U \cup C \cup A_0$. ☐

For stable theories, we can turn every indiscernible sequence into a Morley sequence by increasing the domain of the type.

Proposition 5.3. *Let T be a stable theory, κ an infinite cardinal, and let $(\bar{a}_i)_{i \in I}$ be an infinite indiscernible sequence over U such that $|\bar{a}_i| < \kappa$, for all $i \in I$. There exist a set C of size $|C| < \kappa \oplus \aleph_1$ and a stationary type $\mathfrak{p} \in S^{<\kappa}(U \cup C)$ such that $(\bar{a}_i)_{i \in I}$ is a $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$-Morley sequence for $\mathfrak{p}$ over $U \cup C$.*

Proof. We have seen in Corollary E5.4.13 and Corollary E5.4.14 that, for every formula $\varphi(\bar{x}; \bar{c})$ the set

$$[\![\varphi(\bar{a}_i; \bar{c})]\!]_{i \in I} = \{\, i \in I \mid \mathbb{M} \vDash \varphi(\bar{a}_i; \bar{c}) \,\}$$

is either finite or cofinite and that, for every set $C \subseteq \mathbb{M}$, the type

$$\mathrm{Av}_1((\bar{a}_i)_i/UC) = \{\, \varphi(\bar{x}) \mid \varphi \text{ a formula over } U \cup C \text{ such that } [\![\varphi(\bar{a}_i)]\!]_{i\in I} \text{ is cofinite} \,\}$$

is complete. According to Proposition E5.4.12, there exists, for every formula $\varphi(\bar{x};\bar{y})$ over U, a finite constant $k(\varphi) < \omega$ such that, for all $\bar{c} \subseteq \mathbb{M}$,

$$\varphi(\bar{x};\bar{c}) \in \mathrm{Av}_1((\bar{a}_i)_i/UC) \quad \text{iff} \quad \big|I \smallsetminus [\![\varphi(\bar{a}_i;\bar{c})]\!]_{i\in I}\big| \leq k(\varphi)\,.$$

Choose an injective function $\mu : \omega \to I$ and set $I_0 := \mathrm{rng}\,\mu$. It follows that

$$\begin{aligned} & \varphi(\bar{x};\bar{c}) \in \mathrm{Av}_1((\bar{a}_i)_i/UC) \\ \text{iff} \quad & \big|[\![\varphi(\bar{a}_{\mu(n)};\bar{c})]\!]_{n<2k(\varphi)+1}\big| > k(\varphi) \\ \text{iff} \quad & \mathbb{M} \vDash \bigvee\Big\{ \bigwedge_{n\in K} \varphi(\bar{a}_{\mu(n)};\bar{c}) \;\Big|\; K \subseteq [2k(\varphi)+1]\,,\ |K| = k(\varphi)+1 \Big\}\,. \end{aligned}$$

Consequently, the type $\mathrm{Av}_1((\bar{a}_i)_i/UC)$ is definable over $\bar{a}[I_0]$, for every $C \subseteq \mathbb{M}$. For $C \subseteq \mathbb{M}$, fix a tuple $\bar{a}(C)$ realising $\mathrm{Av}_1((\bar{a}_i)_i/UC\bar{a}[I_0])$. For every C, we have

$$\bar{a}(C) \equiv_{U\bar{a}[I_0]} \bar{a}(\varnothing) \quad \text{and} \quad \bar{a}(C) \overset{\mathrm{df}}{\bigvee}_{\bar{a}[I_0]} UC\,.$$

Consequently,

$$\bar{a}(\varnothing)\ {}^*(\overset{\mathrm{df}}{\bigvee})_{\bar{a}[I_0]}\ U$$

By Propositions 3.7 and 2.5 it follows that the type

$$\mathfrak{p} := \mathrm{tp}(\bar{a}(\varnothing)/U\bar{a}[I_0]) = \mathrm{Av}_1((\bar{a}_i)_i/U\bar{a}[I_0])$$

is stationary. Since the tuple $\bar{a}_i$ realises $\mathrm{Av}_1((\bar{a}_i)_i/U\bar{a}[I_0]\bar{a}[<i])$, for $i \in I \smallsetminus I_0$,

$$\bar{a}_i\ {}^*(\overset{\mathrm{df}}{\bigvee})_{\bar{a}[I_0]}\ U\bar{a}[I_0]\bar{a}[<i] \quad \text{implies} \quad \bar{a}_i \mathop{\downarrow}^{\mathrm{f}}_{\bar{a}[I_0]} U\bar{a}[I_0]\bar{a}[<i]\,,$$

by Propositions 1.5 and F3.1.12. Therefore, it follows by Lemma 5.1 that $(\bar{a}_i)_{i\in I\smallsetminus I_0}$ is a $\mathop{\downarrow}^{\mathrm{f}}$-Morley sequence for $\mathfrak{p}$ over $U \cup \bar{a}[I_0]$. ☐

As an application, it follows that every indiscernible sequence can be turned into an indiscernible sequence over a larger set if we remove some of its elements.

Lemma 5.4. *Let T be a stable theory and $(\bar a_i)_{i\in I}$ and indiscernible sequence over U with $|\bar a_i| < \aleph_0$. For every set $C \subseteq \mathbb{M}$ there exists a set $I_0 \subseteq I$ of size $|I_0| \leq |C| \otimes \mathrm{loc}_0(\mathop{\downarrow}^{\mathrm{f}})$ such that $(\bar a_i)_{i\in I\setminus I_0}$ is indiscernible over $U \cup C \cup \bar a[I_0]$.*

Proof. We have seen in Proposition 5.3 that there exist a set $U' \supseteq U$ and a stationary type $\mathfrak{p}$ over U' such that $(\bar a_i)_{i\in I}$ is a Morley sequence for $\mathfrak{p}$ over U'. For every finite $C_0 \subseteq C$, we can find a set $J(C_0) \subseteq I$ of size $|J(C_0)| < \mathrm{loc}_0(\mathop{\downarrow}^{\mathrm{f}})$ such that

$$C_0 \mathop{\downarrow}^{\mathrm{f}}_{U'\bar a[J(C_0)]} \bar a[I]\,.$$

Setting $I_0 := \bigcup\{\, J(C_0) \mid C_0 \subseteq C \text{ finite}\,\}$ it follows that

$$|I_0| \leq |C| \otimes \mathrm{loc}_0(\mathop{\downarrow}^{\mathrm{f}}) \quad \text{and} \quad C \mathop{\downarrow}^{\mathrm{f}}_{U'\bar a[I_0]} \bar a[I]\,.$$

Consequently, we can use Lemma 5.2 to show that $(\bar a_i)_{i\in I\setminus I_0}$ is a Morley sequence over $U'C\bar a[I_0]$. In particular, it is indiscernible over $UC\bar a[I_0]$. □

In totally transcendental theories, it is particularly simple to find Morley sequences.

Definition 5.5. Let $\surd$ be an abstract independence relation. A family $(A_i)_{i\in I}$ of sets is $\surd$-*independent* over U if

$$A_k \surd_U \bigcup_{i\neq k} A_i\,, \quad \text{for all } k \in I\,.$$

Lemma 5.6. *Let T be a totally transcendental theory and $\mathfrak{p} \in S^{<\omega}(U)$ a type. Every set $I \subseteq \mathfrak{p}^{\mathbb{M}}$ that is $\mathop{\downarrow}^{\mathrm{f}}$-independent over U has a finite partition $I = I_0 \cup \dots \cup I_{n-1}$ such that each I_i is totally indiscernible over U.*

Proof. We have seen in Corollary 4.8 (b) that $\mathrm{mult}_{\downarrow^{\mathrm{f}}}(\mathfrak{p}) < \aleph_0$. Thus, $\mathfrak{p}$ has only finitely many free extensions $\mathfrak{q}_0, \dots, \mathfrak{q}_{n-1}$ over $\mathrm{acl}^{\mathrm{eq}}(U)$. By Lemma 5.1, each set $I_i := I \cap \mathfrak{q}_i^{\mathbb{M}}$ forms a Morley sequence over U. In particular, it is totally indiscernible. □

6. The stability spectrum

The *stability spectrum* of a theory T is the class of all cardinals κ such that T is κ-stable. In this section, we will compute the stability spectrum from two parameters: $\mathrm{fc}(\downarrow^{\mathrm{f}})$ and $\mathrm{st}(T)$. Recall that $\mathrm{fc}(\downarrow^{\mathrm{f}})$ is the least cardinal κ such that there is no $\downarrow^{\mathrm{f}}$-forking chain of length κ for a finite set. The cardinal $\mathrm{st}(T)$ is defined as follows.

Definition 6.1. Let T be a complete theory. $\mathrm{st}(T)$ is the minimal infinite cardinal κ such that T is κ-stable. If there is no such cardinal, we set $\mathrm{st}(T) := \infty$.

The following technical lemma contains the main ingredients to determine the stability spectrum of a theory.

Lemma 6.2. *Let T be a stable theory and κ a cardinal.*

(a) *If $\kappa < \kappa^{<\mathrm{fc}(\downarrow^{\mathrm{f}})}$, then T is not κ-stable.*

(b) $\mathrm{fc}(\downarrow^{\mathrm{f}}) \leq |T|^+$.

(c) $|S^{<\omega}(U)| \leq \mathrm{st}(T) \leq 2^{|T|}$, *for every set U of size $|U| \leq \mathrm{st}(T)$.*

(d) $\mathrm{fc}(\downarrow^{\mathrm{f}}) \oplus \mathrm{mult}(\downarrow^{\mathrm{f}}) \leq \mathrm{st}(T)$.

(e) *If $\kappa \geq \mathrm{st}(T)$ and $\kappa = \kappa^{<\mathrm{fc}(\downarrow^{\mathrm{f}})}$, then T is κ-stable.*

Proof. (a) Let μ be the least cardinal with $\kappa^\mu > \kappa$. Then $\kappa < \kappa^{<\mathrm{fc}(\downarrow^{\mathrm{f}})}$ implies that $\mu < \mathrm{fc}(\downarrow^{\mathrm{f}})$. Hence, there exist a finite tuple $\bar a$ and a $\downarrow^{\mathrm{f}}$-forking chain $(\bar b_\alpha)_{\alpha<\mu}$ for $\bar a$ over $\varnothing$ of length μ. We construct a tree $(\bar c_\eta)_{\eta \in \kappa^{\leq \mu}}$ as follows. We start with $\bar c_{\langle\rangle} := \bar b_0$. For the inductive step, suppose that $\bar c_\eta$ is already defined for all $\eta \in \kappa^{<\mu}$ with $|\eta| < \alpha$ and set

$$C_\alpha := \bigcup \{ \bar c_\eta \mid \eta \in \kappa^{<\alpha} \}.$$

If α is a limit ordinal, we choose, for every $\eta \in \kappa^\alpha$, a tuple $\bar{c}_\eta$ with

$$\bar{c}_\eta \bar{c}[<\eta] \equiv \bar{b}_\alpha \bar{b}[<\alpha] \quad \text{and} \quad \bar{c}_\eta \mathop{\downarrow}^{\mathrm{f}}_{\bar{c}[<\eta]} C_\alpha \,.$$

For the successor step, suppose that $\alpha = \beta + 1$. For each $\eta \in \kappa^\beta$, we choose

- a tuple $\bar{d}$ such that $\bar{d}\bar{c}[\leq\eta] \equiv \bar{b}_\alpha \bar{b}[<\alpha]$,
- a $\mathop{\downarrow}^{\mathrm{f}}$-Morley sequence $(\bar{c}'_i)_{i<\kappa}$ for $\operatorname{tp}(\bar{d}/\bar{c}[\leq\eta])$ over $\bar{c}[\leq\eta]$, and
- a sequence $(\bar{c}''_i)_{i<\kappa}$ such that

$$\bar{c}''[<\kappa] \equiv_{\bar{c}[\leq\eta]} \bar{c}'[<\kappa] \quad \text{and} \quad \bar{c}''[<\kappa] \mathop{\downarrow}^{\mathrm{f}}_{\bar{c}[\leq\eta]} C_\alpha \,.$$

Then we set $\bar{c}_{\eta\alpha} := \bar{c}''_\alpha$, for $\alpha < \kappa$.

Having constructed the tree $(\bar{c}_\eta)_{\eta\in\kappa^{<\mu}}$, we set $U := \bigcup_{\eta\in\kappa^{<\mu}} \bar{c}_\eta$. Then $|U| = \kappa^{<\mu} = \kappa$. For each $\zeta \in \kappa^\mu$, let $\bar{a}_\zeta$ be a tuple such that

$$\bar{a}_\zeta \bar{c}[<\zeta] \equiv \bar{a}\bar{b}[<\mu] \quad \text{and} \quad \bar{a}_\zeta \mathop{\downarrow}^{\mathrm{f}}_{\bar{c}[<\zeta]} U \,.$$

We claim that

$$\bar{a}_\xi \not\equiv_U \bar{a}_\zeta \,, \quad \text{for } \xi \neq \zeta \,.$$

This implies that $|S^{<\omega}(U)| \geq \kappa^\mu > \kappa = |U|$. Hence, T is not κ-stable.

It remains to prove the claim. Given $\xi \neq \zeta$, let η be the longest common prefix of ξ and ζ and let $\alpha \neq \beta$ be the indices such that $\eta\alpha \prec \xi$ and $\eta\beta \prec \zeta$. We start by showing that

$$\bar{c}[\leq\zeta_0] \mathop{\downarrow}^{\mathrm{f}}_{\bar{c}[\leq\eta]} \bar{c}_{\eta\alpha} \,, \quad \text{for all } \zeta_0 \prec \zeta \,.$$

The proof is by induction on $|\zeta_0|$. Note that we have

$$\bar{c}_{\eta\beta} \mathop{\downarrow}^{\mathrm{f}}_{\bar{c}[\leq\eta]} \bar{c}_{\eta\alpha}$$

by choice of $\bar{c}_{\eta\beta}$ and $\bar{c}_{\eta\alpha}$. By (NOR) this implies that

$$\bar{c}[\leq\eta\beta] \mathop{\downarrow}^{\mathrm{f}}_{\bar{c}[\leq\eta]} \bar{c}_{\eta\alpha} \,.$$

Hence, the claim holds for $\zeta_0 \leq \eta\beta$. For the inductive step, let $\zeta_0 > \eta\beta$. Then

$$\bar{c}_{\zeta_0} \mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}_{\bar{c}[<\zeta_0]} C_{|\zeta_0|} \quad \text{implies} \quad \bar{c}_{\zeta_0} \mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}_{\bar{c}[<\zeta_0]} \bar{c}_{\eta\alpha} .$$

By inductive hypothesis, we furthermore have

$$\bar{c}[<\zeta_0] \mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}_{\bar{c}[\leq\eta]} \bar{c}_{\eta\alpha} .$$

Hence, the claim follows by transitivity.

Having proved the claim, it follows by finite character that

$$\bar{c}[<\zeta] \mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}_{\bar{c}[\leq\eta]} \bar{c}_{\eta\alpha} .$$

Since $\bar{a}_\zeta \mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}_{\bar{c}[<\zeta]} \bar{c}_{\eta\alpha}$ this implies by transitivity that

$$\bar{a}_\zeta \mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}_{\bar{c}[\leq\eta]} \bar{c}_{\eta\alpha} .$$

On the other hand,

$$\bar{a} \not\mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}_{\bar{b}[\leq\alpha]} \bar{b}_{\alpha+1} \quad \text{implies} \quad \bar{a}_\xi \not\mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}_{\bar{c}[\leq\eta]} \bar{c}_{\eta\alpha} .$$

Consequently, $\bar{a}_\xi \not\equiv_{\bar{c}[\leq\eta\alpha]} \bar{a}_\zeta$.

(b) follows from Theorem F3.2.18.

(c) For the upper bound, it is sufficient to note that, according to Theorem C3.5.17, $(2^{|T|})^{|T|} = 2^{|T|}$ implies that T is $2^{|T|}$-stable.

For the lower bound, let U be a set of size $|U| \leq \mathrm{st}(T)$. Fixing some set $A \supseteq U$ of size $|A| = \mathrm{st}(T)$, it follows by $\mathrm{st}(T)$-stability of T that

$$|S^{<\omega}(U)| \leq |S^{<\omega}(A)| = |A| = \mathrm{st}(T) .$$

(d) We start by showing that $\mathrm{fc}(\mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}) \leq \mathrm{st}(T)$. Note that

$$\kappa < \mathrm{fc}(\mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}) \quad \text{implies} \quad \kappa < \kappa^\kappa \leq \kappa^{<\mathrm{fc}(\mathop{\smile\!\!\!\!\!|}^{\mathrm{f}})} .$$

Therefore, it follows by (a) that T is not κ-stable for $\kappa < \mathrm{fc}(\mathop{\smile\!\!\!\!\!|}^{\mathrm{f}})$. Thus, $\mathrm{st}(T) \geq \mathrm{fc}(\mathop{\smile\!\!\!\!\!|}^{\mathrm{f}})$.

It therefore remains to prove that $\operatorname{mult}(\downarrow^{\mathrm{f}}) \leq \operatorname{st}(T)$. Let $\mathfrak{p} \in S^{<\omega}(U)$. By Proposition F2.3.24, there exists a set $U_o \subseteq U$ of size $|U_o| < \operatorname{loc}_o(\downarrow^{\mathrm{f}}) \leq \operatorname{fc}(\downarrow^{\mathrm{f}})$ such that $\mathfrak{p}|_{U_o} \leq \mathfrak{p}$. Since every free extension of $\mathfrak{p}$ is also a free extension of $\mathfrak{p}|_{U_o}$, it follows that $\operatorname{mult}(\mathfrak{p}|_{U_o}) \geq \operatorname{mult}(\mathfrak{p})$. As T is $\operatorname{st}(T)$-stable,

$$|U_o| < \operatorname{fc}(\downarrow^{\mathrm{f}}) \leq \operatorname{st}(T) \quad \text{implies} \quad |S^{\leq\omega}(U_o)| \leq \operatorname{st}(T)\,.$$

Consequently, there exists a model $\mathfrak{M}$ of size $\operatorname{st}(T)$ that contains U_o. We claim that $|S^{<\omega}(M)| \geq \operatorname{mult}(\mathfrak{p}|_{U_o})$. As T is $\operatorname{st}(T)$-stable, it then follows that

$$\operatorname{mult}(\mathfrak{p}) \leq \operatorname{mult}(\mathfrak{p}|_{U_o}) \leq |S^{<\omega}(M)| \leq \operatorname{st}(T)\,.$$

To prove the claim, consider a set $C \supseteq U_o$ and let $(\mathfrak{q}_\alpha)_{\alpha<\lambda}$ be a sequence of distinct free extensions of $\mathfrak{p}|_{U_o}$ over C. For each $\alpha < \lambda$, choose a free extension $\mathfrak{q}^+_\alpha \geq \mathfrak{q}_\alpha$ over $C \cup M$ and set $\mathfrak{r}_\alpha := \mathfrak{q}^+_\alpha|_M$. If we can show that $\mathfrak{r}_\alpha \neq \mathfrak{r}_\beta$, for $\alpha \neq \beta$, it will follow that $|S^{<\omega}(M)| \geq \lambda$, as desired.

Hence, suppose that $\mathfrak{r}_\alpha = \mathfrak{r}_\beta$. Since $\mathfrak{q}^+_\alpha$ and $\mathfrak{q}^+_\beta$ are free extensions of the stationary type $\mathfrak{r}_\alpha = \mathfrak{r}_\beta$, it follows that $\mathfrak{q}^+_\alpha = \mathfrak{q}^+_\beta$. In particular, $\mathfrak{q}_\alpha = \mathfrak{q}_\beta$, which implies that $\alpha = \beta$.

(e) Let $\kappa \geq \operatorname{st}(T)$ be a cardinal such that $\kappa^{<\operatorname{fc}(\downarrow^{\mathrm{f}})} = \kappa$ and let U be a set of size $|U| \leq \kappa$. By Proposition F2.3.24, we can find, for every type $\mathfrak{p} \in S^{<\omega}(U)$, a set $U_o \subseteq U$ of size $|U_o| < \operatorname{loc}_o(\downarrow^{\mathrm{f}}) \leq \operatorname{fc}(\downarrow^{\mathrm{f}})$ such that $\mathfrak{p}|_{U_o} \leq \mathfrak{p}$. As T is $\operatorname{st}(T)$-stable,

$$|U_o| < \operatorname{fc}(\downarrow^{\mathrm{f}}) \leq \operatorname{st}(T) \quad \text{implies} \quad |S^{\leq\omega}(U_o)| \leq \operatorname{st}(T)\,.$$

Consequently, it follows as in Theorem 4.5 that

$$\begin{aligned} \left|S^{<\omega}(U)\right| &\leq |U|^{<\operatorname{fc}(\downarrow^{\mathrm{f}})} \otimes \operatorname{st}(T) \otimes \operatorname{mult}(\downarrow^{\mathrm{f}}) \\ &\leq \kappa^{<\operatorname{fc}(\downarrow^{\mathrm{f}})} \otimes \operatorname{st}(T) \otimes \operatorname{mult}(\downarrow^{\mathrm{f}}) = \kappa\,, \end{aligned}$$

where the last equality follows by (c) and our choice of κ. □

Combining statements (a) and (d) of Lemma 6.2 we obtain the following description of the stability spectrum.

Theorem 6.3. *A stable theory T is κ-stable if, and only if,*

$$\kappa = \mathrm{st}(T) \oplus \kappa^{<\mathrm{fc}(\mathop{\downarrow}^{\mathrm{f}})}.$$

Proof. ($\Leftarrow$) follows by Lemma 6.2 (d). For ($\Rightarrow$), suppose that T is κ-stable. By definition, this implies that $\kappa \geq \mathrm{st}(T)$. Furthermore, it follows by Lemma 6.2 (a) that $\kappa \geq \kappa^{<\mathrm{fc}(\mathop{\downarrow}^{\mathrm{f}})}$. Since the converse inequality $\kappa \leq \kappa^{<\mathrm{fc}(\mathop{\downarrow}^{\mathrm{f}})}$ is trivial, the claim follows. □

Let us consider a subclass of stable theories where the stability spectrum is particularly simple.

Definition 6.4. A complete first-order theory T is called *supersimple* if $\mathrm{loc}(\mathop{\not\downarrow}^{\mathrm{f}}) \leq \aleph_0$. If T is supersimple and stable, we call it *superstable.*

Note that it follows by Theorem F2.4.17 that every supersimple theory is simple.

Theorem 6.5. *Let T be a complete first-order theory. The following conditions are equivalent.*

(1) *T is supersimple.*

(2) $\mathrm{fc}(\mathop{\not\downarrow}^{\mathrm{f}}) \leq \aleph_0$

(3) $\mathrm{loc}_0(\mathop{\not\downarrow}^{\mathrm{f}}) \leq \aleph_0$

Proof. (2) $\Leftrightarrow$ (3) follows by Proposition F2.3.24 and (1) $\Leftrightarrow$ (3) follows by Lemma F2.3.20. □

Theorem 6.6. *Let T be a complete first-order theory. The following conditions are equivalent.*

(1) *T is superstable.*

(2) *T is κ-stable if, and only if, $\kappa \geq \mathrm{st}(T)$.*

(3) *There is a cardinal λ such that T is κ-stable for all $\kappa \geq \lambda$.*

Proof. (2) $\Rightarrow$ (3) is trivial.

(1) $\Rightarrow$ (2) According to Theorem 6.3, T is κ-stable if, and only if,

$$\kappa \geq \mathrm{st}(T) \quad \text{and} \quad \kappa = \kappa^{<\aleph_0}.$$

As the second condition is vacuously true, the claim follows.

(3) $\Rightarrow$ (1) Fix a cardinal $\kappa \geq \lambda$ with $\mathrm{cf}\,\kappa = \aleph_0$. As $\kappa \geq \lambda \geq \mathrm{st}(T)$, it follows by Theorem 6.3 that $\kappa = \kappa^{<\mathrm{fc}(\downfree^{\mathrm{f}})}$. Since $\kappa^{\aleph_0} = \kappa^{\mathrm{cf}\,\kappa} > \kappa$, this implies that $\mathrm{fc}(\downfree^{\mathrm{f}}) \leq \aleph_0$. Hence, the claim follows by Theorem 6.5. □

Corollary 6.7. *Every $\aleph_0$-stable theory is superstable.*

Proof. If T is $\aleph_0$-stable, then $\mathrm{fc}(\downfree^{\mathrm{f}}) \leq \mathrm{st}(T) = \aleph_0$ and it follows by Theorem 6.5 that T is supersimple. □

We conclude this section by noting that, for countable theories, the characterisation in Theorem 6.3 leaves only four possibilities.

Theorem 6.8. *Every countable complete theory T satisfies exactly one of the following conditions:*

(1) *T is totally transcendental. Then*

$$T \text{ is } \kappa\text{-stable} \quad \text{iff} \quad \kappa \geq \aleph_0.$$

(2) *T is superstable, but not totally transcendental. Then*

$$T \text{ is } \kappa\text{-stable} \quad \text{iff} \quad \kappa \geq 2^{\aleph_0}.$$

(3) *T is stable, but not superstable. Then*

$$T \text{ is } \kappa\text{-stable} \quad \text{iff} \quad \kappa = \kappa^{\aleph_0}.$$

(4) *T is unstable.*

Proof. Let T be a stable theory. If T is $\aleph_0$-stable, we have seen in Theorem C3.5.18 that T is totally transcendental and κ-stable for all infinite κ.

Hence, suppose that T is not $\aleph_0$-stable. Then there exists a countable set $U \subseteq \mathbb{M}$ with $|S^{<\omega}(U)| > \aleph_0$. According to Corollary B5.7.5, this implies that $|S^{<\omega}(U)| = 2^{\aleph_0}$. Consequently, $\mathrm{st}(T) \geq 2^{\aleph_0}$. By Lemma 6.2 (c), it follows that $\mathrm{st}(T) = 2^{\aleph_0}$.

Furthermore, we have $\mathrm{fc}(\mathop{\downarrow}^{\mathrm{f}}) \leq \aleph_1$, according to Lemma 6.2 (b). If $\mathrm{fc}(\mathop{\downarrow}^{\mathrm{f}}) = \aleph_0$, then T is superstable and it follows by Theorem 6.6 that

$$T \text{ is } \kappa\text{-stable} \quad \text{iff} \quad \kappa \geq \mathrm{st}(T) = 2^{\aleph_0}\,.$$

If $\mathrm{fc}(\mathop{\downarrow}^{\mathrm{f}}) = \aleph_1$, Theorem 6.3 implies that T is κ-stable if, and only if, $\kappa \geq \mathrm{st}(T) = 2^{\aleph_0}$ and $\kappa = \kappa^{\aleph_0}$. Note that $\kappa = \kappa^{\aleph_0}$ implies $2^{\aleph_0} \leq \kappa^{\aleph_0} = \kappa$. Hence, the first condition is superfluous and

$$T \text{ is } \kappa\text{-stable} \quad \text{iff} \quad \kappa = \kappa^{\aleph_0}\,. \qquad \square$$

G2. *Models of stable theories*

1. *Isolation relations*

In this chapter, we study the structure of models of a stable theory. Our main tool will be a generalisation of the notion of a construction which we introduced in Section E3.4. This generalisation is based on the notion of a so-called isolation relation.

Definition 1.1. (a) A ternary relation $\surd$ on small subsets of $\mathbb{M}$ is an *isolation relation* if it is an abstract independence relation satisfying the axioms (INV), (BMON), and

(RSH) *Right Shift.*

$$AC \surd_U B \quad \text{and} \quad C \surd_U AB \quad \text{implies} \quad A \surd_U BC .$$

If $\bar{a} \surd_U B$, we say that $\operatorname{tp}(\bar{a}/B)$ is $\surd$-*isolated* over U. For $U = B$, we sometimes drop the subscript and abbreviate

$$A \surd_U U \quad \text{by} \quad A \surd U .$$

(b) The *left base-monotonicity cardinal* $\operatorname{lbm}(\surd)$ of an isolation relation $\surd$ is the least cardinal κ such that there are sets A, B, C, U with

$$|C| \leq \kappa , \quad AC \surd_U B , \quad \text{and} \quad A \not\surd_{UC} B .$$

If such a cardinal does not exist, we set $\operatorname{lbm}(\surd) = \infty$.

Thus, the difference between an isolation relation and a preforking relation is that we have dropped (DEF) while we have provided a converse to Lemma F2.2.4 by (RSH).

Lemma 1.2. *Every symmetric preforking relation is an isolation relation with* $\mathrm{lbm}(\mathop{\smile\hspace{-0.9em}\mid}) = \infty$.

Proof. By symmetry and base monotonicity it follows that $\mathrm{lbm}(\mathop{\smile\hspace{-0.9em}\mid}) = \infty$. Hence, we only have to prove (RSH). Let $AC \mathop{\smile\hspace{-0.9em}\mid}_U B$ and $C \mathop{\smile\hspace{-0.9em}\mid}_U A$. Then $B \mathop{\smile\hspace{-0.9em}\mid}_U AC$ and Lemma F2.2.4 implies that $BC \mathop{\smile\hspace{-0.9em}\mid}_U A$. Hence, $A \mathop{\smile\hspace{-0.9em}\mid}_U BC$. □

Our main example of an isolation relation will be the relation $\sqrt[\mathrm{at}]{\;}$. To illustrate the concept, we also introduce three variants.

Definition 1.3. Let κ be an infinite cardinal.

$$\begin{aligned}
A \sqrt[\mathrm{at}\,\kappa]{\;}_U B \quad &:\text{iff} \quad \text{for every finite } \bar a \subseteq A\,,\ \text{there exists a set} \\
&\qquad \Phi \subseteq \mathrm{tp}(\bar a/U) \text{ of size } |\Phi| < \kappa \text{ such that} \\
&\qquad\quad \mathbb{M} \vDash \Phi(\bar a') \ \Rightarrow\ \bar a' \equiv_{UB} \bar a\,. \\
A \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{wo}}_U B \quad &:\text{iff} \quad A' \equiv_U A \qquad\quad \Rightarrow\ A' \equiv_{UB} A\,. \\
A \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{a}}_U B \quad &:\text{iff} \quad A' \equiv_{\mathrm{acl}^{\mathrm{eq}}(U)} A \ \Rightarrow\ A' \equiv_{UB} A\,.
\end{aligned}$$

Lemma 1.4. (a) $\sqrt[\mathrm{at}]{\;} \subseteq \sqrt[\mathrm{at}\,\kappa]{\;} \subseteq \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{wo}} \subseteq \sqrt[\mathrm{d}]{\;}$

(b) $\sqrt[\mathrm{at}]{\;}$ *is an isolation relation with* $\mathrm{lbm}(\sqrt[\mathrm{at}]{\;}) \geq \aleph_0$.

(c) $\sqrt[\mathrm{at}\,\kappa]{\;}$ *satisfies all axioms of an isolation relation except for* (RSH). *For regular cardinals κ, we have* $\mathrm{lbm}(\sqrt[\mathrm{at}\,\kappa]{\;}) \geq \kappa$.

(d) $\mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{wo}}$ *is a symmetric isolation relation with* $\mathrm{lbm}(\mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{wo}}) = \infty$.

Proof. (a) The first two inclusions follow immediately from the definitions. For the last one, suppose that $\bar a \mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{wo}}_U \bar b$. To prove that $\bar a \sqrt[\mathrm{d}]{\;}_U \bar b$, let $(\bar b_n)_{n<\omega}$ be an indiscernible sequence over U with $\bar b_0 = \bar b$. According to Lemma F3.1.3, it is sufficient to find a tuple $\bar a' \equiv_{U\bar b} \bar a$ such that

$$\bar b_m \equiv_{U\bar a'} \bar b_n\,, \qquad \text{for all } m, n < \omega\,.$$

We will show in (d) that $\downarrow^{\mathrm{wo}}$ is symmetric. Hence, we also have $\bar b \downarrow^{\mathrm{wo}}_U \bar a$ and

$$\bar b_n \equiv_U \bar b \quad \text{implies} \quad \bar b_n \equiv_{U\bar a} \bar b\,, \quad \text{for all } n < \omega\,.$$

Consequently, the tuple $\bar a$ itself has the desired properties.

(b) We have already seen in Lemma F2.3.3 that $\sqrt[\mathrm{at}]{}$ is an abstract independence relation satisfying (INV) and (BMON). The fact that $\mathrm{lbm}(\sqrt[\mathrm{at}]{}) \geq \aleph_0$ follows by (c) for $\kappa = \aleph_0$. Hence, the only axiom that remains to be verified is (RSH).

Suppose that $AC \sqrt[\mathrm{at}]{}_U B$ and $C \sqrt[\mathrm{at}]{}_U AB$. To check that $A \sqrt[\mathrm{at}]{}_U BC$, consider a finite tuple $\bar a \subseteq A$. Since $AC \sqrt[\mathrm{at}]{}_U B$, there is a formula $\varphi(\bar x)$ over U isolating $\mathrm{tp}(\bar a/UB)$. We claim that φ also isolates $\mathrm{tp}(\bar a/UBC)$. Let $\bar a'$ be a tuple satisfying φ. Then $\bar a' \equiv_{UB} \bar a$ and we have to show that $\bar a' \equiv_{UBC} \bar a$. Given a finite tuple $\bar c \subseteq C$, we fix some tuple $\bar c'$ such that

$$\bar a'\bar c \equiv_{UB} \bar a\bar c'\,.$$

Since $\bar c \sqrt[\mathrm{at}]{}_U AB$, $\bar c' \equiv_U \bar c$ implies that $\bar c' \equiv_{UB\bar a} \bar c$. Hence,

$$\bar a'\bar c \equiv_{UB} \bar a\bar c' \equiv_{UB} \bar a\bar c\,.$$

We have shown that

$$\bar a' \equiv_{UB\bar c} \bar a\,, \quad \text{for all finite } \bar c \subseteq C\,.$$

Consequently, $\bar a' \equiv_{UBC} \bar a$.

(c) (INV) and (FIN) follow immediately from the definition.

(BMON) Suppose that $\bar a \sqrt[\mathrm{at}\,\kappa]{}_U BC$. By (FIN) we may assume that $\bar a$ is finite. Hence, there exists a set $\Phi(\bar x) \subseteq \mathrm{tp}(\bar a/U)$ of size $|\Phi| < \kappa$ such that

$$\mathbb{M} \vDash \Phi(\bar a') \quad \text{implies} \quad \bar a' \equiv_{UBC} \bar a\,.$$

Since $\Phi \subseteq \mathrm{tp}(\bar a/UC)$, the same set shows that $\bar a \sqrt[\mathrm{at}\,\kappa]{}_{UC} B$.

(MON) Suppose that $\bar a\bar c \mathop{\sqrt[\mathrm{at}\,\kappa]{}}_U BD$. We claim that $\bar a \mathop{\sqrt[\mathrm{at}\,\kappa]{}}_U B$. Again we may assume that $\bar a$ and $\bar c$ are finite. According to the definition, there exists a set $\Phi(\bar x, \bar y) \subseteq \mathrm{tp}(\bar a\bar c/U)$ of size $|\Phi| < \kappa$ such that

$$\mathbb{M} \vDash \Phi(\bar a', \bar c') \quad \text{implies} \quad \bar a'\bar c' \equiv_{UBD} \bar a\bar c\,.$$

We claim that the set

$$\Psi(\bar x) := \left\{ \exists \bar y \bigwedge \Phi_0 \;\middle|\; \Phi_0 \subseteq \Phi \text{ finite} \right\}$$

is the desired witness for $\bar a \mathop{\sqrt[\mathrm{at}\,\kappa]{}}_U B$. Hence, suppose that $\mathbb{M} \vDash \Psi(\bar a')$. By definition of Ψ it follows that, for every finite subset $\Phi_0 \subseteq \Phi$, we can find some tuple $\bar c'$ with $\mathbb{M} \vDash \Phi_0(\bar a', \bar c')$. By compactness, this implies that there is some tuple $\bar c'$ with $\mathbb{M} \vDash \Phi(\bar a', \bar c')$. Consequently, $\bar a'\bar c' \equiv_{UBD} \bar a\bar c$. In particular, we have $\bar a' \equiv_{UB} \bar a$.

(NOR) Suppose that $A \mathop{\sqrt[\mathrm{at}\,\kappa]{}}_U B$. To show that $AU \mathop{\sqrt[\mathrm{at}\,\kappa]{}}_U BU$, let $\bar a \subseteq A$ and $\bar c = \langle c_0, \dots, c_{n-1}\rangle \subseteq U$ be finite. There exists a set $\Phi(\bar x) \subseteq \mathrm{tp}(\bar a/U)$ of size $|\Phi| < \kappa$ such that

$$\mathbb{M} \vDash \Phi(\bar a') \quad \text{implies} \quad \bar a' \equiv_{UBC} \bar a\,.$$

Setting

$$\Psi(\bar x, \bar y) := \Phi(\bar x) \cup \{y_0 = c_0, \dots, y_{n-1} = c_{n-1}\}$$

it follows that

$$\mathbb{M} \vDash \Psi(\bar a', \bar c') \quad \text{implies} \quad \bar a' \equiv_{UBC} \bar a \quad \text{and} \quad \bar c' = \bar c\,.$$

In particular, we have $\bar a'\bar c' \equiv_{UB} \bar a\bar c$.

(LRF) Let $A, B \subseteq \mathbb{M}$. To show that $A \mathop{\sqrt[\mathrm{at}\,\kappa]{}}_A B$, consider a finite tuple $\bar a = \langle a_0, \dots, a_{n-1}\rangle \subseteq A$. We set

$$\Phi(\bar x) := \{x_0 = a_0, \dots, x_{n-1} = a_{n-1}\}\,.$$

Then $\mathbb{M} \vDash \Phi(\bar a')$ implies that $\bar a' = \bar a$. In particular, we have $\bar a' \equiv_{AB} \bar a$.

(LTR) Suppose that $A_0A_1A_2 \mathrel{{}^{\mathrm{at}\,\kappa}\!\surd_{A_0A_1}} B$ and $A_0A_1 \mathrel{{}^{\mathrm{at}\,\kappa}\!\surd_{A_0}} B$. By (NOR) it is sufficient to prove that $A_1A_2 \mathrel{{}^{\mathrm{at}\,\kappa}\!\surd_{A_0}} B$. Hence, let $\bar a_1 \subseteq A_1$ and $\bar a_2 \subseteq A_2$ be finite. By assumption, there are sets $\Phi(\bar x_1, \bar x_2) \subseteq \mathrm{tp}(\bar a_1 \bar a_2/A_0A_1)$ and $\Psi(\bar x_1) \subseteq \mathrm{tp}(\bar a_1/A_0)$ of size $|\Phi|, |\Psi| < \kappa$ such that

$$\begin{aligned} &\mathbb{M} \vDash \Phi(\bar a_1', \bar a_2') \quad \text{implies} \quad \bar a_1' \bar a_2' \equiv_{BA_0A_1} \bar a_1 \bar a_2\,, \\ \text{and}\quad &\mathbb{M} \vDash \Psi(\bar a_1') \quad \text{implies} \quad \bar a_1' \equiv_{BA_0} \bar a_1\,. \end{aligned}$$

Let $U \subseteq A_1$ be the set of parameters from A_1 that are used in Φ and let Φ' be the set of formulae obtained from Φ by replacing each parameter $c \in U$ by a variable y_c. For every finite $\bar c \subseteq U$, there exists a set $\Gamma_{\bar c}(\bar y) \subseteq \mathrm{tp}(\bar c/A_0)$ of size $|\Gamma_{\bar c}| < \kappa$ such that

$$\mathbb{M} \vDash \Psi(\bar c') \quad \text{implies} \quad \bar c' \equiv_{BA_0} \bar c\,.$$

Suppose that $\bar x_1 = \langle x_1^0, \dots, x_1^{n-1} \rangle$ and $\bar a_1 = \langle a_1^0, \dots, a_1^{n-1} \rangle$. We set

$$\begin{aligned} \Xi(\bar x_1, \bar x_2, (y_c)_{c \in U}) := \Phi' \cup \Psi \cup \bigcup_{\bar c \subseteq U} \Gamma_{\bar c} \\ \cup \{x_1^0 = y_{a_1^0}, \dots, x_1^{n-1} = y_{a_1^{n-1}}\}\,. \end{aligned}$$

Then

$$\mathbb{M} \vDash \Xi(\bar a_1', \bar a_2', \bar c') \quad \text{implies} \quad \bar a_1' \equiv_{BA_0} \bar a_1 \quad \text{and} \quad \bar c' \equiv_{BA_0} \bar c\,,$$

where $\bar c$ is an enumeration of U. Hence, there exists an automorphism $\pi \in \mathrm{Aut}\,\mathbb{M}_{BA_0}$ with $\pi[\bar c'] = \bar c$. By the equations added to Ξ, it follows that $\pi[\bar a_1'] = \bar a_1$. Consequently,

$$\mathbb{M} \vDash \Xi(\bar a_1', \bar a_2', \bar c') \quad \text{implies} \quad \mathbb{M} \vDash \Xi(\pi[\bar a_1'], \pi[\bar a_2'], \pi[\bar c'])\,.$$

Hence, $\mathbb{M} \vDash \Phi(\bar a_1, \pi[\bar a_2'], \bar c)$, which means that

$$\pi[\bar a_2'] \equiv_{BA_0A_1} \bar a_2\,.$$

Consequently,

$$\bar a_1' \bar a_2' \equiv_{BA_0} \bar a_1 \pi[\bar a_2'] \equiv_{BA_0} \bar a_1 \bar a_2\,.$$

To compute $\mathrm{lbm}(\overset{\mathrm{at}\,\kappa}{\surd})$, suppose that κ is regular and $AC \overset{\mathrm{at}\,\kappa}{\surd}_U B$ for $|C| < \kappa$. We have to show that $A \overset{\mathrm{at}\,\kappa}{\surd}_{UC} B$. Hence, let $\bar{a} \subseteq A$ be finite. For every finite tuple $\bar{c} \subseteq C$, there exists a set $\Phi_{\bar{c}}(\bar{x}, \bar{x}') \subseteq \mathrm{tp}(\bar{a}\bar{c}/U)$ of size $|\Phi_{\bar{c}}| < \kappa$ such that

$$\mathbb{M} \vDash \Phi_{\bar{c}}(\bar{a}', \bar{c}') \quad \text{implies} \quad \bar{a}'\bar{c}' \equiv_{UB} \bar{a}\bar{c}\,.$$

We set

$$\Psi(\bar{x}) := \bigcup \{\, \Phi_{\bar{c}}(\bar{x}, \bar{c}) \mid \bar{c} \subseteq C \text{ finite} \,\}\,.$$

Then $|\Psi| < \kappa$ since κ is regular. Furthermore, $\mathbb{M} \vDash \Psi(\bar{a}')$ implies

$$\bar{a}'\bar{c} \equiv_{UB} \bar{a}\bar{c}\,, \quad \text{for all finite } \bar{c} \subseteq C\,.$$

Hence, $\bar{a}'C \equiv_{UB} \bar{a}C$, which implies that $\bar{a}' \equiv_{UBC} \bar{a}$.

(d) (INV) follows immediately form the definition.

(MON) Suppose that $AC \mathop{\downarrow}^{\mathrm{wo}}_U BD$. To show that $A \mathop{\downarrow}^{\mathrm{wo}}_U B$, consider a set $A' \equiv_U A$. Then there exists a set C' such that $A'C' \equiv_U AC$. By assumption, this implies that $A'C' \equiv_{UBD} AC$. In particular, we have $A' \equiv_{UB} A$.

(BMON) Suppose that $A \mathop{\downarrow}^{\mathrm{wo}}_U BC$. To show that $A \mathop{\downarrow}^{\mathrm{wo}}_{UC} B$, consider a set $A' \equiv_{UC} A$. Then $A' \equiv_U A$, which implies that $A' \equiv_{UBC} A'$.

(NOR) Suppose that $A \mathop{\downarrow}^{\mathrm{wo}}_U B$. To show that $AU \mathop{\downarrow}^{\mathrm{wo}}_U BU$, consider sets $A'U' \equiv_U AU$. Then $U' = U$ and $A' \equiv_U A$. Hence, $A' \equiv_{UB} A$, which implies that $A'U' \equiv_{UB} AU$.

(LRF) Let A and B be sets. To show that $A \mathop{\downarrow}^{\mathrm{wo}}_A B$, let $A' \equiv_A A$. Then $A' = A$, which implies that $A' \equiv_{AB} A$.

(SYM) Suppose that $A \mathop{\downarrow}^{\mathrm{wo}}_U B$. To show that $B \mathop{\downarrow}^{\mathrm{wo}}_U A$, consider a set $B' \equiv_U B$. We fix a set A' such that $B'A \equiv_U BA'$. Then $A' \equiv_U A$, which implies that $A' \equiv_{UB} A$. Hence, $B'A \equiv_U BA' \equiv_U BA$, that is, $B' \equiv_{UA} B$.

(LTR) Since we have already proved symmetry, it is sufficient to show that $\mathop{\downarrow}^{\mathrm{wo}}$ is right transitive. Hence, suppose that $A \mathop{\downarrow}^{\mathrm{wo}}_{B_0} B_1$ and $A \mathop{\downarrow}^{\mathrm{wo}}_{B_1} B_2$, for $B_0 \subseteq B_1 \subseteq B_2$. To show that $A \mathop{\downarrow}^{\mathrm{wo}}_{B_0} B_2$, consider a set $A' \equiv_{B_0} A$. Then $A' \equiv_{B_1} A$, which implies that $A' \equiv_{B_2} A$.

(FIN) Suppose that $A_0 \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{wo}}_U B$, for all finite $A_0 \subseteq A$. To show that $A \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{wo}}_U B$, let $A' \equiv_U A$. Fix an automorphism $\pi \in \operatorname{Aut} \mathbb{M}_U$ with $\pi[A'] = A$ and consider a finite subset $A'_0 \subseteq A'$. By assumption, $A'_0 \equiv_U \pi[A'_0]$ implies that $A'_0 \equiv_{UB} \pi[A'_0]$. Consequently, we have

$$A'_0 \equiv_{UB} \pi[A'_0], \quad \text{for all finite } A'_0 \subseteq A'.$$

This implies that $A' \equiv_{UB} \pi[A'] = A$.

(RSH) Suppose that $AC \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{wo}}_U B$ and $C \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{wo}}_U AB$. By symmetry, it follows that $B \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{wo}}_U AC$. Hence, $B \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{wo}}_{UC} A$, which implies that $BC \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{wo}}_{UC} A$. Together with $C \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{wo}}_U A$ it follows by left transitivity that $BC \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{wo}}_U A$. Hence, $A \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{wo}}_U BC$.

Finally, the fact that $\operatorname{lbm}(\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{wo}}) = \infty$ follows by symmetry and base monotonicity. ☐

Exercise 1.1. Show that $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{a}}$ satisfies all axioms of a symmetric isolation relation except for (LTR) and (RSH). Furthermore, $\operatorname{lbm}(\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{a}}) = \infty$.

Remark. One can show that, if the theory in question is stable, the relation $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{a}}$ is also transitive and, thus, an isolation relation.

Recall that we write $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}^{!}_U B$ if $\operatorname{tp}(\bar{a}/UB)$ is the unique free extension of $\operatorname{tp}(\bar{a}/U)$ over $U \cup B$. This relation will be used in Section 5 below. It inherits some, but not all properties from $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{f}}$. In particular, it is a symmetric isolation relation, but not necessarily a forking relation.

Lemma 1.5. *Let T be a stable theory. The relation $\mathop{\smile\!\!\!\!\!\!\,|}^{!}$ is a symmetric isolation relation with $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{wo}} \subseteq \mathop{\smile\!\!\!\!\!\!\,|}^{!} \subseteq \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{f}}$.*

Proof. The second inclusion follows immediately from the definition. For the first one, suppose that $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{wo}}_U B$. By Lemma 1.4, it follows that $\bar{a} \not\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{d}}_U B$. As T is stable, this is equivalent to $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{f}}_U B$. For uniqueness, suppose that $\bar{a}' \equiv_U \bar{a}$ is another tuple with $\bar{a}' \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{f}}_U B$. Then $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{wo}}_U B$ implies that $\bar{a}' \equiv_{UB} \bar{a}$. Consequently, $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}^{!}_U B$.

It remains to prove that $\mathop{\smile\!\!\!\!\!\!\,|}^{!}$ is a symmetric isolation relation.

(INV) follows immediately from the definition.

(MON) Suppose that $\bar{a}_0\bar{a}_1 \mathop{\smile\!\!\!\!\!\!|}^{!}_{U} B$ and let $B_0 \subseteq B$. If $\operatorname{tp}(\bar{a}_0/UB)$ is the unique free extension of $\operatorname{tp}(\bar{a}_0/U)$ over $U \cup B$, then $\operatorname{tp}(\bar{a}_0/UB_0)$ is its unique free extension over $U \cup B_0$. Hence, it is sufficient show that $\bar{a}_0\bar{a}_1 \mathop{\smile\!\!\!\!\!\!|}^{!}_{U} B$ implies $\bar{a}_0 \mathop{\smile\!\!\!\!\!\!|}^{!}_{U} B$. By monotonicity of $\mathop{\smile\!\!\!\!\!\!|}^{f}$, we only need to prove uniqueness.

Consider a tuple $\bar{a}'_0 \equiv_U \bar{a}_0$ with $\bar{a}'_0 \mathop{\smile\!\!\!\!\!\!|}^{f}_{U} B$. We have to show that $\bar{a}'_0 \equiv_{UB} \bar{a}_0$. Choose $\bar{a}'_1$ such that $\bar{a}'_0\bar{a}'_1 \equiv_U \bar{a}_0\bar{a}_1$, and let $\bar{a}''_1$ be a tuple with

$$\bar{a}''_1 \equiv_{U\bar{a}'_0} \bar{a}'_1 \quad \text{and} \quad \bar{a}''_1 \mathop{\smile\!\!\!\!\!\!|}^{f}_{U\bar{a}'_0} B\,.$$

Since $\bar{a}'_0 \mathop{\smile\!\!\!\!\!\!|}^{f}_{U} B$, transitivity implies that

$$\bar{a}'_0\bar{a}''_1 \mathop{\smile\!\!\!\!\!\!|}^{f}_{U} B\,.$$

As $\bar{a}'_0\bar{a}''_1 \equiv_U \bar{a}'_0\bar{a}'_1 \equiv_U \bar{a}_0\bar{a}_1$, it follows that

$$\bar{a}_0\bar{a}_1 \mathop{\smile\!\!\!\!\!\!|}^{!}_{U} B \quad \text{implies} \quad \bar{a}'_0\bar{a}''_1 \equiv_{UB} \bar{a}_0\bar{a}_1\,.$$

In particular, $\bar{a}'_0 \equiv_{UB} \bar{a}_0$.

(NOR) Suppose that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{!}_{U} B$. Then $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{f}_{U} B$ implies $U\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{f}_{U} BU$. Hence, we only need to prove uniqueness. Let $\bar{c}$ be an enumeration of U and suppose that there are tuples $\bar{a}'\bar{c}' \equiv_U \bar{a}\bar{c}$ such that $\bar{a}'\bar{c}' \mathop{\smile\!\!\!\!\!\!|}^{f}_{U} BU$. Then $\bar{c}' = \bar{c}$, and $\bar{a}' \equiv_U \bar{a}$ implies that $\bar{a}' \equiv_{UB} \bar{a}$. Hence, $\bar{a}'\bar{c}' = \bar{a}'\bar{c} \equiv_{UB} \bar{a}\bar{c}$.

(BMON) Suppose that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{!}_{U} BC$. We claim that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{!}_{UC} B$. Since $\mathop{\smile\!\!\!\!\!\!|}^{f}$ is base monotone, we only need to prove uniqueness. Hence, consider a tuple $\bar{a}' \equiv_{UC} \bar{a}$ with $\bar{a}' \mathop{\smile\!\!\!\!\!\!|}^{f}_{UC} B$. Then

$$\bar{a}' \equiv_U \bar{a} \quad \text{and} \quad \bar{a} \mathop{\smile\!\!\!\!\!\!|}^{!}_{U} BC \quad \text{implies} \quad \bar{a}' \equiv_{UBC} \bar{a}\,.$$

(FIN) If $A \mathop{\smile\!\!\!\!\!\!|}^{!}_{U} B$, then (MON) implies that $A_0 \mathop{\smile\!\!\!\!\!\!|}^{!}_{U} B$, for every finite $A_0 \subseteq A$. Conversely, suppose that $A \not\mathop{\smile\!\!\!\!\!\!|}^{!}_{U} B$. If $A \not\mathop{\smile\!\!\!\!\!\!|}^{f}_{U} B$, there is a finite subset $A_0 \subseteq A$ with $A_0 \not\mathop{\smile\!\!\!\!\!\!|}^{f}_{U} B$ and we are done. Hence, suppose that $A \mathop{\smile\!\!\!\!\!\!|}^{f}_{U} B$. Then there is a set $A' \equiv_U A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}^{f}_{U} B$ and $A' \not\equiv_{UB} A$. Let $\pi \in \operatorname{Aut} \mathbb{M}_U$ be an automorphism with $\pi[A] = A'$. Since $A' \not\equiv_{UB} A$, we can find a finite tuple $\bar{a} \subseteq A$ with $\pi(\bar{a}) \not\equiv_{UB} \bar{a}$. As $\pi(\bar{a}) \equiv_U \bar{a}$ and $\pi(\bar{a}) \downarrow^{\mathrm{f}}_U B$, it follows that $\bar{a} \not\downarrow^{!}_U B$.

(SYM) Let $\bar{a} \downarrow^{!}_U \bar{b}$. By symmetry of $\downarrow^{\mathrm{f}}$, we have $\bar{b} \downarrow^{\mathrm{f}}_U \bar{a}$. For uniqueness, consider a tuple $\bar{b}' \equiv_U \bar{b}$ with $\bar{b}' \downarrow^{\mathrm{f}}_U \bar{a}$. There exists an automorphism $\pi \in \operatorname{Aut} \mathbb{M}_U$ with $\pi(\bar{b}') = \bar{b}$. Since $\downarrow^{\mathrm{f}}$ is invariant under automorphisms,

$$\bar{b}' \downarrow^{\mathrm{f}}_U \bar{a} \quad \text{implies} \quad \bar{b} \downarrow^{\mathrm{f}}_U \pi(\bar{a})\,.$$

By symmetry, $\pi(\bar{a}) \downarrow^{\mathrm{f}}_U \bar{b}$. Since $\pi(\bar{a}) \equiv_U \bar{a}$ and $\bar{a} \downarrow^{!}_U \bar{b}$, it follows that

$$\pi(\bar{a}) \equiv_{U \cup \bar{b}} \bar{a}\,.$$

For every formula $\varphi(\bar{x}, \bar{y})$ over U, we therefore have

$$\begin{aligned} \mathbb{M} \vDash \varphi(\bar{a}, \bar{b}) \quad &\text{iff} \quad \mathbb{M} \vDash \varphi(\pi(\bar{a}), \bar{b}) \\ &\text{iff} \quad \mathbb{M} \vDash \varphi(\bar{a}, \pi^{-1}(\bar{b})) \quad \text{iff} \quad \mathbb{M} \vDash \varphi(\bar{a}, \bar{b}')\,. \end{aligned}$$

Consequently, $\bar{b}' \equiv_{U \cup \bar{a}} \bar{b}$.

(LTR) As we already have proved symmetry, it is sufficient to show that, for sets $B_0 \subseteq B_1 \subseteq B_2$,

$$\bar{a} \downarrow^{!}_{B_0} B_1 \quad \text{and} \quad \bar{a} \downarrow^{!}_{B_1} B_2 \quad \text{implies} \quad \bar{a} \downarrow^{!}_{B_0} B_2\,.$$

By transitivity of $\downarrow^{\mathrm{f}}$, we only need to prove uniqueness. Hence, consider a tuple $\bar{a}' \equiv_{B_0} \bar{a}$ with $\bar{a}' \downarrow^{\mathrm{f}}_{B_0} B_2$. Then

$$\bar{a} \downarrow^{!}_{B_0} B_1 \quad \text{implies} \quad \bar{a}' \equiv_{B_1} \bar{a}\,.$$

Hence,

$$\bar{a} \downarrow^{!}_{B_1} B_2 \quad \text{implies} \quad \bar{a}' \equiv_{B_2} \bar{a}\,.$$

(RSH) Suppose that $AC \downarrow^{!}_U B$ and $C \downarrow^{!}_U AB$. Then

$$B \downarrow^{!}_U AC \quad \text{and} \quad C \downarrow^{!}_U A\,,$$

and it follows by Lemma F2.2.4 that $BC \downarrow^{!}_U A$. Thus, $A \downarrow^{!}_U BC$. □

2. Constructions

We can use isolation relations to stratify a structure such that every part is isolated over the previous ones. This leads to a generalised notion of a construction.

Throughout the chapter, we will use the following notation. Given a sequence $(A_\alpha)_{\alpha<\gamma}$ of sets, an ordinal $\alpha \leq \gamma$, and a set $I \subseteq \gamma$, we will write

$$A[I] := \bigcup_{i \in I} A_i\,, \quad A[<\alpha] := \bigcup_{i<\alpha} A_i\,, \quad \text{and} \quad A[\leq\alpha] := \bigcup_{i \leq \alpha} A_i\,.$$

Definition 2.1. Let $\surd$ be a ternary relation on small subsets of $\mathbb{M}$ and $A, U \subseteq \mathbb{M}$.

(a) A $\surd$-*stratification* of A over U is a sequence $\zeta = (B_\alpha)_{\alpha<\gamma}$ of disjoint sets $B_\alpha \subseteq A$ such that $A = B[<\gamma]$ and

$$B_\alpha \surd UB[<\alpha]\,, \quad \text{for all } \alpha < \gamma\,.$$

(b) A $\surd$-stratification ζ is a $\surd$-*construction* if each set B_α is a singleton. In this case, we identify ζ with the corresponding sequence $(b_\alpha)_{\alpha<\gamma}$ of elements $b_\alpha \in B_\alpha$. We say that a set A is $\surd$-*constructible* over U if there exists a $\surd$-construction of A over U.

(c) Let $\zeta = (B_\alpha)_{\alpha<\gamma}$ be a $\surd$-stratification. The *locality cardinal* $\operatorname{loc}(\zeta)$ of ζ is the least cardinal κ such that, for every $\alpha < \gamma$, there exists a set $C_\alpha \subseteq U \cup B[<\alpha]$ of size $|C_\alpha| < \kappa$ such that

$$B_\alpha \surd_{C_\alpha} UB[<\alpha]\,.$$

Remark. In this terminology, the kind of constructions introduced in Section E3.4 are $\overset{\text{at}}{\surd}$-constructions.

We will use $\surd$-stratifications to study the structure of $\surd$-constructible models. To do so, we will frequently be interested in whether a given subset of a $\surd$-constructible set is itself $\surd$-constructible. A simple sufficient condition is given by the notion of a *closed* set.

Definition 2.2. Let $\zeta = (B_\alpha)_{\alpha<\gamma}$ be a $\surd$-stratification of A. A subset $I \subseteq \gamma$ of indices is *ζ-closed* over U if

$$B_\alpha \surd_{UB[I\cap\alpha]} UB[<\alpha]\,, \quad \text{for all } \alpha \in I\,.$$

Similarly, we call a set $C \subseteq A$ *ζ-closed* if it is of the form $C = B[I]$, for some ζ-closed set $I \subseteq \gamma$.

Lemma 2.3. *Let $\surd$ be an isolation relation and $\zeta = (B_\alpha)_{\alpha<\gamma}$ a $\surd$-stratification of A over U. If $I \subseteq \gamma$ is ζ-closed over U, then $(B_\alpha)_{\alpha\in I}$ is a $\surd$-stratification of $B[I]$ over U.*

Proof. Consider an index $\alpha \in I$. As I is ζ-closed, we have

$$B_\alpha \surd_{UB[I\cap\alpha]} UB[<\alpha]\,.$$

By (MON), this implies that

$$B_\alpha \surd_{UB[I\cap\alpha]} UB[I\cap\alpha]\,. \qquad \square$$

In particular, ζ-closed subsets of a $\surd$-constructible set are themselves $\surd$-constructible. Before proving further properties of ζ-closed sets, let us present a lemma with several ways to construct such sets.

Lemma 2.4. *Let $\surd$ be a relation satisfying* (BMON), *$\zeta = (B_\alpha)_{\alpha<\gamma}$ a $\surd$-stratification of A over U, and let $\kappa \geq \operatorname{loc}(\zeta)$ be a regular cardinal.*

(a) *A union of ζ-closed sets is ζ-closed.*

(b) *If $I \subseteq \gamma$ is ζ-closed over U, then so is $I \cap \beta$ for every $\beta \leq \gamma$.*

(c) *Every index $\alpha < \gamma$ is contained in a ζ-closed set $I \subseteq \Downarrow\alpha$ over U of size $|I| < \kappa$.*

(d) *For every set $C \subseteq A$ of size $|C| < \kappa$, there is some ζ-closed set $I \subseteq \gamma$ over U of size $|I| < \kappa$ with $C \subseteq B[I]$.*

Proof. (a) Let $\mathcal{I}$ be a set of ζ-closed sets. To show that $K := \bigcup \mathcal{I}$ is ζ-closed, consider an index $\alpha \in K$. Then $\alpha \in I$, for some ζ-closed set $I \in \mathcal{I}$, and

$$B_\alpha \sqrt{}_{UB[I \cap \alpha]} UB[<\alpha] \quad \text{implies} \quad B_\alpha \sqrt{}_{UB[K \cap \alpha]} UB[<\alpha]$$

by (BMON).

(b) Let I be ζ-closed and fix $\beta \leq \gamma$. For $\alpha \in I \cap \beta$, we have $I \cap \alpha = (I \cap \beta) \cap \alpha$. Hence,

$$B_\alpha \sqrt{}_{UB[I \cap \alpha]} UB[<\alpha] \quad \text{implies} \quad B_\alpha \sqrt{}_{UB[(I \cap \beta) \cap \alpha]} UB[<\alpha] .$$

(c) We prove the claim by induction on α. There exists a set $C_\alpha \subseteq U \cup B[<\alpha]$ of size $|C_\alpha| < \operatorname{loc}(\zeta) \leq \kappa$ such that

$$B_\alpha \sqrt{}_{C_\alpha} UB[<\alpha] .$$

Set $J := \{ \beta < \alpha \mid C_\alpha \cap B_\beta \neq \varnothing \}$. By inductive hypothesis, every index $\beta \in J$ is contained in some ζ-closed set $I_\beta \subseteq \Downarrow\beta$ of size $|I_\beta| < \kappa$. By (a), the union $I := \bigcup_{\beta \in J} I_\beta$ is also ζ-closed. Since

$$C_\alpha \subseteq U \cup B[J] \subseteq U \cup B[I] = U \cup B[I \cap \alpha] ,$$

it follows by (BMON) that

$$B_\alpha \sqrt{}_{C_\alpha} UB[<\alpha] \quad \text{implies} \quad B_\alpha \sqrt{}_{UB[I \cap \alpha]} UB[<\alpha] .$$

Since $I \subseteq \alpha$, this implies that $I \cup \{\alpha\}$ is also ζ-closed. Furthermore, $|I \cup \{\alpha\}| < \kappa$, as κ is regular.

(d) Given $C \subseteq A$, set

$$J := \{ \alpha < \gamma \mid C \cap B_\alpha \neq \varnothing \} .$$

By (c), every $\alpha \in J$ is contained in some ζ-closed set $K_\alpha \subseteq \gamma$ of size $|K_\alpha| < \kappa$. By (a), the union $I := \bigcup_{\alpha < \gamma} K_\alpha$ is ζ-closed. As κ is regular and $|J| \leq |C| < \kappa$, we have $|I| < \kappa$. Since $C \subseteq B[J] \subseteq B[I]$, the claim follows. □

The next proposition collects several properties of ζ-closed sets. In particular, we show that a $\surd$-stratification over U is also a $\surd$-stratification over $U \cup C$, for every ζ-closed set C.

Proposition 2.5. *Let $\surd$ be an isolation relation and $\zeta = (B_\alpha)_{\alpha<\gamma}$ a $\surd$-stratification of A over U.*

(a) *If $I \subseteq \gamma$ is ζ-closed over U, then*

$$B[I] \surd_{UB[I\cap\alpha]} B[<\alpha]\,, \quad \textit{for all } \alpha \leq \gamma\,.$$

(b) *ζ is a $\surd$-stratification of A over $U \cup B[I]$, for every ζ-closed set $I \subseteq \gamma$.*

(c) *If $K \subseteq \gamma$ is a ζ-closed set over U of size $|B[K]| < \mathrm{lbm}(\surd)$, then every set $I \subseteq \gamma$ that is ζ-closed over U is also ζ-closed over $U \cup B[K]$.*

Proof. (a) Fix $\alpha \leq \gamma$. We prove the statement by induction on the minimal ordinal β such that $I \subseteq \beta$. If $I \subseteq \alpha$, then (LRF) and (NOR) imply that

$$B[I] \surd_{UB[I]} B[<\alpha]\,.$$

As $I = I \cap \alpha$, the claim follows.

For the successor step, suppose that $I = \{\beta\} \cup I_0$ where $I_0 \subseteq \beta$ and $\beta \geq \alpha$. Since I is ζ-closed, we have

$$B_\beta \surd_{UB[I\cap\beta]} B[<\beta]$$

which, by (MON), implies that

$$B_\beta \surd_{UB[I_0]B[I\cap\alpha]} B[<\alpha]\,.$$

Furthermore, the set $I_0 = I \cap \beta$ is ζ-closed according to Lemma 2.4 (b). Consequently, the inductive hypothesis yields

$$B[I_0] \surd_{UB[I_0\cap\alpha]} B[<\alpha]\,.$$

Since $I_0 \cap \alpha = I \cap \alpha$, it follows by (NOR) and (LTR) that

$$B_\beta B[I_0] \sqrt{}_{UB[I\cap\alpha]} B[<\alpha] .$$

Finally, suppose that I has no maximal element. We have seen in Lemma 2.4 (b) that $I \cap \beta$ is ζ-closed, for all $\beta < \gamma$. By inductive hypothesis, it therefore follows that

$$B[I \cap \beta] \sqrt{}_{UB[I\cap\alpha]} B[<\alpha] , \quad \text{for all } \beta \in I .$$

Consequently, (FIN) implies that

$$B[I] \sqrt{}_{UB[I\cap\alpha]} B[<\alpha] .$$

(b) We have to show that

$$B_\alpha \sqrt{} UB[I]B[<\alpha] , \quad \text{for all } \alpha < \gamma .$$

Hence, let $\alpha < \gamma$. If $\alpha \in I$, the claim follows by (NOR). Thus, suppose that $\alpha \notin I$. By (a), we have

$$B[I] \sqrt{}_{UB[I\cap(\alpha+1)]} B[\leq\alpha] .$$

Since $\alpha \notin I$, this implies by (BMON) that

$$B[I] \sqrt{}_{UB[<\alpha]} B[<\alpha]B_\alpha .$$

Hence, (BMON) and (NOR) yield

$$UB[I]B_\alpha \sqrt{}_{UB[<\alpha]B_\alpha} B[<\alpha] .$$

As $B_\alpha \sqrt{}_{UB[<\alpha]} B[<\alpha]$, it follows by (NOR) and (LTR) that

$$B[I]B_\alpha \sqrt{}_{UB[<\alpha]} B[<\alpha] .$$

Hence, $B[I] \sqrt{}_{UB[<\alpha]} B[<\alpha]B_\alpha$ implies by (rsh) that

$$B_\alpha \sqrt{}_{UB[<\alpha]} B[I]B[<\alpha] .$$

Consequently, it follows by (bmon) and (nor) that

$$B_\alpha \sqrt{}_{UB[I]B[<\alpha]} UB[I]B[<\alpha] .$$

(c) Let $I, K \subseteq \gamma$ be ζ-closed over U. Then $I \cup K$ is also ζ-closed. Hence, it follows by (a) that

$$B[I \cup K] \sqrt{}_{UB[(I \cup K) \cap \alpha]} B[<\alpha] .$$

For $\alpha \in I$, this implies that

$$B_\alpha B[K] \sqrt{}_{UB[K \cap \alpha]B[I \cap \alpha]} B[<\alpha] .$$

Since $|B[K]| < \mathrm{lbm}(\sqrt{})$, it follows that

$$B_\alpha \sqrt{}_{UB[K]B[I \cap \alpha]} B[<\alpha] ,$$

as desired. □

Corollary 2.6. *Let $\sqrt{}$ be an isolation relation and $\zeta = (B_\alpha)_{\alpha<\gamma}$ a $\sqrt{}$-stratification of A over U. Then $A \sqrt{} UB[<\alpha]$, for all $\alpha \leq \gamma$.*

Proof. Since the set $I := \gamma$ is ζ-closed, this follows immediately from Proposition 2.5 (a). □

We conclude this section by presenting conditions for the existence of a stratification or a construction.

Lemma 2.7. *Let $\sqrt{}$ be an isolation relation, $\zeta = (B_\alpha)_{\alpha<\gamma}$ a $\sqrt{}$-stratification of A over U, and let $(C_\alpha)_{\alpha<\delta}$ be a sequence of subsets of A such that each set C_α is ζ-closed over $U \cup C[<\alpha]$.*

(a) *The union $C[<\delta]$ is ζ-closed over U.*

(b) $(C_\alpha \setminus C[<\alpha])_{\alpha<\delta}$ *is a* $\surd$*-stratification of* $C[<\delta]$ *over* U.

Proof. (a) For each $\beta < \delta$, fix a set $I_\beta \subseteq \gamma$ such that $C_\beta = B[I_\beta]$. We prove that $I[<\delta]$ is ζ-closed over U by induction on δ. Consider an index $\alpha \in I[<\delta]$. Then $\alpha \in I_\beta$ for some ordinal $\beta < \delta$. By inductive hypothesis, $I[<\beta]$ is ζ-closed over U. Hence, it follows by Proposition 2.5 (a) that

$$C[<\beta] \surd_{UB[I[<\beta]\cap\alpha]} B[<\alpha] .$$

By (BMON), this implies that

$$C[<\beta] \surd_{UB[I[<\beta]\cap\alpha]B[I_\beta\cap\alpha]} B[<\alpha] .$$

Furthermore, as C_β is ζ-closed over $U \cup C[<\beta]$, we have

$$B_\alpha \surd_{UC[<\beta]B[I_\beta\cap\alpha]} B[<\alpha] .$$

Therefore, it follows by (LTR) that

$$B_\alpha C[<\beta] \surd_{UB[I[<\beta]\cap\alpha]B[I_\beta\cap\alpha]} B[<\alpha] .$$

By (BMON), this implies that

$$B_\alpha \surd_{UB[I[<\delta]\cap\alpha]} B[<\alpha] .$$

(b) Let $\alpha < \delta$. By (a) and Proposition 2.5 (b), ζ is a $\surd$-stratification of A over $U \cup C[<\alpha]$. As C_α is ζ-closed over $U \cup C[<\alpha]$, it follows therefore by Proposition 2.5 (a) that $C_\alpha \surd UC[<\alpha]$. □

Lemma 2.8. *Let* $\surd$ *be an isolation relation and* $(B_\alpha)_{\alpha<\gamma}$ *a sequence of sets.*

(a) *If every* B_α *is* $\surd$*-constructible over* $U \cup B[<\alpha]$*, then* $B[<\gamma]$ *is* $\surd$*-constructible over* U.

(b) *Let* ζ *be a* $\surd$*-construction of some set* $A \supseteq B[<\gamma]$ *over* U*. If each* B_α *is* ζ*-closed over* U*, then* $B[<\gamma]$ *is* $\surd$*-constructible over* U.

Proof. (a) For each $\alpha < \gamma$, fix a $\surd$-construction $(b^\alpha_i)_{i<\eta(\alpha)}$ of B_α over $U \cup B[<\alpha]$. Set $\delta(\alpha) := \sum_{i<\alpha} \eta(i)$ and let $(a_\beta)_{\beta<\delta(\gamma)}$ be the concatenation of all sequences $(b^\alpha_i)_i$, for $\alpha < \gamma$, that is,

$$a_{\delta(\alpha)+i} := b^\alpha_i\,, \quad \text{for } \alpha < \gamma \text{ and } i < \eta(\alpha)\,.$$

To prove that $(a_\beta)_{\beta<\delta(\gamma)}$ is a $\surd$-construction of $B[<\gamma]$ over U, consider an index $\beta < \delta(\gamma)$. Then $\beta = \delta(\alpha) + i$, for some $\alpha < \gamma$ and $i < \eta(\alpha)$, and

$$b^\alpha_i \;\surd\; UB[<\alpha]b^\alpha[<i] \quad \text{implies} \quad a_\beta \;\surd\; Ua[<\beta]\,.$$

(b) According to Lemma 2.7 (a), B_α is $\surd$-constructible over U. Furthermore, Lemma 2.4 (a) implies that each set of the form $B[<\alpha]$ is ζ-closed over U. Hence, it follows by Proposition 2.5 (b) that B_α is $\surd$-constructible over $U \cup B[<\alpha]$. Consequently, the claim follows by (a). □

Lemma 2.9. *Let $\surd$ be an isolation relation. A set A of size $|A| \leq \operatorname{lbm}(\surd)$ is $\surd$-constructible over a set U if, and only if, $A \;\surd\; U$.*

Proof. ($\Rightarrow$) follows by Corollary 2.6. For ($\Leftarrow$), let $A \;\surd\; U$ and let $\zeta = (a_\alpha)_{\alpha<\kappa}$ be an enumeration of A of length $\kappa := |A|$. We claim that ζ is a $\surd$-construction of A over U. For each $\alpha < \kappa$,

$$A \;\surd\; U \quad \text{implies} \quad a_\alpha a[<\alpha] \;\surd\; U\,.$$

Since $\big|a[<\alpha]\big| < \kappa \leq \operatorname{lbm}(\surd)$, it follows that $a_\alpha \;\surd_{Ua[<\alpha]}\; U$. □

Corollary 2.10. *Let $\surd$ be an isolation relation and let $(B_\alpha)_{\alpha<\gamma}$ be a $\surd$-stratification of A over U where*

$$|B_\alpha| \leq \operatorname{lbm}(\surd)\,, \quad \textit{for all } \alpha < \gamma\,.$$

Then A is $\surd$-constructible over U.

Proof. Since $B_\alpha \;\surd\; UB[<\alpha]$, it follows by Lemma 2.9 that each B_α is $\surd$-constructible over $U \cup B[<\alpha]$. Consequently, the claim follows by Lemma 2.8 (a). □

Clearly, if a set A has a $\surd$-construction of length γ, then $|A| \leq \gamma < |A|^+$. The next lemma can be used to obtain constructions of length exactly $|A|$.

Lemma 2.11. *Let ζ be a $\surd$-construction of A over U. If $\mathrm{loc}(\zeta)^{\mathrm{reg}} \leq |A|$, then A has a $\surd$-construction over U of length $|A|$.*

Proof. Let $(a_\alpha)_{\alpha<\kappa}$ be an enumeration of A of length $\kappa := |A|$. By induction on $\alpha < \kappa$, we can use Lemma 2.4 to choose subsets $B_\alpha \subseteq A$ of size $|B_\alpha| < \mathrm{loc}(\zeta)^{\mathrm{reg}}$ such that $a_\alpha \in B_\alpha$ and B_α is ζ-closed over $U \cup B[<\alpha]$. By Lemma 2.3, each set $B_\alpha \setminus B[<\alpha]$ has a $\surd$-construction $\xi_\alpha = (b_i^\alpha)_{i<\gamma_\alpha}$ of length

$$\gamma_\alpha < |B_\alpha|^+ \leq \mathrm{loc}(\zeta)^{\mathrm{reg}} \leq \kappa\,.$$

We have seen in the proof of Lemma 2.8 (a) that the concetanation of these $\surd$-constructions is a $\surd$-construction of $B[<\kappa] = A$ over U of length $\sum_{\alpha<\kappa} \gamma_\alpha = \kappa$. □

3. Prime models

Using $\surd$-constructions we can generalise the results of Section E3.4 to arbitrary isolation relations. One important property of the relation $\overset{\mathrm{at}}{\surd}$ that is not captured by the notion of an isolation relation is the fact that every model realises all isolated types. When generalising certain results about $\overset{\mathrm{at}}{\surd}$-constructions we have to require this property separately. This leads to the notion of $\surd$-saturation.

Definition 3.1. Let $\surd$ be an isolation relation, κ a cardinal, and $A, U \subseteq \mathbb{M}$ sets.

(a) A is $\surd$-*κ-saturated* if, for all sets $C \subseteq A$ of size $|C| < \kappa$ and every finite set $B \subseteq \mathbb{M}$ with $B \surd C$, there is some set $B' \subseteq A$ with $B' \equiv_C B$.

(b) A is $\surd$-*κ-prime* over U if it is $\surd$-κ-saturated and, for every $\surd$-κ-saturated set $B \supseteq U$, there exists an automorphism $\pi \in \mathrm{Aut}\,\mathbb{M}_U$ with $\pi[A] \subseteq B$.

Our aim is to prove that $\surd$-prime models are unique, up to isomorphism. We start by setting up the required back-and-forth machinery.

Lemma 3.2. *Let A be $\surd$-κ-saturated and $U \subseteq A$ a set of size $|U| < \kappa$. For every $\surd$-constructible set B over U of size $|B| \leq \kappa$, there exists some set $B' \subseteq A$ with $B' \equiv_U B$.*

Proof. Let $\zeta = (b_\alpha)_{\alpha<\gamma}$ be a $\surd$-construction of B over U of length $\gamma \leq \kappa$. We inductively construct a sequence $(b'_\alpha)_{\alpha<\gamma}$ in A such that

$$b'[<\alpha] \equiv_U b[<\alpha], \quad \text{for all } \alpha \leq \gamma.$$

Suppose that we have already defined b'_α, for all $\alpha < \beta$. Fix an element c such that

$$b_\beta b[<\beta] \equiv_U cb'[<\beta].$$

Then

$$b_\beta \surd Ub[<\beta] \quad \text{implies} \quad c \surd Ub'[<\beta].$$

Since A is $\surd$-κ-saturated and $|U| \oplus |\beta| < \kappa$, we can therefore find an element $b'_\beta \in A$ with $b'_\beta \equiv_{Ub'[<\beta]} c$. It follows that

$$b'_\beta b'[<\beta] \equiv_U cb'[<\beta] \equiv_U b_\beta b[<\beta]. \qquad \square$$

Lemma 3.3. *Let $\surd$ be an isolation relation, $\kappa \geq \mathrm{lbm}(\surd)$ an uncountable cardinal, $M, N \subseteq \mathbb{M}$ $\surd$-κ-saturated, $U \subseteq \mathbb{M}$ a set of size $|U| < \kappa$, and let ξ and ζ be $\surd$-constructions of, respectively, M and N over U such that $\mathrm{loc}(\xi)^{\mathrm{reg}}, \mathrm{loc}(\zeta)^{\mathrm{reg}} \leq \mathrm{lbm}(\surd)$. Then*

$$H : M \cong^\kappa_{\mathrm{iso}} N,$$

where H is the set of all elementary maps $f : A \to B$ such that

- *$A \subseteq M$ is a ξ-closed set of size $|A| < \kappa$,*
- *$B \subseteq N$ is a ζ-closed set of size $|B| < \kappa$, and*

- $f \upharpoonright U = \mathrm{id}_U$.

Proof. Set $\lambda := \mathrm{loc}(\xi)^{\mathrm{reg}} \oplus \mathrm{loc}(\zeta)^{\mathrm{reg}}$. By symmetry it is sufficient to check the forth property. Consider a map $f : A \to B$ in H and an element $c \in M$. We will construct an increasing chain of elementary maps $g_n : C_n \to D_n$, for $n < \omega$, such that

- $f \subseteq g_0$ and $c \in \mathrm{dom}(g_0)$,
- $|C_n \smallsetminus C_{n-1}|, |D_n \smallsetminus D_{n-1}| < \lambda$ (where $C_{-1} := A$ and $D_{-1} := B$),
- C_n is $\surd$-constructible over $U \cup C_{n-1}$,
- D_n is $\surd$-constructible over $U \cup D_{n-1}$,
- C_n is ξ-closed over U, for even $n < \omega$, and
- D_n is ζ-closed over U, for odd $n < \omega$.

Then we can set $g := \bigcup_{n<\omega} g_n$. By Lemma 2.4 (a),

$$\mathrm{dom}(g) = \bigcup_{n<\omega} C_n = \bigcup_{n<\omega} C_{2n}$$

is ξ-closed and $\mathrm{rng}(g) = \bigcup_{n<\omega} D_{2n+1}$ is ζ-closed. Furthermore,

$$|\mathrm{dom}(g)| < |A|^+ \oplus \lambda \oplus \aleph_1 \leq \kappa\,.$$

Hence, $g \in H$.

It remains to construct $(g_n)_n$. By Lemma 2.4 (d), we can find a ξ-closed set $C'_0 \subseteq M$ of size $|C'_0| < \lambda$ with $c \in C'_0$. Choose a set $D'_0 \subseteq \mathbb{M}$ with

$$AC'_0 \equiv_U BD'_0\,.$$

Note that it follows by Proposition 2.5 (b) that C'_0 is $\surd$-constructible over $U \cup A$. Consequently, D'_0 is $\surd$-constructible over $U \cup B$. Since N is $\surd$-κ-saturated, we can therefore use Lemma 3.2 to find a set $D''_0 \subseteq N$ with

$$D''_0 \equiv_{UB} D'_0\,.$$

Consequently, $AC'_0 \equiv_U BD''_0$. Let $g_0 : A \cup C'_0 \to B \cup D''_0$ be the corresponding extension of f.

For the successor step, suppose that we have already defined $g_n : C_n \to D_n$. First, consider the case where n is even. As λ is regular, we can use Lemma 2.4 (d) to find a ζ-closed set $D'_{n+1} \subseteq N$ of size $|D'_{n+1}| < \lambda$ with $D_n \setminus D_{n-1} \subseteq D'_{n+1}$. Choose a set $C'_{n+1} \subseteq \mathbb{M}$ with

$$C_n C'_{n+1} \equiv_U D_n D'_{n+1} .$$

By Proposition 2.5 (b), D'_{n+1} is $\surd$-constructible over $U \cup D_{n-1}$. According to Lemma 2.9 this implies that

$$D'_{n+1} \surd_{UD_{n-1}} UD_{n-1} .$$

Since $|D_n \setminus D_{n-1}| < \lambda \leq \operatorname{lbm}(\surd)$, it follows that

$$D'_{n+1} \surd_{UD_n} UD_n .$$

Applying Lemma 2.9 again, we see that the set D'_{n+1} is $\surd$-constructible over $U \cup D_n$. By invariance, it follows that C'_{n+1} is $\surd$-constructible over $U \cup C_n$. Hence, we can use Lemma 3.2 to find a set $C''_{n+1} \subseteq M$ with

$$C''_{n+1} \equiv_{UC_n} C'_{n+1} .$$

Consequently, $C_n C''_{n+1} \equiv_U D_n D'_{n+1}$. Let $g_{n+1} : C_n \cup C''_{n+1} \to D_n \cup D'_{n+1}$ be the corresponding extension of g_n.

If n is odd, we proceed similarly by choosing a ξ-closed set $C_{n+1} \subseteq M$ containing $C_n \setminus C_{n-1}$. □

With the back-and-forth machinery in place we can construct isomorphisms between $\surd$-κ-saturated sets.

Proposition 3.4. *Let $\surd$ be an isolation relation and $A, B \subseteq \mathbb{M}$ $\surd$-κ-saturated sets that are $\surd$-constructible over some set $U \subseteq A$ of size $|U| < \kappa$.*

(a) *If $|A| \leq \kappa$, there exists an automorphism $\pi : \operatorname{Aut} \mathbb{M}_U$ such that $\pi[A] \subseteq B$.*

(b) *If* $|A|, |B| \leq \kappa$, *there exists an automorphism* $\pi : \operatorname{Aut} \mathbb{M}_U$ *such that* $\pi[A] = B$.

Proof. By Lemma 3.3, we have $H : A \equiv^{\kappa}_{\text{iso}} B$. Consequently, we can use Lemma C4.4.10 (a) or (b) to find an elementary embedding $h : A \to B$ such that $h \upharpoonright U = \text{id}_U$ and, in case $|B| \leq \kappa$, such that h is surjective. As $\mathbb{M}$ is strongly κ^+-homogeneous, we can extend h to the desired automorphism π. □

Corollary 3.5. *Let U be a set of size $|U| < \kappa$. Up to isomorphism, there is at most one $\surd$-κ-saturated set A of size $|A| \leq \kappa$ that is $\surd$-constructible over U.*

Proof. If A and B are $\surd$-κ-saturated sets of size at most κ that are $\surd$-constructible over U, we can use Proposition 3.4 (b) to find an automorphism mapping A to B. □

Corollary 3.6. *Let A be a $\surd$-κ-saturated set of size $|A| \leq \kappa$ that is $\surd$-constructible over a set $U \subseteq A$ of size $|U| < |A|$. Then A is $\surd$-κ-prime over U.*

Proof. If $B \supseteq U$ is $\surd$-κ-saturated and $\surd$-constructible over U, we can use Proposition 3.4 (a) to find the desired automorphism mapping A to a subset of B. □

Having proved that $\surd$-κ-saturated sets are unique, it remains to show that such sets exists. For the relation $\overset{\text{at}}{\surd}$ we will prove below that every model is $\overset{\text{at}}{\surd}$-$\kappa$-saturated. For isolation relations $\surd$ that satisfy the extension axiom, we have the following lemma.

Lemma 3.7. *Let $\surd$ be an isolation relation satisfying* (EXT) *such that* $\text{lbm}(\surd) \geq \aleph_0$, *and let κ be an infinite cardinal. For every set U, there exists some $\surd$-κ-saturated set A that is $\surd$-constructible over U.*

Proof. We construct an increasing sequence $(A_\alpha)_{\alpha<\kappa^+}$ of sets $A_\alpha \subseteq \mathbb{M}$ such that each A_α is $\surd$-constructible over $U \cup A[<\alpha]$ and, for every

$\alpha < \kappa^+$, every set $C \subseteq A[<\alpha]$ of size $|C| < \kappa$, and every finite set $B \subseteq \mathbb{M}$, there is some set $B' \subseteq A[\leq\alpha]$ with $B' \equiv_C B$.

We start with $A_0 := U$. For the inductive step, suppose that we have already defined A_α, for all $\alpha < \beta$. To find A_β, we fix an enumeration $\langle C_\alpha, \mathfrak{p}_\alpha \rangle_{\alpha<\gamma}$ of all pairs $\langle C, \mathfrak{p} \rangle$ where $C \subseteq A[<\beta]$ has size $|C| < \kappa$ and $\mathfrak{p} \in S^{<\omega}(C)$ is a type such that $\bar{a} \surd C$, for every realisation $\bar{a}$ of $\mathfrak{p}$. By induction on $\alpha < \gamma$, we choose finite tuples $\bar{b}_\alpha \in M^{<\omega}$ as follows. Let $\bar{b}'_\alpha$ be a realisation of $\mathfrak{p}_\alpha$. Then $\bar{b}'_\alpha \surd C$ implies, by (EXT), that there is some tuple $\bar{b}_\alpha \equiv_C \bar{b}'_\alpha$ with

$$\bar{b}_\alpha \surd_C A[<\beta]\bar{b}[<\alpha] .$$

We set $A_\beta := A[<\beta] \cup \bar{b}[<\gamma]$. Note that $(\bar{b}_\alpha \smallsetminus \bar{b}[<\alpha])_{\alpha<\gamma}$ is a $\surd$-stratification of A_β over $A[<\beta]$. Since $\mathrm{lbm}(\surd) \geq \aleph_0$, it follows by Corollary 2.10 that A_β is $\surd$-constructible over $A[<\beta]$.

Having constructed $(A_\alpha)_{\alpha<\kappa^+}$, we claim that the union $A := A[<\kappa^+]$ has the desired properties. Since every set A_α is $\surd$-constructible over $U \cup A[<\alpha]$, it follows by Lemma 2.8 (a) that A is $\surd$-constructible over U. To show that it is also $\surd$-κ-saturated, let $C \subseteq A$ be a set of size $|C| < \kappa$ and let $\bar{b} \subseteq \mathbb{M}^{<\omega}$ be a tuple with $\bar{b} \surd C$. As κ^+ is regular, there is some index $\alpha < \kappa^+$ such that $C \subseteq A[<\alpha]$. Since the pair $\langle C, \mathrm{tp}(\bar{b}/C) \rangle$ appears in the sequence used in the construction of A_α, it follows that there is some $\bar{b}' \subseteq A_\alpha$ with $\bar{b}' \equiv_C \bar{b}$. □

4. $\overset{\text{at}}{\surd}$-constructible models

In this section we take a closer look at $\overset{\text{at}}{\surd}$-constructible sets. We have already seen in Section E3.4 that a model which is $\overset{\text{at}}{\surd}$-constructible over some set U is atomic over U, prime over U, and unique up to isomorphisms over U. These facts also follow from the general results we have have derived in the present chapter once we have shown that models are always $\overset{\text{at}}{\surd}$-saturated.

Proposition 4.1. *Every model $M \subseteq \mathbb{M}$ is $\sqrt[\mathrm{at}]{}$-$|M|^+$-saturated.*

Proof. Suppose that $\bar{a} \sqrt[\mathrm{at}]{} U$ where $\bar{a} \in \mathbb{M}^{<\omega}$ and $U \subseteq M$. Let $\varphi(\bar{x})$ be a formula over U isolating $\mathrm{tp}(\bar{a}/U)$. Then

$$\mathbb{M} \vDash \varphi(\bar{a}) \quad \text{implies} \quad \mathbb{M} \vDash \exists \bar{x} \varphi(\bar{x}) \quad \text{implies} \quad \mathfrak{M} \vDash \exists \bar{x} \varphi(\bar{x}) .$$

Hence, there is some tuple $\bar{a}' \in M^{<\omega}$ with $\mathfrak{M} \vDash \varphi(\bar{a}')$. By choice of φ this implies that $\mathrm{tp}(\bar{a}'/U) = \mathrm{tp}(\bar{a}/U)$. □

Example. Consider the theory T of the structure $\mathfrak{C} := \langle 2^\omega, (P_n)_{n<\omega} \rangle$ where

$$P_n := \{ \alpha \in 2^\omega \mid \alpha(n) = 1 \} .$$

For this theory, we have

$$\begin{aligned} & A \sqrt[\mathrm{d}]{}_U B && \text{iff} && A \cap B \subseteq U , \\ \text{and} \quad & A \sqrt[\mathrm{at}]{}_U B && \text{iff} && A \subseteq U . \end{aligned}$$

Furthermore,

$$|S^{<\omega}(U)| \leq |U| \oplus |S^{<\omega}(\varnothing)| = |U| \oplus 2^{\aleph_0} ,$$

since

$$\mathrm{tp}(\bar{a}/\bar{a} \cap U) \vDash \mathrm{tp}(\bar{a}/U) , \quad \text{for all } \bar{a}, U \subseteq \mathbb{M} .$$

Consequently, the theory T is superstable with $\mathrm{st}(T) = 2^{\aleph_0}$, and a set A is $\sqrt[\mathrm{at}]{}$-constructible over U if, and only if, $A \subseteq U$. In particular, no model of T is $\sqrt[\mathrm{at}]{}$-constructible over $\varnothing$.

For stable theories one can show that subsets of $\sqrt[\mathrm{at}]{}$-constructible sets are again $\sqrt[\mathrm{at}]{}$-constructible. We start with three technical lemmas.

Lemma 4.2. *Let $\downfree$ be a symmetric preforking relation. If $A \downfree_U B$ then, for every set $D \subseteq \mathbb{M}$, there exists a set $C \subseteq A$ of size $|C| < \mathrm{loc}(\downfree) \oplus |D|^+$ such that*

$$A \downfree_{UC} BD\,.$$

Proof. By right locality, we can choose a set $C \subseteq A$ of size

$$|C| < \mathrm{loc}(\downfree) \oplus |D|^+ \quad \text{such that} \quad D \downfree_{UBC} UBA\,.$$

It follows that $D \downfree_{UBC} A$. Furthermore, $A \downfree_U B$ implies $B \downfree_{UC} A$. By transitivity it follows that $BD \downfree_{UC} A$. □

Lemma 4.3. *Let $\downfree$ be a symmetric preforking relation, $\sqrt{\ }$ an isolation relation, and let $\kappa \geq \mathrm{loc}(\downfree)$ a be regular cardinal. Let ζ be a $\sqrt{\ }$-construction of some set A over U with* $\mathrm{loc}(\zeta) \leq \kappa$ *and let $C \subseteq A$ be a subset. For every ζ-closed set $B \subseteq A$ with*

$$B \downfree_{U(C \cap B)} C$$

and every set $D \subseteq A$ of size $|D| < \kappa$, there exists a ζ-closed set $B_+ \supseteq B \cup D$ such that

$$|B_+ \setminus B| < \kappa \quad \textit{and} \quad B_+ \downfree_{U(C \cap B_+)} C\,.$$

Proof. Let $(d_\alpha)_{\alpha<\gamma}$ be an enumeration of D. Starting with $B_0 := B$, we construct an increasing chain $(B_\alpha)_{\alpha<\gamma}$ of ζ-closed sets such that

$$d_\alpha \in B_{\alpha+1}\,, \quad |B_\alpha \setminus B| < \kappa\,, \quad \text{and} \quad B_\alpha \downfree_{U(C \cap B_\alpha)} C\,.$$

Then we can set $B_+ := \bigcup_{\alpha<\gamma} B_\alpha$.

For the successor step, suppose that we have already defined B_α. By Lemma 2.4 (c), there exists a ζ-closed set $Z \subseteq A$ of size $|Z| < \kappa$ containing d_α. We can choose a set $W \subseteq U \cup B_\alpha \cup C$ of size

$$|W| < \mathrm{loc}(\downfree) \oplus |Z|^+ \leq \kappa \quad \text{such that} \quad Z \downfree_W UB_\alpha C\,.$$

It follows that $Z \downarrow_{UB_\alpha(W \cap C)} C$. Furthermore,

$$B_\alpha \downarrow_{U(C \cap B_\alpha)} C \quad \text{implies} \quad B_\alpha \downarrow_{U(C \cap B_\alpha)(C \cap Z)(C \cap W)} C .$$

With transitivity it follows that

$$B_\alpha Z(W \cap C) \downarrow_{U(C \cap B_\alpha)(C \cap Z)(C \cap W)} C .$$

Setting $B_{\alpha+1} := B_\alpha \cup Z \cup (W \cap C)$, we have

$$|B_{\alpha+1} \smallsetminus B| \leq |B_\alpha \smallsetminus B| \oplus |Z| \oplus |W \cap C| < \kappa ,$$

as desired.

For the limit step, let δ be a limit ordinal and suppose that B_α is already defined for all $\alpha < \delta$. Then we set $B_\delta := \bigcup_{\alpha<\delta} B_\alpha$. □

Lemma 4.4. *Let T be a stable theory.*

$$A \overset{\text{at}}{\surd} UB \quad \textit{and} \quad A \downarrow^{\mathrm{f}}_U B \quad \textit{implies} \quad A \overset{\text{at}}{\surd} U .$$

Proof. Let $\bar a \subseteq A$ be finite. We have to show that $\operatorname{tp}(\bar a/U)$ is isolated. Since $\bar a \overset{\text{at}}{\surd} UB$, the type $\mathfrak{p} := \operatorname{tp}(\bar a/UB)$ is isolated, that is, the set $\{\mathfrak{p}\}$ is open in $\mathfrak{S}^{\bar s}(UB)$. By assumption, $\mathfrak{p}$ does not fork over U. Hence, $\mathfrak{p} \in F^{\bar s}(UB/U)$ in the notation of the Open Mapping Theorem. According to that theorem, the restriction map $\mathfrak{F}^{\bar s}(UB/U) \to \mathfrak{S}^{\bar s}(U)$ is open. Consequently, the image $\{\mathfrak{p}|_U\}$ is open in $\mathfrak{S}^{\bar s}(U)$ and $\mathfrak{p}|_U$ is isolated. □

Using these lemmas, we can show that subsets of constructible sets are again constructible.

Theorem 4.5. *Let T be a stable theory with $\operatorname{fc}(\downarrow^{\mathrm{f}}) \leq \aleph_1$ and let A be a $\overset{\text{at}}{\surd}$-constructible set over U. Every subset $C \subseteq A$ is also $\overset{\text{at}}{\surd}$-constructible over U.*

Proof. Let ζ be a $\sqrt[\mathrm{at}]{}$-construction of A over U. Since

$$\begin{aligned} &\mathrm{loc}(\zeta) \leq \mathrm{loc}_0(\sqrt[\mathrm{at}]{}) \leq \aleph_0 \\ \text{and}\quad &\mathrm{loc}(\downarrow^{\mathrm{f}}) \leq \mathrm{loc}_0(\downarrow^{!})^{\mathrm{reg}} = \mathrm{fc}(\downarrow^{\mathrm{f}})^{\mathrm{reg}} \leq \aleph_1\,, \end{aligned}$$

we can use Lemma 4.3 to inductively construct an increasing sequence $(B_\alpha)_{\alpha<\gamma}$ of sets $B_\alpha \subseteq A$ such that

- $B_0 = \emptyset$ and $B[<\gamma] = A$,
- each B_α is ζ-closed over $U \cup B[<\alpha]$,
- $|B_\alpha \setminus B[<\alpha]| \leq \aleph_0$,
- $B[<\alpha] \downarrow^{\mathrm{f}}_{U(C\cap B[<\alpha])} C$.

Set $C_\alpha := C \cap (B_\alpha \setminus B[<\alpha])$, for $\alpha < \gamma$. Then $|C_\alpha| \leq \aleph_0 = \mathrm{lbm}(\sqrt[\mathrm{at}]{})$ and, by Corollary 2.10, it is sufficient to prove that $(C_\alpha)_{\alpha<\gamma}$ is a $\sqrt[\mathrm{at}]{}$-stratification of C over U.

Hence, let $\alpha < \gamma$. Since B_α is ζ-closed over $U \cup B[<\alpha]$, it follows by Lemma 2.8 (a) that $(B_\alpha)_{\alpha<\gamma}$ is a $\sqrt[\mathrm{at}]{}$-stratification of A over U. Hence,

$$B_\alpha \sqrt[\mathrm{at}]{}\; UB[<\alpha]\,, \quad \text{which implies that} \quad C_\alpha \sqrt[\mathrm{at}]{}\; UB[<\alpha]\,.$$

Since $C_\alpha \downarrow^{\mathrm{f}}_{UC[<\alpha]} B[<\alpha]$, Lemma 4.4 implies that $C_\alpha \sqrt[\mathrm{at}]{}\; UC[<\alpha]$, as desired. □

Corollary 4.6. *Let T be a stable theory with* $\mathrm{fc}(\downarrow^{\mathrm{f}}) \leq \aleph_1$ *and let $\mathfrak{M}$ be a $\sqrt[\mathrm{at}]{}$-constructible model over U. Then $\mathfrak{M}$ is the unique prime model of T over U.*

Proof. By Corollary 3.6 $\mathfrak{M}$ is prime over U. For uniqueness, let $\mathfrak{N}$ be another prime model over U. Then there exists an elementary embedding $h : \mathfrak{N} \to \mathfrak{M}$. By Theorem 4.5, $\mathfrak{N}$ is $\sqrt[\mathrm{at}]{}$-constructible over U. Hence, it follows by Corollary 3.5 that $\mathfrak{N} \cong \mathfrak{M}$. □

Finally, let us take a look at $\sqrt[\mathrm{at}]{}$-constructible sets in totally transcendental theories. We will prove below that a model of such a theory is prime if, and only if, it is $\sqrt[\mathrm{at}]{}$-constructible. We also give a characterisation in terms of the length of indiscernible sequences. One direction is contained in the following proposition.

Proposition 4.7. *Let T be totally transcendental and let A be $\sqrt[\mathrm{at}]{}$-constructible over U. Every indiscernible sequence over U that is contained in A is countable.*

Proof. Let $\zeta = (a_\alpha)_{\alpha<\gamma}$ be a $\sqrt[\mathrm{at}]{}$-construction of A over U and suppose that A contains an uncountable indiscernible sequence $(c_\alpha)_{\alpha<\omega_1}$ over U. Note that $\mathrm{loc}(\zeta) \leq \mathrm{loc}_0(\sqrt[\mathrm{at}]{}) = \aleph_0$. Hence, we can use Lemma 2.4 (c) to fix, for every $\alpha < \omega_1$, a finite ζ-closed set $B_\alpha \subseteq A$ over U with $c_\alpha \in B_\alpha$. By Lemma E5.3.11 there exists, for every $\alpha < \omega_1$, an ordinal $\delta_\alpha < \omega_1$ such that $(c_\beta)_{\delta_\alpha \leq \beta < \omega_1}$ is indiscernible over $U \cup B[<\alpha]$. For $\alpha < \omega_1$, set

$$\gamma^\alpha_0 := \alpha\,, \quad \gamma^\alpha_{n+1} := \delta_{\gamma^\alpha_n}\,, \quad \text{and} \quad \gamma^\alpha_* := \sup_{n<\omega} \gamma^\alpha_n\,.$$

Note that $\gamma^\alpha_* < \omega_1$, for all $\alpha < \omega_1$. Since $(c_\beta)_{\beta \geq \gamma^\alpha_*}$ is indiscernible over $U \cup B[<\gamma^\alpha_n]$, for all $n < \omega$, it follows that it is indiscernible over

$$\bigcup_{n<\omega} (U \cup B[<\gamma^\alpha_n]) = U \cup B[<\gamma^\alpha_*]\,.$$

Set

$$D_0 := \{\, \gamma^\alpha_* \mid \alpha < \omega_1 \,\} \quad \text{and} \quad D := \{\, \sup I \mid I \subseteq D_0\,,\ \sup I < \omega_1 \,\}\,.$$

The set D is closed by definition and it is unbounded since $\gamma^\alpha_* \geq \alpha$, for all α. By Lemma A4.6.8 (a) it follows in particular that D is stationary. Furthermore, for every $\delta \in D$, the suffix $(c_\alpha)_{\delta \leq \alpha < \omega_1}$ is indiscernible over $U \cup B[<\delta]$. By Lemma 2.4 and Proposition 2.5, $B[<\delta]$ is ζ-closed and ζ is a $\sqrt[\mathrm{at}]{}$-construction over $U \cup B[<\delta]$. By Corollary 2.6, it follows that

$$A \ \sqrt[\mathrm{at}]{}\ UB[<\delta]\,, \quad \text{which implies that} \quad c_\delta \ \sqrt[\mathrm{at}]{}\ UB[<\delta]\,.$$

Since $\mathrm{loc}(\sqrt[\mathrm{at}]{}) = \aleph_0$, we can fix, for every $\delta \in D$, a finite set $W_\delta \subseteq U \cup B[<\delta]$ such that

$$c_\delta \sqrt[\mathrm{at}]{}_{W_\delta} UB[<\delta] .$$

By the Theorem of Fodor, there exist an index $\delta_0 \in D$ and a stationary set $E \subseteq D$ such that $W_\varepsilon \subseteq B[<\delta_0]$, for all $\varepsilon \in E$. Fix two ordinals $\varepsilon < \eta$ in E. By indiscernibility, we have $c_\varepsilon \equiv_{U \cup B[<\delta_0]} c_\eta$. But $W_\eta \subseteq U \cup B[<\delta_0]$ implies that

$$\mathrm{tp}(c_\eta / UB[<\delta_0]) \vDash \mathrm{tp}(c_\eta / UB[<\delta_0] c_\varepsilon) .$$

Hence, $c_\eta \neq c_\varepsilon$ implies that $c_\varepsilon \neq c_\varepsilon$. A contradiction. □

For the converse, we need several lemmas.

Definition 4.8. Let $\mathfrak{M}$ be a model. A set $A \subseteq M$ is *invariant* over $U \subseteq M$ if, for all finite tuples $\bar a, \bar b \in M^{<\omega}$,

$$\bar a \equiv_U \bar b \quad \text{implies} \quad \bar a \subseteq A \Leftrightarrow \bar b \subseteq A .$$

Lemma 4.9. *Let T be a totally transcendental theory, $\mathfrak{M}$ a model of T, and let $U \subseteq M$ be a set such that $M \sqrt[\mathrm{at}]{} U$. Then $M \sqrt[\mathrm{at}]{} UC$ for every set $C \subseteq M$ that is invariant over U.*

Proof. Let $\bar a \in M^{<\omega}$. Then the type $\mathrm{tp}(\bar a / U)$ is isolated by some formula $\varphi(\bar x)$ over U. We have seen in Lemma E3.4.12 that the isolated types in $\mathfrak{S}^{<\omega}(UC)$ are dense. Consequently, we can find some isolated type

$$\mathfrak{p} \in \langle \varphi \rangle_{\mathfrak{S}^{<\omega}(UC)} .$$

Let $\psi(\bar x, \bar d)$ be a formula over U isolating $\mathfrak{p}$ with $\bar d \subseteq C$ and fix a tuple $\bar b$ realising $\mathfrak{p}$. Then $\mathfrak{M} \vDash \varphi(\bar b)$ implies that $\bar a \equiv_U \bar b$. Consequently, we can find some tuple $\bar c$ with $\bar a \bar c \equiv_U \bar b \bar d$. Then $\bar d \subseteq C$ implies $\bar c \subseteq C$ by invariance of C over U. Furthermore,

$$\mathfrak{M} \vDash \psi(\bar b; \bar d) \quad \text{implies} \quad \mathfrak{M} \vDash \psi(\bar a; \bar c) .$$

We claim that $\psi(\bar{x};\bar{c})$ isolates $\mathrm{tp}(\bar{a}/UC)$. Let $\vartheta(\bar{x};\bar{c}') \in \mathrm{tp}(\bar{a}/UC)$. Fix some tuple $\bar{d}'$ such that

$$\bar{a}\bar{c}\bar{c}' \equiv_U \bar{b}\bar{d}\bar{d}' .$$

By invariance of C over U, it follows that $\bar{d}' \subseteq U \cup C$. Since $\psi(\bar{x};\bar{d})$ isolates $\mathrm{tp}(\bar{b}/UC)$ and $\mathfrak{M} \vDash \vartheta(\bar{a};\bar{c}')$ implies $\mathfrak{M} \vDash \vartheta(\bar{b};\bar{d}')$, we have

$$T(U \cup \bar{d}\bar{d}') \vDash \psi(\bar{x};\bar{d}) \to \vartheta(\bar{x};\bar{d}') .$$

Consequently,

$$T(U \cup \bar{c}\bar{c}') \vDash \psi(\bar{x};\bar{c}) \to \vartheta(\bar{x};\bar{c}') ,$$

as desired. □

Lemma 4.10. *Let T be a totally transcendental theory, $\mathfrak{M}, \mathfrak{N}$ models of T, $\mathfrak{p} \in S^1(U)$ a type over $U \subseteq M$, and $U \subseteq C \subseteq M$ a set that is invariant over U. If $M \overset{\mathrm{at}}{\bigvee} U$ and M does not contain an uncountable indiscernible sequence over U, then every elementary map $f : C \to N$ can be extended to an elementary map $C \cup \mathfrak{p}^{\mathfrak{M}} \to N$.*

Proof. We prove the claim by induction on $\alpha := \mathrm{rk}_M(\mathfrak{p})$. If $\alpha = 0$, then $\mathfrak{p}^{\mathfrak{M}} \subseteq \mathrm{acl}(U)$ and the claim holds trivially.

For the inductive step, suppose that we have proved the statement already for all types of Morley rank less than α. Let $I \subseteq \mathfrak{p}^{\mathfrak{M}}$ be a maximal set such that

$$a \mathop{\smile\!\!\!\!\!|}^{\mathrm{f}}_U I \setminus \{a\} , \quad \text{for all } a \in I .$$

According to Lemma G1.5.6, we can partition I into finitely many totally indiscernible sequences. By assumption, each of them is countable. Hence, so is I. Let $(a_n)_{n<\omega}$ be an enumeration of I and set

$$C_n := C \cup \{ c \in M \mid \mathrm{rk}_M(c/Ua[<n]) < \alpha \} , \quad \text{for } n < \omega .$$

Then $(C_n)_{n<\omega}$ forms an increasing chain starting with $C_0 = C$. Furthermore, every element $b \in \mathfrak{p}^{\mathfrak{M}} \setminus \bigcup_{n<\omega} C_n$ satisfies

$$\mathrm{rk}_{\mathrm{M}}(b/UI) = \mathrm{rk}_{\mathrm{M}}(b/U)\,.$$

By Corollary G1.4.8 (a), this implies that $b \perp^{\mathrm{f}}_U I$. By maximality of I, it therefore follows that

$$\mathfrak{p}^{\mathfrak{M}} \subseteq \bigcup_{n<\omega} C_n\,.$$

Note that $M \overset{\text{at}}{\bigvee} U$ implies $M \overset{\text{at}}{\bigvee} Ua[<n]$ since $\mathrm{lbm}(\overset{\text{at}}{\bigvee}) \geq \aleph_0$. As C_n is invariant over $U \cup a[<n]$, it follows by Lemma 4.9 that $M \overset{\text{at}}{\bigvee} C_n$. We prove by induction on n, that f can be extended to C_n. For $n = 0$, we have $C_0 = C$ and there is nothing to do. For the inductive step, suppose that we have already extended f to C_n. We first extend f to $C_n \cup \{a_n\}$. Since $M \overset{\text{at}}{\bigvee} C_n$, there exists some formula $\varphi(x;\bar{c})$ with parameters $\bar{c} \subseteq C_n$ isolating $\mathrm{tp}(a_n/C_n)$. Then

$$\mathfrak{M} \vDash \exists x \varphi(x;\bar{c}) \quad \text{implies} \quad \mathfrak{N} \vDash \exists x \varphi(x; f(\bar{c}))\,.$$

Hence, there exists some element $b \in N$ such that we can extend f by setting $f(a_n) := b$. Having done so we obtain the desired extension of f to C_{n+1} by applying the inductive hypothesis on α to C_n (for U) and $C_n \cup \{a_n\}$ (for C). □

We obtain the following characterisation of prime models of totally transcendental theories.

Theorem 4.11. *Let T be a totally transcendental theory and U a set of parameters.*

(a) *T has a unique prime model over U.*

(b) *For a model $\mathfrak{M}$ of T, the following statements are equivalent:*

  (1) *$\mathfrak{M}$ is prime over U*

  (2) *$\mathfrak{M}$ is $\overset{\text{at}}{\bigvee}$-constructible over U.*

(3) *$\mathfrak{M}$ is atomic over U and M does not contain an uncountable indiscernible sequence over U.*

Proof. (a) The existence and uniqueness of a prime model were already proved in Theorem E3.4.14.

(b) (2) ⇒ (1) was shown in Proposition E3.4.3.

(1) ⇒ (3) Suppose that $\mathfrak{M}$ is prime over U. By Proposition E3.4.13, there exists a model $\mathfrak{A}$ that is $\overset{\mathrm{at}}{\surd}$-constructible over U. As $\mathfrak{M}$ is prime over U, we can find an elementary embedding $h : \mathfrak{M} \to \mathfrak{A}$ fixing U. By Corollary 2.6, we have $A \overset{\mathrm{at}}{\surd} U$. Since $h[M] \subseteq A$, it follows by invariance that $M \overset{\mathrm{at}}{\surd} U$. Thus, $\mathfrak{M}$ is atomic over U.

Furthermore, every indiscernible sequence in M is mapped by h to an indiscernible sequence in A. We have seen in Proposition 4.7 that A only contains countable indiscernible sequences over U. Hence, so does M.

(3) ⇒ (2) Let $\mathfrak{N}$ be an arbitrary model of $T(U)$. To find the desired elementary embedding $\mathfrak{M}_U \to \mathfrak{N}$, we choose a maximal elementary map $f : C \to N$ with a domain $C \subseteq M$ that is invariant over U. We claim that $C = M$. For a contradiction, suppose otherwise. Then there is some element $a \in M \setminus C$. By Lemma 4.9, we have $M \overset{\mathrm{at}}{\surd} C$. Hence, setting $\mathfrak{p} := \mathrm{tp}(a/C)$ we can use Lemma 4.10 to extend f to an elementary map $C \cup \mathfrak{p}^{\mathfrak{M}} \to N$. This contradicts the maximality of f. ☐

5. *Strongly independent stratifications*

In contrast to $\overset{\mathrm{at}}{\surd}$, the isolation relation $\downarrow^{!}$ admits arbitrarily large constructible sets. Consequently, the length of $\downarrow^{!}$-constructions is unbounded. But we will show that there are $\downarrow^{!}$-stratifications of bounded length such that every enumeration refining them is a $\downarrow^{!}$-construction.

Unique free extensions

As a preliminary step, we start with computing $\mathrm{fc}(\downarrow^{!})$. We employ the following characterisation of $\downarrow^{!}$ which is a variant of Lemma G1.3.6.

Definition 5.1. For sets $A, B, U \subseteq \mathbb{M}$, we define

$$A \perp^{\mathrm{do}}_U B \quad :\text{iff} \quad \begin{array}{l}\text{every relation that is definable over both}\\ A \cup U \text{ and } B \cup U \text{ is already definable over } U\,.\end{array}$$

If $A \perp^{\mathrm{do}}_U B$, we call A and B *definably orthogonal* over U.

Proposition 5.2. *Let T be a stable theory. Then*

$$A \downarrow^{\mathrm{f}}_U C \quad \textit{and} \quad C \perp^{\mathrm{do}}_U \mathrm{acl}^{\mathrm{eq}}(U) \quad \textit{implies} \quad A \downarrow^{!}_U C\,.$$

Proof. For a contradiction, suppose that $C \perp^{\mathrm{do}}_U \mathrm{acl}^{\mathrm{eq}}(U)$ and there are tuples $\bar{a}$ and $\bar{b}$ such that

$$\bar{a} \downarrow^{\mathrm{f}}_U C\,, \quad \bar{b} \downarrow^{\mathrm{f}}_U C\,, \quad \text{and} \quad \bar{a} \equiv_U \bar{b}\,,$$

but $\mathfrak{p} := \mathrm{tp}(\bar{a}/UC)$ and $\mathfrak{q} := \mathrm{tp}(\bar{b}/UC)$ are different. By Lemma **??**, we can find some $\chi \in \mathrm{FE}(U)$ such that

$$\mathfrak{p}(\bar{x}) \cup \mathfrak{q}(\bar{y}) \vDash \neg\chi(\bar{x}, \bar{y})\,.$$

Let E be the set of those equivalence classes $[\bar{c}]_\chi$ containing some tuple $\bar{c}' \in [\bar{c}]_\chi$ realising $\mathfrak{p}$. Then E is finite and we can choose an enumeration $[\bar{c}_0]_\chi, \dots, [\bar{c}_{m-1}]_\chi$ of E where each representative $\bar{c}_i$ realises $\mathfrak{p}$. We set

$$\varphi(x) := \bigvee_{i<m} \chi(\bar{x}, \bar{c}_i)\,.$$

Then $\mathbb{M} \vDash \varphi(\bar{a})$, while

$$\mathfrak{p}(\bar{x}) \cup \mathfrak{q}(\bar{y}) \vDash \neg\chi(\bar{x}, \bar{y}) \quad \text{implies} \quad \mathbb{M} \vDash \neg\chi(\bar{c}_i, \bar{b})\,,$$

for all $i < m$. Hence, $\mathbb{M} \vDash \varphi(\bar{a}) \wedge \neg\varphi(\bar{b})$ and it is sufficient to prove that φ is equivalent to a formula over U.

Since $\varphi^{\mathbb{M}} \equiv \bigvee_{i<m} \iota_\chi \bar{x} = [\bar{c}_i]_\chi$ is definable over $\mathrm{acl}^{\mathrm{eq}}(U)$ (recall that the structure $\mathbb{M}^{\mathrm{eq}}$ is equipped with projection functions $\iota_\chi : \mathbb{M}^{\bar{s}} \to \mathbb{M}_\chi$), it is sufficient to show that it is also definable over C. By Theorem E2.1.11 we only have to check that $\pi[\varphi^{\mathbb{M}}] = \varphi^{\mathbb{M}}$, for all $\pi \in \mathrm{Aut}\,\mathbb{M}_C$. Hence, consider an automorphism $\pi \in \mathrm{Aut}\,\mathbb{M}_C$. Then π can be extended to an automorphism of $\mathbb{M}^{\mathrm{eq}}_C$ and, therefore, it induces a permutation on the equivalence classes of χ. Since $\pi[\mathfrak{p}^{\mathbb{M}}] = \mathfrak{p}^{\mathbb{M}}$ it follows that π induces a permutation of E. Hence, we have

$$\begin{aligned}\pi[\varphi^{\mathbb{M}}] = \bigcup_{i<m} \pi[\chi(\bar{x}, \bar{c}_i)]^{\mathbb{M}} &= \bigcup_{i<m} \chi(\bar{x}, \pi(\bar{c}_i))^{\mathbb{M}} \\ &= \bigcup_{i<m} \chi(\bar{x}, \bar{c}_i)^{\mathbb{M}} = \varphi^{\mathbb{M}},\end{aligned}$$

as desired. □

Proposition 5.3. *Let T be a stable theory.*

(a) $\mathrm{fc}(\downarrow^{\mathrm{f}}) \leq \mathrm{fc}(\downarrow^{!}) \leq |T|^+$.

(b) $\mathrm{fc}(\downarrow^{!}) \leq \mathrm{fc}(\downarrow^{\mathrm{f}})^{\mathrm{reg}} \oplus \mathrm{mult}(\downarrow^{\mathrm{f}})^+$.

(c) *If T is $\aleph_0$-stable, then* $\mathrm{fc}(\downarrow^{!}) \leq \aleph_0$.

Proof. (a) For the lower bound, note that $\downarrow^{!} \subseteq \downarrow^{\mathrm{f}}$ implies that every $\downarrow^{\mathrm{f}}$-forking chain is also a $\downarrow^{!}$-forking chain. For the upper bound, we prove that $\mathrm{loc}_0(\downarrow^{!}) \leq |T|^+$. Since $|T|^+$ is regular, it then follows by Proposition F2.3.24 that $\mathrm{fc}(\downarrow^{!}) \leq \mathrm{loc}_0(\downarrow^{!})^{\mathrm{reg}} \leq |T|^+$.

Hence, consider a finite set A and an arbitrary set B. We construct an increasing sequence $(U_n)_{n<\omega}$ of subsets $U_n \subseteq B$ of size $|U_n| \leq |T|$ such that the union $U := \bigcup_{n<\omega} U_n$ is a set of size $|U| \leq |T|$ with $A \downarrow^{!}_U B$.

We start with some set $U_0 \subseteq B$ of size $|U_0| < \mathrm{fc}(\downarrow^{\mathrm{f}}) \leq |T|^+$ such that $A \downarrow^{\mathrm{f}}_{U_0} B$. For the inductive step, suppose that U_n is already defined. Note that there are at most

$$|T^{\mathrm{eq}}| \oplus |\mathrm{acl}^{\mathrm{eq}}(U_n)| \leq |T^{\mathrm{eq}}| \oplus |U_n| = |T|$$

formulae over $\mathrm{acl}^{\mathrm{eq}}(U_n)$ and, consequently, at most that many relations that are definable over both B and $\mathrm{acl}^{\mathrm{eq}}(U_n)$. Consequently, we can choose a set $C_n \subseteq B$ of size $|C_n| \leq |T|$ such that every relation definable over both B and $\mathrm{acl}^{\mathrm{eq}}(U_n)$ is definable over C_n. Setting $U_{n+1} := U_n \cup C_n$, it follows that

$$B \perp^{\mathrm{do}}_{U_{n+1}} \mathrm{acl}^{\mathrm{eq}}(U_n) \quad \text{and} \quad |U_{n+1}| = |U_n| \oplus |C_n| \leq |T| .$$

To see that $U := \bigcup_{n<\omega} U_n$ has the desired properties, first note that every relation definable over $\mathrm{acl}^{\mathrm{eq}}(U) = \bigcup_{n<\omega} \mathrm{acl}^{\mathrm{eq}}(U_n)$ is definable over $\mathrm{acl}^{\mathrm{eq}}(U_n)$, for some n, and hence over $U_{n+1} \subseteq U$. Consequently,

$$B \perp^{\mathrm{do}}_U \mathrm{acl}^{\mathrm{eq}}(U)$$

and it follows by Proposition 5.2 that $A \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_U B$ implies $A \mathop{\smile\!\!\!\!\!\mid}^{!}_U B$.

(b), (c) We prove both bounds simultaneously. Let κ be the least regular cardinal such that

$$\mathrm{mult}_{\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}}(\mathfrak{p}) < \kappa , \quad \text{for all types } \mathfrak{p} .$$

Then $\mathrm{mult}(\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}) \leq \kappa \leq \mathrm{mult}(\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}})^+$. Furthermore, for an $\aleph_0$-stable theory, we have $\mathrm{fc}(\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}})^{\mathrm{reg}} \leq \mathrm{st}(T)^{\mathrm{reg}} = \aleph_0$ and we have seen in Corollary G1.4.8 that $\kappa \leq \aleph_0$. Therefore, both (b) and (c) follow if we can prove that

$$\mathrm{fc}(\mathop{\smile\!\!\!\!\!\mid}^{!}) \leq \mathrm{fc}(\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}})^{\mathrm{reg}} \oplus \kappa .$$

For a contradiction, suppose that we can find some $\mathop{\smile\!\!\!\!\!\mid}^{!}$-forking chain $(B_\alpha)_{\alpha<\gamma}$ for some finite tuple $\bar a$ over $\varnothing$ whose length is $\gamma := \mathrm{fc}(\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}})^{\mathrm{reg}} \oplus \kappa$. Let $I \subseteq \gamma$ be the set of all indices $\alpha < \gamma$ such that $\bar a \not\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_{B[<\alpha]} B_\alpha$. Then $(B_\alpha)_{\alpha\in I}$ is a $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}$-forking chain for $\bar a$ over $\varnothing$ since

$$A \not\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_{B[<\alpha]} B_\alpha \quad \text{implies} \quad A \not\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_{B[I]\cap B[<\alpha]} B_\alpha , \quad \text{for all } \alpha \in I .$$

Hence, $|I| < \mathrm{fc}(\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}) \leq \gamma$. As γ is regular, it follows that $I \subseteq \beta_0$, for some index $\beta_0 < \gamma$. Note that $\beta_0 + \gamma = \gamma$. Hence, replacing $(B_\alpha)_{\alpha<\gamma}$ by the subsequence $(B_\alpha)_{\beta_0\leq\alpha<\gamma}$ and setting $U := B[<\beta_0]$, we may assume that

$$\bar a \not\mathop{\smile\!\!\!\!\!\mid}^{!}_{UB[<\alpha]} B_\alpha \quad \text{and} \quad \bar a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_{UB[<\alpha]} B_\alpha , \quad \text{for all } \alpha < \gamma .$$

We construct a sequence $(\bar{c}_\alpha)_{\alpha<\gamma}$ of tuples as follows. As

$$\bar{a} \not\mathrel{\raise0.2ex\hbox{$\downarrow$}}^{!}_{UB[<\alpha]} B_\alpha \quad \text{and} \quad \bar{a} \mathrel{\raise0.2ex\hbox{$\downarrow$}}^{\mathrm{f}}_{UB[<\alpha]} B_\alpha\,,$$

there exists a tuple $\bar{c}'_\alpha$ such that

$$\bar{c}'_\alpha \mathrel{\raise0.2ex\hbox{$\downarrow$}}^{\mathrm{f}}_{UB[<\alpha]} B_\alpha\,, \quad \bar{c}'_\alpha \equiv_{UB[<\alpha]} \bar{a}\,, \quad \text{and} \quad \bar{c}'_\alpha \not\equiv_{UB[<\alpha]B_\alpha} \bar{a}\,.$$

We choose some tuple $\bar{c}_\alpha \equiv_{UB[<\alpha]B_\alpha} \bar{c}'_\alpha$ such that

$$\bar{c}_\alpha \mathrel{\raise0.2ex\hbox{$\downarrow$}}^{\mathrm{f}}_{UB[<\alpha]} B[<\gamma]\,.$$

By a straightforward induction on α, one can show that $\bar{a} \mathrel{\raise0.2ex\hbox{$\downarrow$}}^{\mathrm{f}}_{U} B[<\alpha]$. Consequently, we have $\bar{c}_\alpha \mathrel{\raise0.2ex\hbox{$\downarrow$}}^{\mathrm{f}}_{U} B[<\alpha]$ and it follows by transitivity that $\bar{c}_\alpha \mathrel{\raise0.2ex\hbox{$\downarrow$}}^{\mathrm{f}}_{U} B[<\gamma]$. Since

$$\bar{c}_\alpha \not\equiv_{UB[<\alpha+1]} \bar{a} \equiv_{UB[<\alpha+1]} c_{\alpha'}\,, \quad \text{for all } \alpha < \alpha' < \gamma\,,$$

the types $\mathrm{tp}(\bar{c}_\alpha/UB[<\gamma])$ are distinct $\mathrel{\raise0.2ex\hbox{$\downarrow$}}^{\mathrm{f}}$-free extensions of $\mathrm{tp}(\bar{a}/U)$. By choice of κ, it follows that $\gamma < \kappa$. A contradiction. $\square$

Strongly independent stratifications

Instead of working with single $\mathrel{\raise0.2ex\hbox{$\downarrow$}}^{!}$-constructions, we will work with families of them that are encoded by a stratification for the following relation.

Definition 5.4. Let $\mathfrak{M}$ be the model of a stable theory. We define

$$A \overset{\mathrm{si}}{\surd}_U B \quad :\text{iff} \quad a \mathrel{\raise0.2ex\hbox{$\downarrow$}}^{!}_{U} B \cup (A \setminus \{a\})\,, \quad \text{for all } a \in A\,.$$

A set A is *strongly independent* over U if $A \overset{\mathrm{si}}{\surd} U$.

Note that $A \overset{\mathrm{si}}{\surd}_U B$ implies that every enumeration of A is a $\mathrel{\raise0.2ex\hbox{$\downarrow$}}^{!}$-construction over U. It follows that every $\overset{\mathrm{si}}{\surd}$-stratification can be refined to a whole family of $\mathrel{\raise0.2ex\hbox{$\downarrow$}}^{!}$-constructions.

Lemma 5.5. *If there exists a* $\overset{\mathrm{si}}{\surd}$*-stratification* ζ *of* A *over* U*, then* A *is* $\downarrow^{!}$*-constructible over* U*.*

Proof. Suppose that $\zeta = (B_\beta)_{\beta<\gamma}$. By Lemma 2.8 (a), it is sufficient to prove that each set B_β is $\downarrow^{!}$-constructible over $U \cup B[<\beta]$. Hence, let $(b_\alpha)_{\alpha<\delta}$ be an arbitrary enumeration of B_β. Then

$$B_\beta \overset{\mathrm{si}}{\surd} UB[<\beta] \quad \text{implies} \quad b_\alpha \downarrow^{!}_{UB[<\beta]} b[<\alpha]\,, \quad \text{for all } \alpha < \delta. \square$$

Lemma 5.6. *In a stable theory, the relation* $\overset{\mathrm{si}}{\surd}$ *satisfies all axioms of an isolation relation except for* (LTR).

Proof. (INV) follows immediately form the definition.

(MON) Suppose that $AC \overset{\mathrm{si}}{\surd}_U BD$. Then

$$a \downarrow^{!}_U BD \cup ((A \cup C) \smallsetminus \{a\})\,, \quad \text{for all } a \in A \cup C\,,$$

which implies that

$$a \downarrow^{!}_U B \cup (A \smallsetminus \{a\})\,, \quad \text{for all } a \in A\,.$$

Hence, $A \overset{\mathrm{si}}{\surd}_U B$.

(BMON) Suppose that $A \overset{\mathrm{si}}{\surd}_U BC$. Then

$$a \downarrow^{!}_U BC \cup (A \smallsetminus \{a\})\,, \quad \text{for all } a \in A\,,$$

which implies that

$$a \downarrow^{!}_{UC} B \cup (A \smallsetminus \{a\})\,, \quad \text{for all } a \in A\,.$$

Hence, $A \overset{\mathrm{si}}{\surd}_{UC} B$.

(NOR) Suppose that $A \overset{\mathrm{si}}{\surd}_U B$. Then

$$a \downarrow^{!}_U B \cup (A \smallsetminus \{a\})\,, \quad \text{for all } a \in A\,,$$

which implies that

$$a \downarrow^{!}_{U} UB \cup (A \setminus \{a\})\,, \quad \text{for all } a \in A\,.$$

Since, by (NOR),

$$c \downarrow^{!}_{U} UB \cup (A \setminus \{c\})\,, \quad \text{for all } c \in U\,,$$

it follows that $AU \sqrt[\mathrm{si}]{}_{U}\, BU$.

(LRF) Let A and B be sets. Since

$$a \downarrow^{!}_{A} B \cup (A \setminus \{a\})\,, \quad \text{for all } a \in A\,,$$

it follows that $A \sqrt[\mathrm{si}]{}_{A}\, B$.

(FIN) Suppose that $A_0 \sqrt[\mathrm{si}]{}_{U}\, B$, for all finite $A_0 \subseteq A$. To show that $A \sqrt[\mathrm{si}]{}_{U}\, B$, consider an element $a \in A$. For every finite $A_0 \subseteq A \setminus \{a\}$,

$$A_0 a \sqrt[\mathrm{si}]{}_{U}\, B \quad \text{implies} \quad a \downarrow^{!}_{U} BA_0\,.$$

Since $\downarrow^{!}$ is a symmetric relation with finite character, it follows that

$$a \downarrow^{!}_{U} B(A \setminus \{a\})\,.$$

(RSH) Suppose that $AC \sqrt[\mathrm{si}]{}_{U}\, B$ and $C \sqrt[\mathrm{si}]{}_{U}\, AB$. In order to show that $A \sqrt[\mathrm{si}]{}_{U}\, BC$, we consider an element $a \in A$. Then

$$AC \sqrt[\mathrm{si}]{}_{U}\, B \quad \text{implies} \quad a \downarrow^{!}_{U} B \cup ((A \cup C) \setminus \{a\})\,.$$

If $a \notin C$, then $B \cup ((A \cup C) \setminus \{a\}) = B \cup C \cup (A \setminus \{a\})$ and we are done. Hence, suppose that $a \in C$. Then

$$C \sqrt[\mathrm{si}]{}_{U}\, AB \quad \text{implies} \quad a \downarrow^{!}_{U} AB \cup (C \setminus \{a\})\,.$$

As $AB \cup (C \setminus \{a\}) = BC \cup (A \setminus \{a\})$, the claim follows. □

The next lemma is our main tool to construct $\sqrt[\mathrm{si}]{}$-stratifications.

Lemma 5.7. *Let T be a stable theory and $c \in \mathbb{M}$. Then*

$$A \overset{\mathrm{si}}{\surd}_U B \quad \text{and} \quad c \downarrow^{!}_U AB \quad \text{implies} \quad Ac \overset{\mathrm{si}}{\surd}_U B\,.$$

Proof. Let $a \in A$ and set $A_0 := A \setminus \{a\}$. We have to show that

$$a \downarrow^{!}_U BA_0c\,.$$

By symmetry, $c \downarrow^{!}_{UBA_0} a$ implies $a \downarrow^{!}_{UBA_0} c$. Since $a \downarrow^{!}_U BA_0$, it follows by transitivity that $a \downarrow^{!}_U BA_0c$, as desired. □

We are finally able to prove that $\overset{\mathrm{si}}{\surd}$-stratifications always exist and that their length can be bounded.

Theorem 5.8. *Let T be a stable theory. Every set $A \subseteq \mathbb{M}$ has a $\overset{\mathrm{si}}{\surd}$-stratification $\zeta = (B_\alpha)_{\alpha<\gamma}$ over $\varnothing$ of length $\gamma \leq \mathrm{fc}(\downarrow^{!})$.*

Proof. Set $\kappa := \mathrm{fc}(\downarrow^{!})$. By induction on α, we choose a sequence $(B_\alpha)_{\alpha<\kappa}$ of disjoint subsets $B_\alpha \subseteq A$ as follows. Suppose that B_α has already been defined for all $\alpha < \beta$. Since the union of an increasing chain of strongly independent sets is again strongly independent, we can use the Lemma of Zorn to find a maximal subset $B_\beta \subseteq A \setminus B[<\beta]$ such that

$$B_\beta \overset{\mathrm{si}}{\surd} B[<\beta]\,.$$

The sequence $(B_\alpha)_{\alpha<\kappa}$ defined in this way is a $\overset{\mathrm{si}}{\surd}$-stratification of $B[<\kappa]$ over $\varnothing$.

It remains to prove that $B[<\kappa] = A$. For a contradiction, suppose that there is some element $a \in A \setminus B[<\kappa]$. By definition of $\kappa = \mathrm{fc}(\downarrow^{!})$ there exists some index $\alpha < \kappa$ such that $a \downarrow^{!}_{B[<\alpha]} B_\alpha$. Since $B_\alpha \overset{\mathrm{si}}{\surd}_{B[<\alpha]} B[<\alpha]$, it follows by Lemma 5.7 that

$$B_\alpha a \overset{\mathrm{si}}{\surd}_{B[<\alpha]} B[<\alpha]\,.$$

This contradicts the maximality of B_α. □

By the special nature of the relation $\overset{\mathrm{si}}{\surd}$, $\overset{\mathrm{si}}{\surd}$-stratifications can always be refined.

Definition 5.9. Let $(B_\alpha)_{\alpha<\gamma}$ and $(C_\alpha)_{\alpha<\delta}$ be partitions of a set A. We call $(B_\alpha)_{\alpha<\gamma}$ a *refinement* of $(C_\alpha)_{\alpha<\delta}$ if there exists an increasing function $f : \gamma \to \delta$ such that

$$B_\alpha \subseteq C_{f(\alpha)}\,, \quad \text{for all } \alpha < \gamma\,.$$

Lemma 5.10. *Let $\zeta = (C_\alpha)_{\alpha<\delta}$ be a $\overset{\mathrm{si}}{\surd}$-stratification of A over U. Every refinement $(B_\alpha)_{\alpha<\gamma}$ of ζ is also a $\overset{\mathrm{si}}{\surd}$-stratification of A over U.*

Proof. Let $f : \gamma \to \delta$ be the function such that $B_\alpha \subseteq C_{f(\alpha)}$. To show that $B_\alpha \overset{\mathrm{si}}{\surd} B[<\alpha]$, we consider an element $a \in B_\alpha$. Since

$$C[<f(\alpha)] \subseteq B[<\alpha] \quad \text{and} \quad B[\leq\alpha] \subseteq C[\leq f(\alpha)]\,,$$

it follows by monotonicity of $\downarrow^{!}$ that

$$a \downarrow^{!}_{UC[<f(\alpha)]} C[\leq f(\alpha)] \setminus \{a\}$$

implies $a \downarrow^{!}_{UB[<\alpha]} B[\leq\alpha] \setminus \{a\}$. □

When considering $\overset{\mathrm{si}}{\surd}$-stratifications as families of $\downarrow^{!}$-constructions, we need to modify the notion of a closed set as follows.

Definition 5.11. Let $\zeta = (B_\alpha)_{\alpha<\gamma}$ be a $\overset{\mathrm{si}}{\surd}$-stratification of A over U.

(a) For every $\alpha < \gamma$ and every element $a \in B_\alpha$, let $W(a) \subseteq U \cup B[<\alpha]$. We call the family $(W(a))_{a\in A}$ a *system of bases* for ζ if

$$a \downarrow^{!}_{W(a)} U \cup (B[\leq\alpha] \setminus \{a\})\,, \quad \text{for all } \alpha < \gamma \text{ and all } a \in B_\alpha\,.$$

(b) Let $(W(a))_{a\in A}$ be a system of bases for ζ. A set $C \subseteq A$ is *W-closed* if $W(a) \subseteq U \cup C$, for all $a \in C$.

Lemma 5.12. *Let T be a stable theory and let $\zeta = (B_\alpha)_{\alpha<\gamma}$ be a $\overset{\mathrm{si}}{\bigvee}$-stratification of a A over U. There exists a system of bases $(W(a))_{a\in A}$ for ζ such that $|W(a)| < \mathrm{loc}_0(\downarrow^!)$, for all $a \in A$.*

Proof. For every $a \in B_\alpha$, we choose a set $W(a) \subseteq U \cup B[<\alpha]$ of size $|W(a)| < \mathrm{loc}_0(\downarrow^!)$ such that

$$a \downarrow^!_{W(a)} UB[<\alpha] .$$

Since $a \downarrow^!_{UB[<\alpha]} U \cup (B[\leq\alpha] \setminus \{a\})$, it follows by transitivity that

$$a \downarrow^!_{W(a)} U \cup (B[\leq\alpha] \setminus \{a\}) .$$ □

Lemma 5.13. *Let $(B_\alpha)_{\alpha<\gamma}$ be a $\overset{\mathrm{si}}{\bigvee}$-stratification of A over U with system of bases $(W(a))_{a\in A}$. If $C \subseteq A$ is W-closed, then $(B_\alpha \cap C)_{\alpha<\gamma}$ is a $\overset{\mathrm{si}}{\bigvee}$-stratification of C over U.*

Proof. Set $C_\alpha := B_\alpha \cap C$, for $\alpha < \gamma$. Consider an element $a \in C_\alpha$. We have to show that $a \downarrow^!_{UC[<\alpha]} C[\leq\alpha] \setminus \{a\}$. Note that $W(a) \subseteq U \cup C[<\alpha]$ and

$$a \downarrow^!_{W(a)} U \cup (B[\leq\alpha] \setminus \{a\}) ,$$

by choice of $W(a)$. By monotonicity, it follows that

$$a \downarrow^!_{UC[<\alpha]} U \cup (C[\leq\alpha] \setminus \{a\}) ,$$

as desired. □

6. Representations

We have introduced indiscernible systems at the end of Section E5.3. Intuitively, if $a : I \to M$ in an indiscernible system over $\mathfrak{I}$, the structure $\mathfrak{M}$ is at least as complicated as $\mathfrak{I}$. If $a : I \to M$ is surjective, the converse is also true to some extend. In this section, we will characterise theories T by classes $\mathcal{C}$ of structures such that, every model $\mathfrak{M}$ of T has a bijective indiscernible system $a : I \to M$ with $\mathfrak{I} \in \mathcal{C}$.

Definition 6.1. Let $\mathcal{C}$ be a class of structures.

(a) A *representation* of a structure $\mathfrak{M}$ in $\mathcal{C}$ is a bijective indiscernible system $r : \mathfrak{I} \to \mathfrak{M}$, for some $\mathfrak{J} \in \mathcal{C}$. If the system $r : \mathfrak{I} \to \mathfrak{M}$ is only QF-indiscernible, i.e., if

$$\operatorname{atp}(\bar{\imath}) = \operatorname{atp}(\bar{k}) \quad \text{implies} \quad \operatorname{atp}(r[\bar{\imath}]) = \operatorname{atp}(r[\bar{k}]),$$

we call r a *quantifier-free representation.*

(b) We say that a theory T has $\mathcal{C}$*-representations* if every model $\mathfrak{M}$ of T has a representation in $\mathcal{C}$. Similarly, we say that a class $\mathcal{K}$ has *quantifier-free* $\mathcal{C}$*-representations* if every structure $\mathfrak{M} \in \mathcal{K}$ has a quantifier-free representation in $\mathcal{C}$.

First, let us note that representations are closed under composition.

Lemma 6.2. *If T has $\mathcal{K}$-representations and $\mathcal{K}$ has quantifier-free $\mathcal{C}$-representations, then T has $\mathcal{C}$-representations.*

Lemma 6.3. *If $\mathcal{C} \subseteq \mathcal{K}$, then $\mathcal{C}$ has quantifier-free $\mathcal{K}$-representations.*

In this section we will characterise stable theories in terms of representations in the following classes.

Definition 6.4. Let κ and λ be cardinals.

(a) $\Upsilon_{\kappa\lambda}$ is the signature consisting of unary predicates P_α, for $\alpha < \kappa$, and unary function symbols f_α, for $\alpha < \lambda$.

(b) We denote the class of all $\Upsilon_{\kappa\lambda}$-structures by $\operatorname{Un}(\kappa, \lambda)$ and the subclass consisting of all structures that are *locally finite* by $\operatorname{Lf}(\kappa, \lambda)$. Finally, $\operatorname{Wf}(\kappa, \lambda) \subseteq \operatorname{Un}(\kappa, \lambda)$ is the subclass of all structures such that the inverse R^{-1} of the relation $R := \bigcup_{\alpha<\lambda} f_\alpha$ is *well-founded.*

First, let us give some simple relationships between these classes.

Lemma 6.5. *Let κ and λ be infinite cardinals.*

(a) $\operatorname{Wf}(\kappa, \lambda)$ *has quantifier-free* $\operatorname{Un}(\kappa, \lambda)$*-representations.*

(b) *If $n < \aleph_0$,* $\operatorname{Wf}(\kappa, n)$ *has quantifier-free* $\operatorname{Lf}(\kappa, n)$*-representations.*

(c) *If* $\kappa_0 \leq \kappa$ *and* $\lambda_0 \leq \lambda$, $\mathrm{Un}(\kappa_0, \lambda_0)$ *has quantifier-free* $\mathrm{Un}(\kappa, \lambda)$*-representations.*

(d) $\mathrm{Un}(\kappa, \lambda)$ *has quantifier-free* $\mathrm{Un}(0, \lambda \oplus \kappa)$*-representations.*

(e) $\mathrm{Wf}(\kappa, \lambda)$ *has quantifier-free* $\mathrm{Wf}(0, \lambda \oplus \kappa)$*-representations.*

Proof. (a) follows by Lemma 6.3.

(b) If n is finite, every structure in $\mathrm{Wf}(\kappa, n)$ is locally finite. Therefore $\mathrm{Wf}(\kappa, n) \subseteq \mathrm{Lf}(\kappa, n)$ and the claim follows again by Lemma 6.3.

(c) For $\mathfrak{M} \in \mathrm{Un}(\kappa_0, \lambda_0)$, we construct a quantifier-free representation $r : I \to M$ where $\mathfrak{I} \in \mathrm{Un}(\kappa, \lambda)$ is the expansion of $\mathfrak{M}$ by the following functions and relations:

$$R_\alpha := \varnothing\,, \quad \text{for } \kappa_0 \leq \alpha < \kappa\,,$$
$$f_\alpha := \mathrm{id}\,, \quad \text{for } \lambda_0 \leq \alpha < \lambda\,.$$

It follows that the identity function $\mathrm{id} : M \to M$ is a quantifier-free indiscernible system over $\mathfrak{I}$.

(d) Consider a structure $\mathfrak{M} \in \mathrm{Un}(\kappa, \lambda)$. If $|M| \leq 1$, let $\mathfrak{I} \in \mathrm{Un}(0, \lambda \oplus \kappa)$ be the unique structure of size $|I| = |M|$ and let $r : I \to M$ be the corresponding bijection. Then $r : I \to M$ is a quantifier-free indiscernible system.

If $|M| > 1$, we proceed as follows. Choosing distinct elements $0, 1 \in M$ we construct the structure $\mathfrak{I}$ with universe $I := M$ and functions

$$f_\alpha^{\mathfrak{I}}(a) := f_\alpha^{\mathfrak{M}}(a)\,, \qquad \text{for } \alpha < \lambda\,,$$
$$g_\beta^{\mathfrak{I}}(a) := \begin{cases} 0 & \text{if } a \notin P_\beta\,, \\ 1 & \text{if } a \in P_\beta\,, \end{cases} \qquad \text{for } \beta < \kappa\,,$$
$$h^{\mathfrak{I}}(a) := 0\,.$$

Note that $\mathfrak{I}$ has $\kappa \oplus \lambda \oplus 1$ functions. Hence, after renaming the functions we obtain a structure in $\mathrm{Un}(0, \lambda \oplus \kappa)$. The identity $\mathrm{id} : I \to M$ is a quantifier-free indiscernible system.

(e) Let $\mathfrak{M} \in \mathrm{Wf}(\kappa, \lambda)$. If $|M| \leq 1$, we proceed as in (d). If $|M| \geq 2$, we modify the construction in (d) as follows. Since $\mathfrak{M} \in \mathrm{Wf}(\kappa, \lambda)$, we can choose $0, 1 \in M$ such that $f_\alpha^{\mathfrak{M}}(0) = 0$ and $f_\alpha^{\mathfrak{M}}(1) \in \{0, 1\}$, for all α. Let $\mathfrak{I}$ be the structure with universe M and functions

$$\begin{aligned}
f_\alpha^{\mathfrak{I}}(a) &:= f_\alpha^{\mathfrak{M}}(a)\,, && \text{for } \alpha < \lambda\,,\\
g_\beta^{\mathfrak{I}}(a) &:= \begin{cases} 0 & \text{if } a \notin P_\beta \text{ or } a = 0\,,\\ 1 & \text{if } a \in P_\beta \text{ and } a \neq 0\,, \end{cases} && \text{for } \beta < \kappa\,,\\
h^{\mathfrak{I}}(a) &:= 0\,.
\end{aligned}$$

After a suitable renaming of the functions, we obtain a structure $\mathfrak{I} \in \mathrm{Wf}(0, \lambda \oplus \kappa)$. Again $\mathrm{id} : I \to M$ is the desired quantifier-free indiscernible system. □

Representable theories are stable

We start by showing that theories represented in one of the above classes are stable. The key argument is contained in the following two Ramsey results.

Lemma 6.6. *Let $\mathfrak{M} \in \mathrm{Un}(\kappa, \lambda)$, for infinite cardinals κ and λ, and let $n < \omega$. For every sequence $(\bar{a}^\alpha)_{\alpha<\mu}$ where $\bar{a}^\alpha \in M^n$ and $\mu := (2^{\kappa\oplus\lambda})^+$, there exists an infinite subset $I \subseteq \mu$ such that the subsequence $(\bar{a}^\alpha)_{\alpha\in I}$ is totally* QF*-indiscernible.*

Proof. Every finitely generated substructure of $\mathfrak{M}$ has size at most λ. By Lemma B1.1.5, there are, up to isomorphism, at most $2^{\kappa\oplus\lambda}$ such substructures. Since $\mu > 2^{\kappa\oplus\lambda}$, there exists a subset $I_0 \subseteq \mu$ of size $|I_0| = \mu$ such that

$$\langle \langle\!\langle \bar{a}^\alpha \rangle\!\rangle_{\mathfrak{M}}, \bar{a}^\alpha \rangle \cong \langle \langle\!\langle \bar{a}^\beta \rangle\!\rangle_{\mathfrak{M}}, \bar{a}^\beta \rangle\,, \quad \text{for all } \alpha, \beta \in I_0\,.$$

For each $\alpha \in I_0$, we fix an enumeration $\bar{b}^\alpha = (b_i^\alpha)_{i<\gamma}$, without repetitions, of $\langle\!\langle \bar{a}^\alpha \rangle\!\rangle_{\mathfrak{M}}$ such that, for all $\alpha, \beta \in I_0$, the map $b_i^\alpha \mapsto b_i^\beta$, $i < \gamma$, induces an isomorphism

$$\langle \langle\!\langle \bar a^\alpha \rangle\!\rangle_{\mathfrak{M}}, \bar a^\alpha \rangle \to \langle \langle\!\langle \bar a^\beta \rangle\!\rangle_{\mathfrak{M}}, \bar a^\beta \rangle .$$

For every $\nu < \mu$, we have $\nu^{<\lambda^+} = \nu^\lambda \leq (2^{\kappa \oplus \lambda})^\lambda = 2^{\kappa \oplus \lambda} < \mu$. Hence, we can use Lemma A4.6.11 to find a subset $U \subseteq M$ and a subset $I_1 \subseteq I_0$ of size $|I_1| = \mu$ such that

$$\bar b^\alpha \cap \bar b^\beta = U , \quad \text{for all } \alpha \neq \beta \text{ in } I_1 .$$

Since $\mu > 2^\lambda$, there exists a subset $I_2 \subseteq I_1$ of size $|I_2| = \mu$ and a set $K \subseteq \lambda$ such that, for all $\alpha \in I_2$,

$$K = \{ i < \gamma \mid b^\alpha_i \in U \} .$$

We claim that the sequence $(\bar a^i)_{i \in I_2}$ is totally QF-indiscernible. Let $\varphi(\bar x)$ be an atomic formula. We have to show that

$$\mathfrak{M} \vDash \varphi(\bar a[\bar\alpha]) \leftrightarrow \varphi(\bar a[\bar\beta]) , \quad \text{for all } \bar\alpha, \bar\beta \subseteq I_2 .$$

First, we consider the case where $\varphi = (s = t)$ is an equation. Then there are indices $\xi_0, \dots, \xi_{m-1}, \eta_0, \dots, \eta_{n-1} < \lambda$ and variables x, y such that

$$s(x) = f_{\xi_{m-1}} \cdots f_{\xi_0} x \quad \text{and} \quad t(y) = f_{\eta_{n-1}} \cdots f_{\eta_0} y .$$

For each component $i < n$ of $\bar a^\alpha$, there are indices i' and i'' such that $b^\alpha_{i'} = s(a^\alpha_i)$ and $b^\alpha_{i''} = t(a^\alpha_i)$. Since we have chosen $\bar b^\alpha$ without repetitions, it follows that

$$\begin{aligned} \mathfrak{M} \vDash s(a^\alpha_i) = t(a^\beta_k) \quad &\text{iff} \quad b^\alpha_{i'} = b^\beta_{k''} \\ &\text{iff} \quad i' = k'' \text{ and } (\alpha = \beta \text{ or } i' \in K) . \end{aligned}$$

The latter condition is invariant under permutations of I_2.

It remains to consider the case where $\varphi = P_\zeta t$ for $t = f_{\xi_{m-1}} \cdots f_{\xi_0} x$. Again we can find, for every component i of $\bar{a}^\alpha$ an index i' such that $b^\alpha_{i'} = t(a^\alpha_i)$. Therefore,

$$\mathfrak{M} \vDash P_\zeta(t(a^\alpha_i)) \quad \text{iff} \quad b^\alpha_{i'} \in P_\zeta \,.$$

The latter condition does not depend on α, since the substructures induced by the tuples $\bar{b}^\alpha$ are isomorphic. □

Lemma 6.7. *Let $\mathfrak{M} \in \mathrm{Lf}(\kappa, \lambda)$, let $\mu > \kappa, \lambda$ be an uncountable cardinal, and $n < \omega$. For every sequence $(\bar{a}^\alpha)_{\alpha<\mu}$ of n-tuples $\bar{a}^\alpha \in M^n$ of length μ and every set $U \subseteq M$ of size $|U| < \mu$, there exists a subset $I \subseteq \mu$ of size $|I| = \mu$ such that*

$$\mathrm{atp}(\bar{a}^\alpha/U) = \mathrm{atp}(\bar{a}^\beta/U)\,, \qquad \textit{for all } \alpha, \beta \in I\,.$$

Proof. Since each substructure $\langle\!\langle \bar{a}^\alpha \rangle\!\rangle_{\mathfrak{M}}$ is finite and there are at most $\kappa \oplus \lambda \oplus \aleph_0 < \mu$ finite substructures, there is a subset $I_0 \subseteq \mu$ of size $|I_0| = \mu$ such that

$$\langle \langle\!\langle \bar{a}^\alpha \rangle\!\rangle_{\mathfrak{M}}, \bar{a}^\alpha \rangle \cong \langle \langle\!\langle \bar{a}^\beta \rangle\!\rangle_{\mathfrak{M}}, \bar{a}^\beta \rangle\,, \quad \text{for all } \alpha, \beta \in I_0\,.$$

Let $V := \langle\!\langle U \rangle\!\rangle_{\mathfrak{M}}$. Since there are at most $|V|^{<\omega} = |V| \oplus \aleph_0 < \mu$ finite subsets of V, we can find a subset $I_1 \subseteq I_0$ of size $|I_1| = \mu$ and a finite set $W \subseteq V$ such that

$$\langle\!\langle \bar{a}^\alpha \rangle\!\rangle_{\mathfrak{M}} \cap V = W\,, \quad \text{for all } \alpha \in I_1\,.$$

Let $\bar{c}$ be an enumeration of W. There exists a subset $I_2 \subseteq I_1$ of size $|I_2| = \mu$ such that

$$\langle \langle\!\langle \bar{a}^\alpha \rangle\!\rangle_{\mathfrak{M}}, \bar{a}^\alpha, \bar{c} \rangle \cong \langle \langle\!\langle \bar{a}^\beta \rangle\!\rangle_{\mathfrak{M}}, \bar{a}^\beta, \bar{c} \rangle\,, \quad \text{for all } \alpha, \beta \in I_2\,.$$

It follows that

$$\mathrm{atp}(\bar{a}^\alpha/V) = \mathrm{atp}(\bar{a}^\beta/V)\,, \quad \text{for all } \alpha, \beta \in I_2\,. \qquad \square$$

Using these two lemmas we can show that theories with representations in one of the above classes are stable.

Proposition 6.8. *Let T be a complete first-order theory.*

(a) *If T has $\mathrm{Un}(\kappa, \lambda)$-representations, for some cardinals κ, λ, it is stable.*

(b) *If T has $\mathrm{Lf}(\kappa, \kappa)$-representations, it is λ-stable, for all $\lambda \geq \kappa \oplus \aleph_0$.*

(c) *If T has $\mathrm{Lf}(\kappa, \lambda)$-representations, it is superstable with $\mathrm{st}(T) \leq \kappa \oplus \lambda \oplus \aleph_0$.*

Proof. (a) With out loss of generality, we may assume that κ and λ are infinite. For a contradiction, assume that T is unstable. Then we can use Theorem E5.3.13 to find a model $\mathfrak{M}$ containing an infinite indiscernible sequence $(\bar{a}^i)_{i \in I}$ that is not totally indiscernible. By Lemma E5.3.9, we can find such a sequence where $I = \mu$, for $\mu := (2^{\kappa \oplus \lambda})^+$. Let $r : \mathfrak{I} \to \mathfrak{M}$ be a representation of $\mathfrak{M}$ in $\mathrm{Un}(\kappa, \lambda)$ and set $\bar{b}^i := r^{-1}(\bar{a}^i)$. By Lemma 6.6, there exists an infinite subset $I_0 \subseteq I$ such that the subsequence $(\bar{b}^i)_{i \in I_0}$ is totally QF-indiscernible. It follows that $(\bar{a}^i)_{i \in I_0}$ is totally indiscernible. As $\mathrm{Av}(\bar{a}^i)_{i \in I_0} = \mathrm{Av}(\bar{a}^i)_{i \in I}$, the sequence $(\bar{a}^i)_{i \in I}$ is also totally indiscernible. A contradiction.

(b) For a contradiction, suppose that $|S^{\bar{s}}(U)| > \lambda$, for some finite tuple $\bar{s}$ of sorts and some set U of parameters of size $|U| = \lambda \geq \kappa \oplus \aleph_0$. Fix a sequence $(\bar{a}^\alpha)_{\alpha < \lambda^+}$ such that

$$\mathrm{tp}(\bar{a}^\alpha / U) \neq \mathrm{tp}(\bar{a}^\beta / U)\,, \quad \text{for all } \alpha \neq \beta\,.$$

Let $r : \mathfrak{I} \to \mathfrak{M}$ be a representation of $\mathfrak{M}$ in $\mathrm{Lf}(\kappa, \kappa)$. We set

$$V := \langle\!\langle r^{-1}[U] \rangle\!\rangle_{\mathfrak{I}} \quad \text{and} \quad \bar{b}^\alpha := r^{-1}(\bar{a}^\alpha)\,, \quad \text{for } \alpha < \lambda^+.$$

By Lemma 6.7, there exists a subset $I \subseteq \lambda^+$ of size $|I| = \lambda^+$ such that

$$\mathrm{atp}(\bar{b}^\alpha / V) = \mathrm{atp}(\bar{b}^\beta / V)\,, \quad \text{for all } \alpha, \beta \in I\,.$$

Since r is a representation, it follows that

$$\operatorname{tp}(\bar{a}^{\alpha}/U) = \operatorname{tp}(\bar{a}^{\beta}/U)\,, \quad \text{for all } \alpha, \beta \in I\,,$$

in contradiction to our choice of $(\bar{a}^{\alpha})_{\alpha}$.

(c) According to Theorem G1.6.6, a theory T is superstable if, and only if, there exists some cardinal λ such that T is κ-stable, for all $\kappa \geq \lambda$. Consequently, the claim follows from (b). □

Stable theories have representations

For the converse statements, we employ $\overset{\mathrm{si}}{\surd}$-stratifications. The following two technical lemmas contain the key argument.

Lemma 6.9. *Let T be stable, $\zeta = (B_{\alpha})_{\alpha<\gamma}$ a $\overset{\mathrm{si}}{\surd}$-stratification of some set M over $\varnothing$ such that $|B_0| = |B_1| = 1$, and let $(W(a))_{a\in M}$ be a system of bases for ζ.*

Suppose that $p : C \to D$ is a bijective function with $C, D \subseteq M$ satisfying the following conditions:

- *$p[B_{\alpha}] \subseteq B_{\alpha}$, for all $\alpha < \gamma$.*
- *C is W-closed.*
- *$p[W(a)] = W(p(a))$, for all $a \in C$.*
- *$p[\operatorname{tp}(a/W(a))] = \operatorname{tp}(p(a)/W(p(a)))$, for all $a \in C$.*

Then p is an elementary map.

Proof. Set $C_{\alpha} := C \cap B_{\alpha}$ and $D_{\alpha} := D \cap B_{\alpha}$. By assumption, $p[C_{\alpha}] = D_{\alpha}$. We will show by induction on $\alpha < \gamma$ that

$$p[\operatorname{tp}(\bar{a}/C[<\alpha])] = \operatorname{tp}(p(\bar{a})/D[<\alpha])\,, \quad \text{for all finite } \bar{a} \subseteq C_{\alpha}\,.$$

For $\alpha < 2$, the claim holds trivially since $|B_{\alpha}| = 1$. For $\alpha \geq 2$, we prove the statement by induction on $|\bar{a}|$. Hence, suppose that $\bar{a} = b\bar{c}$ and that we have already shown that

$$p[\operatorname{tp}(\bar{c}/C[<\alpha])] = \operatorname{tp}(p(\bar{c})/D[<\alpha])\,.$$

By assumption, we have

$$W(p(b)) = p[W(b)]\,,$$
$$\text{and}\quad \operatorname{tp}\big(p(b) \,/\, W(p(b))\big) = p\big(\operatorname{tp}(b/W(b))\big)\,.$$

Since $b \mathop{\downarrow}^{!}_{W(b)} B[\leq\alpha] \setminus \{b\}$, the type $\mathfrak{p} := \operatorname{tp}(b/C[<\alpha]\bar{c})$ is the unique free extension of $\operatorname{tp}(b/W(b))$. As we have already shown that the map $p \upharpoonright C[<\alpha]\bar{c}$ is elementary, it follows that the image $p(\mathfrak{p})$ does not fork over $p[W(b)] = W(p(b))$. Since

$$\operatorname{tp}(p(b)/W(p(b))) = \operatorname{tp}(p(b)/p[W(b)]) = p(\operatorname{tp}(b/W(b)))$$

and $p(b) \mathop{\downarrow}^{!}_{W(p(b))} D[<\alpha]p(\bar{c})$, it follows that

$$p(\mathfrak{p}) = \operatorname{tp}\big(p(b) \,/\, D[<\alpha]p(\bar{c})\big)\,.$$

Consequently,

$$p\big(\operatorname{tp}(b\bar{c}/C[<\alpha])\big) = \operatorname{tp}\big(p(b\bar{c})/D[<\alpha]\big)\,. \qquad \square$$

Lemma 6.10. *Let $\mathfrak{M}$ be a structure, $\zeta = (B_\alpha)_{\alpha<\gamma}$ a $\sqrt[\text{si}]{}$-stratification of M over $\varnothing$ such that $|B_0| = |B_1| = 1$, and let $(W(a))_{a\in M}$ be a system of bases for ζ. If*

$$\mathfrak{I} := \big\langle M,\ (f_\alpha)_{\alpha<\kappa},\ (P_\alpha)_{\alpha<\lambda},\ (Q_\alpha)_{\alpha<\mu}\big\rangle$$

is a structure such that

- $\{\, f_\alpha(a) \mid \alpha < \kappa \,\} = \{a\} \cup W(a)\,,$ *for all* $a \in M\,,$
- $a \in P_\beta \Leftrightarrow b \in P_\beta$, *for all* $\beta < \lambda$, *implies that* $a \in B_\alpha \Leftrightarrow b \in B_\alpha$, *for all* $\alpha < \gamma\,,$
- $a \in Q_\beta \Leftrightarrow b \in Q_\beta$, *for all* $\beta < \mu$, *implies that*

$$a(f_\alpha(a))_{\alpha<\kappa} \equiv b(f_\alpha(b))_{\alpha<\kappa}\,,$$

then the identity map $\mathrm{id} : M \to M$ *is a representation of* $\mathfrak{M}$ *in* $\mathfrak{I}$.

Proof. Let $\bar a, \bar b \subseteq M$ be finite tuples such that

$$\langle \langle\!\langle \bar a \rangle\!\rangle_{\mathfrak{I}}, \bar a \rangle \cong \langle \langle\!\langle \bar b \rangle\!\rangle_{\mathfrak{I}}, \bar b \rangle .$$

We have to show that $\bar a \equiv \bar b$ in $\mathfrak{M}$. Let $p : \langle\!\langle \bar a \rangle\!\rangle_{\mathfrak{I}} \to \langle\!\langle \bar b \rangle\!\rangle_{\mathfrak{I}}$ be an isomorphism with $p(\bar a) = \bar b$. It is sufficient to prove that p satisfies the conditions of Lemma 6.9.

By assumption on the predicates P_β we have

$$p(c) \in B_\alpha \quad \text{iff} \quad c \in B_\alpha .$$

Hence, $p[B_\alpha] \subseteq B_\alpha$. Furthermore, if $c \in \operatorname{dom}(p) = \langle\!\langle \bar a \rangle\!\rangle_{\mathfrak{I}}$, then

$$\{c\} \cup W(c) = \{ f_\alpha(c) \mid \alpha < \kappa \} \subseteq \langle\!\langle \bar a \rangle\!\rangle_{\mathfrak{I}} = \operatorname{dom}(p) ,$$

and

$$\begin{aligned} p[W(c)] &= \{ p(f_\alpha(c)) \mid \alpha < \kappa,\ f_\alpha(c) \neq c \} \\ &= \{ f_\alpha(p(c)) \mid \alpha < \kappa,\ f_\alpha(p(c)) \neq p(c) \} \\ &= W(p(c)) . \end{aligned}$$

Hence, $\operatorname{dom}(p)$ is W-closed and $p[W(c)] = W(p(c))$.

Finally, for $c \in \operatorname{dom}(p)$, it follows by assumption on the predicates Q_α that

$$c\bar d \equiv p(c\bar d) ,$$

where $\bar d$ is an enumeration of $W(c)$. Hence,

$$p[\operatorname{tp}(c/W(c))] = \operatorname{tp}(p(c)/W(p(c))) . \qquad \square$$

Using these two lemmas we can construct representations for stable theories.

Proposition 6.11. *Let T be a stable theory and set* $\kappa := \min\{\operatorname{st}(T), |T|\}$ *and* $\lambda := \min\{\operatorname{fc}(\mathop{\smile\!\!\!\!\!\mid}^{!}), |T|\}$.

(a) *T has* $\operatorname{Wf}(\kappa, \lambda)$*-representations.*

(b) *If* $\operatorname{fc}(\mathop{\smile\!\!\!\!\!\mid}^{!}) \leq \aleph_0$, *$T$ has* $\operatorname{Lf}(\kappa, \aleph_0)$*-representations.*

Proof. Let $\mathfrak{M}$ be a model of T. By Theorem 5.8, there exists a $\sqrt[\mathrm{i}]{}$-stratification $\zeta = (B_n)_{n<\gamma}$ of M over $\varnothing$ of length $\gamma \leq \mathrm{fc}(\downarrow^!)$. Since

$$\mathrm{fc}(\downarrow^!) \leq \mathrm{fc}(\downarrow^{\mathrm{f}}) \oplus \mathrm{mult}(\downarrow^{\mathrm{f}})^+ \leq \mathrm{st}(T)^+ \quad \text{and} \quad \mathrm{fc}(\downarrow^!) \leq |T|^+,$$

it follows that $\gamma \leq \kappa^+$. Taking a suitable refinement of ζ we may assume by Lemma 5.10 that $|B_0| = |B_1| = 1$. By Lemma 5.12, there exists a system of bases $(W(a))_{a\in M}$ with

$$|W(a)| < \mathrm{loc}_0(\downarrow^!) \leq \mathrm{fc}(\downarrow^!) \leq |T|^+.$$

Consequently, $|W(a)| \leq \lambda$.

We define $\mathfrak{I} := \langle M, (f_\alpha)_{\alpha<\lambda}, (P_\alpha)_{\alpha<\kappa}, (Q_\alpha)_{\alpha<\kappa}\rangle$ as follows. We choose the functions $(f_\alpha)_{\alpha<\lambda}$ such that, for every $a \in M$, $(f_\alpha(a))_{\alpha<\lambda}$ is an enumeration of $\{a\} \cup W(a)$. Fixing an injective function $h : 2^\kappa \to \mathscr{P}(\kappa)$, we define

$$P_\alpha := \bigcup \{ B_\beta \mid \alpha \in h(\beta) \}.$$

(Note that $\gamma \leq \kappa^+ \leq 2^\kappa$.) As there are at most $\mathrm{st}(T)$ many types over the empty set and there are only $|T|$ formulae, we can fix an enumeration $(\varphi_\alpha(x))_{\alpha<\kappa}$ of all formulae (up to logical equivalence) of the form

$$\varphi(x; f_{\beta_0}x, \dots, f_{\beta_{n-1}}x),$$

where φ is a formula over the signature of T, $n < \omega$, and $\beta_0, \dots, \beta_{n-1} < \kappa$. We set

$$Q_\alpha := \{ a \in M \mid \langle \mathfrak{M}, (f_\alpha)_\alpha \rangle \vDash \varphi_\alpha(a) \}.$$

It is straightforward to check that the structure $\mathfrak{I}$ satisfies the conditions of Lemma 6.10. Hence, the identity function $\mathrm{id} : M \to M$ is a representation of $\mathfrak{M}$ in $\mathfrak{I}$.

Finally, note that $\mathfrak{I} \in \mathrm{Wf}(\kappa, \lambda)$ since,

$$a \in B_\alpha \quad \text{implies} \quad f_\beta(a) \in \{a\} \cup B[<\alpha].$$

If $\mathrm{fc}(\downarrow^!) \leq \aleph_0$, the sets $\{ f_\alpha(a) \mid \alpha < \kappa \} = W(a) \cup \{a\}$ are finite, for all $a \in M$. By the Lemma of Kőnig, it follows that $\mathfrak{I} \in \mathrm{Lf}(\kappa, \aleph_0)$. $\square$

The following two theorems summarise the results of this section.

Theorem 6.12 (Cohen, Shelah). *Let T be a complete first-order theory. The following conditions are equivalent:*

(1) *T is stable.*

(2) *T has* $\mathrm{Un}(\kappa, \lambda)$*-representations, for some cardinals κ and λ.*

(3) *T has* $\mathrm{Wf}(0, |T|)$*-representations.*

(4) *T has* $\mathrm{Wf}(|T|, |T|)$*-representations.*

Proof. (2) ⇒ (1) has been shown in Proposition 6.8 (a), the implications (4) ⇒ (3) ⇒ (2) follow from Lemmas 6.5 and 6.2, and (1) ⇒ (4) follows by Proposition 6.11. □

Theorem 6.13 (Cohen, Shelah). *Let T be a complete first-order theory. The following conditions are equivalent:*

(1) *T is $\aleph_0$-stable.*

(2) *T has* $\mathrm{Lf}(\aleph_0, \aleph_0)$*-representations.*

Proof. (2) ⇒ (1) follows by Proposition 6.8 (b) and (1) ⇒ (2) follows by Proposition 6.11. □

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Symbol Index

Chapter A1

$\mathbb{S}$	universe of sets, 5
$a \in b$	membership, 5
$a \subseteq b$	subset, 5
HF	hereditary finite sets, 7
$\bigcap A$	intersection, 11
$A \cap B$	intersection, 11
$A \setminus B$	difference, 11
$\mathrm{acc}(A)$	accumulation, 12
$\mathrm{fnd}(A)$	founded part, 13
$\bigcup A$	union, 21
$A \cup B$	union, 21
$\wp(A)$	power set, 21
$\mathrm{cut}\, A$	cut of A, 22

Chapter A2

$\langle a_0, \dots, a_{n-1} \rangle$	tuple, 27
$A \times B$	cartesian product, 27
$\mathrm{dom}\, f$	domain of f, 28
$\mathrm{rng}\, f$	range of f, 29
$f(a)$	image of a under f, 29
$f : A \to B$	function, 29
B^A	set of all functions $f : A \to B$, 29
id_A	identity function, 30
$S \circ R$	composition of relations, 30
$g \circ f$	composition of functions, 30
R^{-1}	inverse of R, 30
$R^{-1}(a)$	inverse image, 30
$R\|_C$	restriction, 30
$R \upharpoonright C$	left restriction, 31
$R[C]$	image of C, 31
$(a_i)_{i \in I}$	sequence, 37
$\prod_i A_i$	product, 37
pr_i	projection, 37
$\bar{a}$	sequence, 38
$\dot{\bigcup}_i A_i$	disjoint union, 38
$A \dot{\cup} B$	disjoint union, 38
in_i	insertion map, 39
$\mathfrak{A}^{\mathrm{op}}$	opposite order, 40
$\Downarrow X$	initial segment, 41
$\Uparrow X$	final segment, 41
$\downarrow X$	initial segment, 41
$\uparrow X$	final segment, 41
$[a, b]$	closed interval, 41
(a, b)	open interval, 41
$\max X$	greatest element, 42
$\min X$	minimal element, 42
$\sup X$	supremum, 42

$\inf X$	infimum, 42
$\mathfrak{A} \cong \mathfrak{B}$	isomorphism, 44
$\mathrm{fix}\, f$	fixed points, 48
$\mathrm{lfp}\, f$	least fixed point, 48
$\mathrm{gfp}\, f$	greatest fixed point, 48
$[a]_\sim$	equivalence class, 54
$A/{\sim}$	set of $\sim$-classes, 54
$\mathrm{TC}(R)$	transitive closure, 55

Chapter A3

a^+	successor, 59
$\mathrm{ord}(\mathfrak{A})$	order type, 64
On	class of ordinals, 64
On_0	von Neumann ordinals, 69
$\rho(a)$	rank, 73
$A^{<\infty}$	functions $\downarrow\alpha \to A$, 74
$\mathfrak{A} + \mathfrak{B}$	sum, 85
$\mathfrak{A} \cdot \mathfrak{B}$	product, 86
$\mathfrak{A}^{(\mathfrak{B})}$	exponentiation of well-orders, 86
$\alpha + \beta$	ordinal addition, 89
$\alpha \cdot \beta$	ordinal multiplication, 89
$\alpha^{(\beta)}$	ordinal exponentiation, 89

Chapter A4

$\lvert A \rvert$	cardinality, 113
∞	cardinality of proper classes, 113
Cn	class of cardinals, 113
$\aleph_\alpha$	aleph alpha, 115
$\kappa \oplus \lambda$	cardinal addition, 116
$\kappa \otimes \lambda$	cardinal multiplication, 116
κ^λ	cardinal exponentiation, 116
$\sum_i \kappa_i$	cardinal sum, 122
$\prod_i \kappa_i$	cardinal product, 122
$\mathrm{cf}\, \alpha$	cofinality, 124
$\beth_\alpha$	beth alpha, 127
$({<}\kappa)^\lambda$	$\sup_\mu \mu^\lambda$, 128
$\kappa^{<\lambda}$	$\sup_\mu \kappa^\mu$, 128

Chapter B1

$R^{\mathfrak{A}}$	relation of $\mathfrak{A}$, 149
$f^{\mathfrak{A}}$	function of $\mathfrak{A}$, 149
$A^{\bar{s}}$	$A_{s_0} \times \cdots \times A_{s_n}$, 151
$\mathfrak{A} \subseteq \mathfrak{B}$	substructure, 152
$\mathrm{Sub}(\mathfrak{A})$	substructures of $\mathfrak{A}$, 152
$\mathfrak{Sub}(\mathfrak{A})$	substructure lattice, 152
$\mathfrak{A}\vert_X$	induced substructure, 152
$\langle\!\langle X \rangle\!\rangle_{\mathfrak{A}}$	generated substructure, 154
$\mathfrak{A}\vert_\Sigma$	reduct, 155
$\mathfrak{A}\vert_T$	restriction to sorts in T, 155
$\mathfrak{A} \cong \mathfrak{B}$	isomorphism, 156
$\ker f$	kernel of f, 158
$h(\mathfrak{A})$	image of h, 162
$\mathcal{C}^{\mathrm{obj}}$	class of objects, 162
$\mathcal{C}(\mathfrak{a}, \mathfrak{b})$	morphisms $\mathfrak{a} \to \mathfrak{b}$, 162
$g \circ f$	composition of morphisms, 162
$\mathrm{id}_{\mathfrak{a}}$	identity, 163
$\mathcal{C}^{\mathrm{mor}}$	class of morphisms, 163
$\mathfrak{Set}$	category of sets, 163
$\mathfrak{Hom}(\Sigma)$	category of homomorphisms, 163
$\mathfrak{Hom}_s(\Sigma)$	category of strict homomorphisms, 163

$\mathfrak{Emb}(\Sigma)$ category of embeddings, 163
$\mathfrak{Set}_*$ category of pointed sets, 163
$\mathfrak{Set}^2$ category of pairs, 163
$\mathcal{C}^{\mathrm{op}}$ opposite category, 166
F^{op} opposite functor, 168
$(F \downarrow G)$ comma category, 170
$F \cong G$ natural isomorphism, 172
$\mathrm{Cong}(\mathfrak{A})$ set of congruence relations, 176
$\mathfrak{Cong}(\mathfrak{A})$ congruence lattice, 176
$\mathfrak{A}/\sim$ quotient, 180

Chapter B2

$|x|$ length of a sequence, 189
$x \cdot y$ concatenation, 189
$\preceq$ prefix order, 189
$\leq_{\mathrm{lex}}$ lexicographic order, 189
$|v|$ level of a vertex, 192
$\mathrm{frk}(v)$ foundation rank, 194
$a \sqcap b$ infimum, 197
$a \sqcup b$ supremum, 197
a^* complement, 200
$\mathfrak{L}^{\mathrm{op}}$ opposite lattice, 206
$\mathrm{cl}_\downarrow(X)$ ideal generated by X, 206
$\mathrm{cl}_\uparrow(X)$ filter generated by X, 206
$\mathfrak{B}_2$ two-element boolean algebra, 210
$\mathrm{ht}(a)$ height of a, 218
$\mathrm{rk}_{\mathrm{P}}(a)$ partition rank, 222
$\deg_{\mathrm{P}}(a)$ partition degree, 226

Chapter B3

$T[\Sigma, X]$ finite Σ-terms, 231
t_v subterm at v, 232
$\mathrm{free}(t)$ free variables, 235
$t^{\mathfrak{A}}[\beta]$ value of t, 235
$\mathfrak{T}[\Sigma, X]$ term algebra, 236
$t[x/s]$ substitution, 238
$\mathfrak{SigVar}$ category of signatures and variables, 239
$\mathfrak{Sig}$ category of signatures, 240
$\mathfrak{Var}$ category of variables, 240
$\mathfrak{Term}$ category of terms, 240
$\mathfrak{A}|_\mu$ μ-reduct of $\mathfrak{A}$, 241
$\mathrm{Str}[\Sigma]$ class of Σ-structures, 241
$\mathrm{Str}[\Sigma, X]$ class of all Σ-structures with variable assignments, 241
$\mathfrak{StrVar}$ category of structures and assignments, 241
$\mathfrak{Str}$ category of structures, 241
$\prod_i \mathfrak{A}^i$ direct product, 243
$[\![\varphi]\!]$ set of indices, 245
$\bar{a} \sim_{\mathfrak{u}} \bar{b}$ filter equivalence, 245
$\mathfrak{u}|_J$ restriction of $\mathfrak{u}$ to J, 246
$\prod_i \mathfrak{A}^i/\mathfrak{u}$ reduced product, 246
$\mathfrak{A}^{\mathfrak{u}}$ ultrapower, 247
$\varinjlim D$ directed colimit, 255
$\varinjlim D$ colimit of D, 257
$\varprojlim D$ directed limit, 260
$f * \mu$ componentwise composition for cocones, 262
$G[\mu]$ image of a cocone under a functor, 265
$\mathfrak{Z}_n$ partial order of an alternating path, 276

$\mathfrak{Z}_n^{\perp}$ partial order of an alternating path, 276
$f \mathrel{\wedge\!\!\wedge} g$ alternating-path equivalence, 277
$[f]_F^{\wedge\!\wedge}$ alternating-path equivalence class, 277
$s * t$ componentwise composition of links, 280
π_t projection along a link, 281
in_D inclusion link, 281
$D[t]$ image of a link under a functor, 284
$\mathrm{Ind}_{\mathcal{P}}(\mathcal{C})$ inductive $\mathcal{P}$-completion, 285
$\mathrm{Ind}_{\mathrm{all}}(\mathcal{C})$ inductive completion, 285

Chapter B4

$\mathrm{Ind}_\kappa^\lambda(\mathcal{C})$ inductive (κ, λ)-completion, 295
$\mathrm{Ind}(\mathcal{C})$ inductive completion, 296
$\circlearrowleft$ loop category, 317
$\|\mathfrak{a}\|$ cardinality in an accessible category, 333
$\mathfrak{Sub}_{\mathcal{K}}(\mathfrak{a})$ category of $\mathcal{K}$-subobjects, 341
$\mathfrak{Sub}_\kappa(\mathfrak{a})$ category of κ-presentable subobjects, 341

Chapter B5

$\mathrm{cl}(A)$ closure of A, 347
$\mathrm{int}(A)$ interior of A, 347
∂A boundary of A, 347
$\mathrm{rk}_{\mathrm{CB}}(x/A)$ Cantor-Bendixson rank, 369
$\mathrm{spec}(\mathfrak{L})$ spectrum of $\mathfrak{L}$, 374
$\langle x \rangle$ basic closed set, 374
$\mathrm{clop}(\mathfrak{S})$ algebra of clopen subsets, 378

Chapter B6

$\mathfrak{Aut}\,\mathfrak{M}$ automorphism group, 390
G/U set of cosets, 390
$\mathfrak{G}/\mathfrak{N}$ factor group, 392
$\mathfrak{Sym}\,\Omega$ symmetric group, 393
ga action of g on a, 394
$G\bar{a}$ orbit of $\bar{a}$, 394
$\mathfrak{G}_{(X)}$ pointwise stabiliser, 395
$\mathfrak{G}_{\{X\}}$ setwise stabiliser, 395
$\langle \bar{a} \mapsto \bar{b} \rangle$ basic open set of the group topology, 399
$\deg p$ degree, 403
$\mathfrak{Idl}(\mathfrak{R})$ lattice of ideals, 404
$\mathfrak{R}/\mathfrak{a}$ quotient of a ring, 406
$\mathrm{Ker}\, h$ kernel, 406
$\mathrm{spec}(\mathfrak{R})$ spectrum, 406
$\bigoplus_i \mathfrak{M}_i$ direct sum, 409
$\mathfrak{M}^{(I)}$ direct power, 409
$\dim \mathfrak{V}$ dimension, 413
$\mathrm{FF}(\mathfrak{R})$ field of fractions, 415
$\mathfrak{K}(\bar{a})$ subfield generated by $\bar{a}$, 418
$p[x]$ polynomial function, 419
$\mathrm{Aut}\,(\mathfrak{L}/\mathfrak{K})$ automorphisms over K, 427
$|a|$ absolute value, 430

Chapter C1

$\mathrm{ZL}[\mathfrak{K}, X]$ Zariski logic, 447
$\models$ satisfaction relation, 448
$\mathrm{BL}(\mathfrak{B})$ boolean logic, 448
$\mathrm{FO}_{\kappa\aleph_0}[\Sigma, X]$ infinitary first-order logic, 449
$\neg\varphi$ negation, 449
$\bigwedge \Phi$ conjunction, 449
$\bigvee \Phi$ disjunction, 449
$\exists x\varphi$ existential quantifier, 449
$\forall x\varphi$ universal quantifier, 449
$\mathrm{FO}[\Sigma, X]$ first-order logic, 449
$\mathfrak{A} \models \varphi[\beta]$ satisfaction, 450
true true, 451
false false, 451
$\varphi \vee \psi$ disjunction, 451
$\varphi \wedge \psi$ conjunction, 451
$\varphi \rightarrow \psi$ implication, 451
$\varphi \leftrightarrow \psi$ equivalence, 451
$\mathrm{free}(\varphi)$ free variables, 454
$\mathrm{qr}(\varphi)$ quantifier rank, 457
$\mathrm{Mod}_L(\Phi)$ class of models, 458
$\Phi \models \varphi$ entailment, 464
$\equiv$ logical equivalence, 464
$\Phi^{\models}$ closure under entailment, 464
$\mathrm{Th}_L(\mathfrak{I})$ L-theory, 465
$\equiv_L$ L-equivalence, 466
$\mathrm{DNF}(\varphi)$ disjunctive normal form, 471
$\mathrm{CNF}(\varphi)$ conjunctive normal form, 471
$\mathrm{NNF}(\varphi)$ negation normal form, 473
$\mathfrak{Logic}$ category of logics, 482
$\exists^\lambda x\varphi$ cardinality quantifier, 485
$\mathrm{FO}_{\kappa\aleph_0}(\mathrm{wo})$ FO with well-ordering quantifier, 486
W well-ordering quantifier, 486
$\mathsf{Q}_{\mathcal{K}}$ Lindström quantifier, 486
$\mathrm{SO}_{\kappa\aleph_0}[\Sigma, \Xi]$ second-order logic, 487
$\mathrm{MSO}_{\kappa\aleph_0}[\Sigma, \Xi]$ monadic second-order logic, 487
$\mathfrak{PO}$ category of partial orders, 492
$\mathfrak{Lb}$ Lindenbaum functor, 492
$\neg\varphi$ negation, 494
$\varphi \vee \psi$ disjunction, 494
$\varphi \wedge \psi$ conjunction, 494
$L|_\Phi$ restriction to Φ, 495
L/Φ localisation to Φ, 495
$\models_\Phi$ consequence modulo Φ, 495
$\equiv_\Phi$ equivalence modulo Φ, 495

Chapter C2

$\mathfrak{Emb}_L(\Sigma)$ category of L-embeddings, 497
$\mathrm{QF}_{\kappa\aleph_0}[\Sigma, X]$ quantifier-free formulae, 498
$\exists\Delta$ existential closure of Δ, 498
$\forall\Delta$ universal closure of Δ, 498
$\exists_{\kappa\aleph_0}$ existential formulae, 498
$\forall_{\kappa\aleph_0}$ universal formulae, 498
$\exists^+_{\kappa\aleph_0}$ positive existential formulae, 498
$\preceq_\Delta$ Δ-extension, 502
$\preceq$ elementary extension, 502

$\Phi^{\models}_{\Delta}$ Δ-consequences of Φ, 525
$\leq_{\Delta}$ preservation of Δ-formulae, 525

Chapter C3

$S(L)$ set of types, 531
$\langle \Phi \rangle$ types containing Φ, 531
$\mathrm{tp}_L(\bar{a}/\mathfrak{M})$ L-type of $\bar{a}$, 532
$S^{\bar{s}}_L(T)$ type space for a theory, 532
$S^{\bar{s}}_L(U)$ type space over U, 532
$\mathfrak{S}(L)$ type space, 537
$f(\mathfrak{p})$ conjugate of $\mathfrak{p}$, 547
$\mathfrak{S}_{\Delta}(L)$ $\mathfrak{S}(L|_{\Delta})$ with topology induced from $\mathfrak{S}(L)$, 561
$\langle \Phi \rangle_{\Delta}$ closed set in $\mathfrak{S}_{\Delta}(L)$, 561
$\mathfrak{p}|_{\Delta}$ restriction to Δ, 564
$\mathrm{tp}_{\Delta}(\bar{a}/U)$ Δ-type of $\bar{a}$, 564

Chapter C4

$\equiv_{\alpha}$ α-equivalence, 581
$\equiv_{\infty}$ ∞-equivalence, 581
$\mathrm{pIso}_{\kappa}(\mathfrak{A}, \mathfrak{B})$ partial isomorphisms, 582
$\bar{a} \mapsto \bar{b}$ map $a_i \mapsto b_i$, 582
$\varnothing$ the empty function, 582
$I_{\alpha}(\mathfrak{A}, \mathfrak{B})$ back-and-forth system, 583
$I_{\infty}(\mathfrak{A}, \mathfrak{B})$ limit of the system, 585
$\cong_{\alpha}$ α-isomorphic, 585
$\cong_{\infty}$ ∞-isomorphic, 585
$m =_k n$ equality up to k, 587
$\varphi^{\alpha}_{\mathfrak{A},\bar{a}}$ Hintikka formula, 590
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$\cong^{\kappa}_{\mathrm{iso}}$ $\infty\kappa$-isomorphic, 603
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(B) boolean closed, 618
(B_+) positive boolean closed, 618
(C) compactness, 618
(CC) countable compactness, 618
(FOP) finite occurrence property, 618
(KP) Karp property, 618
(LSP) Löwenheim-Skolem property, 618
(REL) closed under relativisations, 618

(SUB) closed under substitutions, 618
(TUP) Tarski union property, 618
$\mathrm{hn}_\kappa(L)$ Hanf number, 622
$\mathrm{ln}_\kappa(L)$ Löwenheim number, 622
$\mathrm{wn}_\kappa(L)$ well-ordering number, 622
$\mathrm{occ}(L)$ occurrence number, 622
$\mathrm{pr}_\Gamma(\mathcal{K})$ Γ-projection, 640
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$L_0 \leq^\kappa_{\mathrm{pc}} L_1$ projective reduction, 641
$\mathrm{RPC}_\kappa(L,\Sigma)$ relativised projective L-classes, 645
$L_0 \leq^\kappa_{\mathrm{rpc}} L_1$ relativised projective reduction, 645
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$\liminf f$ least partial fixed point, 662
$\limsup f$ greatest partial fixed point, 663
f_φ function defined by φ, 668
$\mathrm{FO}_{\kappa\aleph_0}(\mathrm{LFP})$ least fixed-point logic, 668
$\mathrm{FO}_{\kappa\aleph_0}(\mathrm{IFP})$ inflationary fixed-point logic, 669
$\mathrm{FO}_{\kappa\aleph_0}(\mathrm{PFP})$ partial fixed-point logic, 669
$\lhd_\varphi$ stage comparison, 679

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DAG theory of divisible torsion-free abelian groups, 710
ODAG theory of ordered divisible abelian groups, 710
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ACF theory of algebraically closed fields, 714
RCF theory of real closed fields, 715

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$\mathrm{H}\forall_\infty[\Sigma, X]$ infinitary universal Horn formulae, 740
$\mathrm{SH}\forall_\infty[\Sigma, X]$ infinitary universal strict Horn formulae, 740
$\mathrm{HO}[\Sigma, X]$ first-order Horn formulae, 740
$\mathrm{SH}[\Sigma, X]$ first-order strict Horn formulae, 740
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$\mathrm{SH}\forall[\Sigma, X]$ first-order universal strict Horn formulae, 740
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$\mathrm{Prod}(\mathcal{K})$ products, 749
$\mathrm{Sub}(\mathcal{K})$ substructures, 749
$\mathrm{Iso}(\mathcal{K})$ isomorphic copies, 749

$\mathrm{Hom}(\mathcal{K})$ weak homomorphic images, 749
$\mathrm{ERP}(\mathcal{K})$ embeddings into reduced products, 749
$\mathrm{QV}(\mathcal{K})$ quasivariety, 749
$\mathrm{Var}(\mathcal{K})$ variety, 749

Chapter D3

(f, g) open cell between f and g, 761
$[f, g]$ closed cell between f and g, 761
$B(\bar{a}, \bar{b})$ box, 762
$\mathrm{Cn}(D)$ continuous functions, 776
$\dim C$ dimension, 777

Chapter E2

$\mathrm{dcl}_L(U)$ L-definitional closure, 819
$\mathrm{acl}_L(U)$ L-algebraic closure, 819
$\mathrm{dcl}_{\mathrm{Aut}}(U)$ Aut-definitional closure, 821
$\mathrm{acl}_{\mathrm{Aut}}(U)$ Aut-algebraic closure, 821
$\mathbb{M}$ the monster model, 829
$A \equiv_U B$ having the same type over U, 830
$\mathfrak{M}^{\mathrm{eq}}$ extension by imaginary elements, 831
$\mathrm{dcl}^{\mathrm{eq}}(U)$ definable closure in $\mathfrak{M}^{\mathrm{eq}}$, 831
$\mathrm{acl}^{\mathrm{eq}}(U)$ algebraic closure in $\mathfrak{M}^{\mathrm{eq}}$, 831
T^{eq} theory of $\mathbb{M}^{\mathrm{eq}}$, 833
$\mathrm{Gb}(\mathfrak{p})$ Galois base, 841

Chapter E3

$I_{\mathrm{cl}}(\mathfrak{A}, \mathfrak{B})$ elementary maps with closed domain and range, 877

Chapter E4

$\mathfrak{p}\mathfrak{Mor}_{\mathcal{K}}(\mathfrak{a}, \mathfrak{b})$ category of partial morphisms, 898
$\mathfrak{a} \sqsubseteq_{\mathcal{K}} \mathfrak{b}$ forth property for objects in $\mathcal{K}$, 899
$\mathfrak{a} \sqsubseteq^{\kappa}_{\mathrm{pres}} \mathfrak{b}$ forth property for κ-presentable objects, 899
$\mathfrak{a} \equiv^{\kappa}_{\mathrm{pres}} \mathfrak{b}$ back-and-forth equivalence for κ-presentable objects, 899
$\mathrm{Sub}_{\kappa}(\mathfrak{a})$ κ-presentable subobjects, 910
$\mathrm{atp}(\bar{a})$ atomic type, 922
$\eta_{\mathfrak{pq}}$ extension axiom, 922
$T[\mathcal{K}]$ extension axioms for $\mathcal{K}$, 922
$T_{\mathrm{ran}}[\Sigma]$ random theory, 922
$\kappa_n(\varphi)$ number of models, 924
$\mathrm{Pr}^n_{\mathfrak{M}}[\mathfrak{M} \vDash \varphi]$ density of models, 924

Chapter E5

$[I]^\kappa$ increasing κ-tuples, 929
$\kappa \to (\mu)^\nu_\lambda$ partition theorem, 929
$\mathrm{pf}(\eta, \zeta)$ prefix of ζ of length $|\eta|$, 934
$\mathfrak{T}_*(\kappa^{<\alpha})$ index tree with small signature, 934
$\mathfrak{T}_n(\kappa^{<\alpha})$ index tree with large signature, 934
$\langle\!\langle X \rangle\!\rangle_n$ substructure generated in $\mathfrak{T}_n(\kappa^{<\alpha})$, 934
$\mathrm{Lvl}(\bar{\eta})$ levels of $\bar{\eta}$, 935
$\approx_*$ equal atomic types in $\mathfrak{T}_*$, 936
$\approx_n$ equal atomic types in $\mathfrak{T}_n$, 936
$\approx_{n,k}$ refinement of $\approx_n$, 936
$\approx_{\omega,k}$ union of $\approx_{n,k}$, 936
$\bar{a}[\bar{\imath}]$ $\bar{a}^{i_0} \dots \bar{a}^{i_{n-1}}$, 945
$\mathrm{tp}_\Delta(\bar{a}/U)$ Δ-type, 945
$\mathrm{Av}((\bar{a}^i)_i/U)$ average type, 947
$[\![\varphi(\bar{a}^i)]\!]$ indices satisfying φ, 956
$\mathrm{Av}_1((\bar{a}^i)_i/C)$ unary average type, 966

Chapter E6

$\mathfrak{Emb}(\mathcal{K})$ embeddings between structures in $\mathcal{K}$, 969
p^F image of a partial isomorphism under F, 972
$\mathrm{Th}_L(F)$ theory of a functor, 975
$\mathfrak{A}^\alpha$ inverse reduct, 979
$\mathcal{R}(\mathfrak{M})$ relational variant of $\mathfrak{M}$, 981
$\mathrm{Av}(F)$ average type, 990

Chapter E7

$\ln(\mathcal{K})$ Löwenheim number, 999
$\mathfrak{A} \leq_{\mathcal{K}} \mathfrak{B}$ $\mathcal{K}$-substructure, 1000
$\mathrm{hn}(\mathcal{K})$ Hanf number, 1007
$\mathcal{K}_\kappa$ structures of size κ, 1008
$I^\kappa_{\mathcal{K}}(\mathfrak{A}, \mathfrak{B})$ $\mathcal{K}$-embeddings, 1012
$\mathfrak{A} \sqsubseteq^\kappa_{\mathcal{K}} \mathfrak{B}$ $I^\kappa_{\mathcal{K}}(\mathfrak{A}, \mathfrak{B}) : \mathfrak{A} \sqsubseteq^\kappa_{\mathrm{iso}} \mathfrak{B}$, 1012
$\mathfrak{A} \equiv^\kappa_{\mathcal{K}} \mathfrak{B}$ $I^\kappa_{\mathcal{K}}(\mathfrak{A}, \mathfrak{B}) : \mathfrak{A} \equiv^\kappa_{\mathrm{iso}} \mathfrak{B}$, 1012

Chapter F1

$\langle\!\langle X \rangle\!\rangle_D$ span of X, 1035
$\dim_{\mathrm{cl}}(X)$ dimension, 1041
$\dim_{\mathrm{cl}}(X/U)$ dimension over U, 1041

Chapter F2

$\mathrm{rk}_\Delta(\varphi)$ Δ-rank, 1077
$\mathrm{rk}^{\bar{s}}_{\mathrm{M}}(\varphi)$ Morley rank, 1077
$\deg^{\bar{s}}_{\mathrm{M}}(\varphi)$ Morley degree of φ, 1080
(MON) Monotonicity, 1088
(NOR) Normality, 1088
(LRF) Left Reflexivity, 1088
(LTR) Left Transitivity, 1088
(FIN) Finite Character, 1089
(SYM) Symmetry, 1089
(BMON) Base Monotonicity, 1089
(SRB) Strong Right Boundedness, 1089

$\mathrm{cl}_{\surd}$ closure operation associated with $\surd$, 1094

(INV) Invariance, 1101

(DEF) Definability, 1101

(EXT) Extension, 1101

$A \overset{\mathrm{df}}{\surd}_U B$ definable over, 1102

$A \overset{\mathrm{at}}{\surd}_U B$ isolated over, 1102

$A \overset{\mathrm{s}}{\surd}_U B$ non-splitting over, 1102

$\mathfrak{p} \leq_{\surd} \mathfrak{q}$ $\surd$-free extension, 1107

$A \overset{\mathrm{u}}{\surd}_U B$ finitely satisfiable, 1108

$\mathrm{Av}(\mathfrak{u}/B)$ average type of $\mathfrak{u}$, 1109

(LLOC) Left Locality, 1113

(RLOC) Right Locality, 1113

$\mathrm{loc}(\surd)$ right locality cardinal of $\surd$, 1113

$\mathrm{loc}_0(\surd)$ finitary right locality cardinal of $\surd$, 1113

κ^{reg} regular cardinal above κ, 1114

$\mathrm{fc}(\surd)$ length of $\surd$-forking chains, 1115

(SFIN) Strong Finite Character, 1115

$\overset{*}{\surd}$ forking relation to $\surd$, 1117

Chapter F3

$A \overset{\mathrm{d}}{\surd}_U B$ non-dividing, 1129

$A \overset{\mathrm{f}}{\surd}_U B$ non-forking, 1129

$A \overset{\mathrm{i}}{\surd}_U B$ globally invariant over, 1138

Chapter F4

$\mathrm{alt}_\varphi(\bar{a}_i)_{i\in I}$ φ-alternation number, 1157

$\mathrm{rk}_{\mathrm{alt}}(\varphi)$ alternation rank, 1157

$\mathrm{in}(\sim)$ intersection number, 1168

$\bar{a} \approx^{\mathrm{ls}}_U \bar{b}$ indiscernible sequence starting with $\bar{a}, \bar{b}, \dots$, 1172

$\bar{a} \equiv^{\mathrm{ls}}_U \bar{b}$ Lascar strong type equivalence, 1172

$\mathrm{CF}((\bar{a}_i)_{i\in I})$ cofinal type, 1198

$\mathrm{Ev}((\bar{a}_i)_{i\in I})$ eventual type, 1203

$\mathrm{rk}_{\mathrm{dp}}(\bar{a}/U)$ dp-rank, 1216

Chapter F5

(LEXT) Left Extension, 1232

$A \overset{\mathrm{fli}}{\surd}_U B$ combination of $\overset{\mathrm{li}}{\surd}$ and $\overset{\mathrm{f}}{\surd}$, 1243

$A \overset{\mathrm{sli}}{\surd}_U B$ strict Lascar invariance, 1243

(WIND) Weak Independence Theorem, 1256

(IND) Independence Theorem, 1257

Chapter G1

$\bar{a} \downarrow^{!}_U B$ unique free extension, 1278

$\mathrm{mult}_{\surd}(\mathfrak{p})$ $\surd$-multiplicity of $\mathfrak{p}$, 1283

$\mathrm{mult}(\surd)$ multiplicity of $\surd$, 1283

$\mathrm{st}(T)$ minimal cardinal T is stable in, 1294

Chapter G2

(RSH)	Right Shift, 1301
$\mathrm{lbm}(\surd)$	left base-monotonicity cardinal, 1301
$A[I]$	$\bigcup_{i\in I} A_i$, 1310
$A[<\alpha]$	$\bigcup_{i<\alpha} A_i$, 1310
$A[\leq\alpha]$	$\bigcup_{i\leq\alpha} A_i$, 1310
$A \perp^{\mathrm{do}}_U B$	definable orthogonality, 1333
$A \overset{\mathrm{si}}{\surd}_U B$	strong independence, 1336
$\Upsilon_{\kappa\lambda}$	unary signature, 1342
$\mathrm{Un}(\kappa, \lambda)$	class of unary structures, 1342
$\mathrm{Lf}(\kappa, \lambda)$	class of locally finite unary structures, 1342

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The Roman and Fraktur alphabets							
A	*a*	𝔄	𝔞	*N*	*n*	𝔑	𝔫
B	*b*	𝔅	𝔟	*O*	*o*	𝔒	𝔬
C	*c*	ℭ	𝔠	*P*	*p*	𝔓	𝔭
D	*d*	𝔇	𝔡	*Q*	*q*	𝔔	𝔮
E	*e*	𝔈	𝔢	*R*	*r*	ℜ	𝔯
F	*f*	𝔉	𝔣	*S*	*s*	𝔖	ſ 𝔰
G	*g*	𝔊	𝔤	*T*	*t*	𝔗	𝔱
H	*h*	ℌ	𝔥	*U*	*u*	𝔘	𝔲
I	*i*	ℑ	𝔦	*V*	*v*	𝔙	𝔳
J	*j*	𝔍	𝔧	*W*	*w*	𝔚	𝔴
K	*k*	𝔎	𝔨	*X*	*x*	𝔛	𝔵
L	*l*	𝔏	𝔩	*Y*	*y*	𝔜	𝔶
M	*m*	𝔐	𝔪	*Z*	*z*	ℨ	𝔷

The Greek alphabet					
Α	*α*	alpha	*Ν*	*ν*	nu
Β	*β*	beta	*Ξ*	*ξ*	xi
Γ	*γ*	gamma	*Ο*	*ο*	omicron
Δ	*δ*	delta	*Π*	*π*	pi
Ε	*ε*	epsilon	*Ρ*	*ρ*	rho
Ζ	*ζ*	zeta	*Σ*	*σ*	sigma
Η	*η*	eta	*Τ*	*τ*	tau
Θ	*ϑ*	theta	*ϒ*	*υ*	upsilon
Ι	*ι*	iota	*Φ*	*ϕ*	phi
Κ	*κ*	kappa	*Χ*	*χ*	chi
Λ	*λ*	lambda	*Ψ*	*ψ*	psi
Μ	*μ*	mu	*Ω*	*ω*	omega