Simple Monadic Theories

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1 Introduction

1.1 Overview

Over the last decades the beginnings of a model theory for monadic second-order logic have emerged. After seminal papers by Büchi [11], Läuchli [35], Rabin [38], and Shelah [43] a thorough investigation of the monadic theory of linear orders was performed by Gurevich and Shelah [32, 33]. General monadic theories and their model theory were studied by Baldwin and Shelah in [1, 45, 46].

A second development advancing the model theory for monadic second-order logic consists in the work on graph grammars initiated by Courcelle. The main subject of this line of work is the study of graph operations that are compatible with monadic second-order theories [19, 21, 23, 27, 36] (see [8] for an overview). Noteworthy recent developments include the *Muchnik iteration* [42, 49, 2, 12, 10] and *set interpretations* [16]. Such operations give rise to graph algebras and the corresponding notions of recognisable sets and equational sets [30, 9]. Furthermore, one can use these operations to define hierarchical decompositions of graphs and the corresponding complexity measures, like tree width, clique width, and partition width [26, 20, 28, 5, 25, 7]. Finally, operations can also be used to construct finite presentations of infinite graphs via regular terms [17, 18, 14, 15, 13, 6, 4].

There is a variant of monadic second-order logic called *guarded second-order logic* [31] which also prominently figures in this line of work. The expressive power of this logic in comparison to monadic second-order logic is investigated in [22, 24, 3].

As monadic second-order logic is more expressive than first-order logic, it is unsurprising that most structures possess an extremely complicated monadic second-order theory. Fortunately, there remain structures where the theory is simple enough for the existence of a structure theory.

The prime example of such a structure is the infinite binary tree which, according to Rabin's theorem, has a decidable monadic theory. Starting from this result we can obtain further structures with a manageable theory by applying operations that preserve decidability of the MSO-theory, like monadic second-order interpretations or disjoint unions. We can also consider other trees than the complete binary one. Although their monadic theories can become highly undecidable there still exists a structure theory for structures interpretable in them (see [7, 5]).

On the other extreme there are structures in which one can define arbitrarily large grids or pairing functions. Their monadic theories are very complex since they can encode arithmetic or even full second-order logic. In particular, there is no hope for a structure theory for such structures.

According to a conjecture of Seese [41] these cases form a dichotomy: either a structure is interpretable in some tree or we can define arbitrarily large grids. For graphs (or structures with relations of arity at most 2) a variant of this conjecture has recently been proved by Courcelle and Oum [29]. But the general case of arbitrary structures is still open.

In the present thesis we consider structures on both sides of this dividing line. In Chapter 2 we will study structures with *first-order* definable pairing functions and their *first-order* model theory. This class of structures can be regarded as an upper approximation of the class in Seese's conjecture. Using tools from first-order model theory we prove that that every structure where there is no such pairing function is tree-like (in a very loose sense defined below). The material in this chapter can be seen as a continuation of the work of Baldwin and Shelah [1, 45, 46].

In Chapter 3 we turn our attention to the structures in the Caucal hierarchy, which can be regarded as an lower approximation of the class in Seese's conjecture. Each structure in this hierarchy has finite partition width and a decidable monadic second-order theory. For graphs there also exists a characterisation in automata-theoretic terms: a graph belongs to the n-th level of the Caucal hierarchy if and only if it can be obtained by contracting ε-transitions from the configuration graph of some higher-order pushdown automaton of level n.

In order to better understand the structure of these graphs we therefore investigate such configuration graphs. Our focus will be on the outdegree of vertices and on the length of paths. We provide operations to decompose and reassemble paths. As a technical tool we derive a pumping lemma for higher-order pushdown automata. These results are taken from [4].

Chapter 2 is organised as follows. We start in Section 2.1 with technical results about indiscernible sequences. In Sections 2.2 and 2.4 we collect properties of structures without definable pairing functions. We study indiscernible sequences in such structures and we show that they are well-behaved. Section 2.5 contains an overview over the notion of finite satisfiability (without stability assumption). In Section 2.6 we finally show that every structure without definable pairing functions has bounded partition width and, hence, is tree-like.

In Sections 3.1 and 3.2 we give basic definitions concerning the Caucal hierarchy and we introduce higher-order pushdown automata. Section 3.3 contains a first result on the structure of graphs in the Caucal hierarchy. We compute a bound on the outdegree of vertices. As an application we show that certain graphs do not belong to a given level of the hierarchy.

The remainder of Chapter 3 contains a detailed study of configuration graphs of higher-order pushdown automata. In Section 3.4 we show how to replace, in all configurations of a given run, the bottom of the stack by another stack content without destroying the property of being a run. Usually this substitution operation can be applied only to parts of a run. Therefore, we introduce in Sections 3.5 and 3.6 two partial orders on runs, the so-called *weak* and *strong domination orders,* that will be used to decompose a given run into such parts. Section 3.7 contains a more detailed investigation of the strong domination order and a proof that it contains arbitrary long chains. Finally, we prove the pumping lemma in Section 3.8.

1.2 Partition width

Let us recall some basic definitions and fix our notation. We write $[n]$ for the set $\{0, \dots, n-1\}$. We tacitly identify tuples $\bar{a} = a_0 \dots a_{n-1} \in A^n$ with functions $[n] \to A$ and frequently we do not distinguish between a tuple $\bar{a}$ and the set $\{a_0, \dots, a_{n-1}\}$ of its components. This allows us to write $\bar{a} \subseteq A$ or $\bar{a}|_I$ for $I \subseteq [n]$. We use the words 'tuple' and 'sequence' synonymously. In particular, tuples may be infinite.

For a set A and an ordinal α, we denote by $A^{<\alpha}$ the set of all sequences of length less than α consisting of elements of A. The *prefix ordering* on $A^{<\alpha}$ is defined by

$$x \leq y \quad :\text{iff} \quad y = xz \text{ for some } z\,.$$

The empty sequence is $\langle\rangle$. The *length* of a sequence $x \in A^{<\alpha}$ is denoted by $|x|$. An (*unlabelled*) *tree* is a partial order of the form $\langle T, \leq\rangle$ where T is a prefix-closed subset of $A^{<\alpha}$ and $\leq$ is the prefix ordering. A Λ-*labelled tree* is a function $t : T \to \Lambda$ where the domain $\operatorname{dom}(t) := T \subseteq A^{<\alpha}$ forms an unlabelled tree.

We start by defining what we consider as 'tree-like'. In the literature several notions have been proposed that measure how much a structure resembles a tree. The most prominent one is *tree width,* which was first introduced by Halin [34] and which plays an important role in the proof of the Graph Minor Theorem by Robertson and Seymour [39]. This measure is closely related to guarded second-order logic. For studying monadic second-order logic more appropriate complexity measures are *clique width,* introduced by Courcelle, Engelfriet, and Rozenberg in [26], and its variant *rank width,* defined by Oum and Seymour [37]. These measures have only been defined for graphs, but there are generalisations of clique width to arbitrary structures. The notion we will use is *partition width* introduced in [7, 5]. Correspondingly we consider a structure to be tree-like if it admits a hierarchical decomposition of the following kind.

Definition 1.2.1. A *partition refinement* of a structure $\mathfrak{M}$ is a system $(U_v)_{v\in T}$ of subsets $U_v \subseteq M$ indexed by a tree $T \subseteq 2^{<\alpha}$ with the following properties:

- $U_{\langle\rangle} = M$,
- for every element $a \in M$, there exists a vertex $v \in T$ with $U_v = \{a\}$,
- $U_v = U_{v0} \uplus U_{v1}$, for all $v \in T$ (where we set $U_w := \varnothing$, for $w \notin T$),
- $U_v = \bigcap_{u<v} U_u$ if $|v|$ is a limit ordinal.

Example. (a) A natural partition refinement for a linear order $\langle A, <\rangle$ consists of a recursive division into intervals.

(b) For a tree $\langle 2^{<\alpha}, \preceq\rangle$, we can take as components all sets of the form $U_v := \{\, x \in 2^{<\alpha} \mid v \preceq x \,\}$ and all singletons.

Clearly, every structure has partition refinements. In order to define when a structure is tree-like we introduce a complexity measure for partition refinements based on the number of types realised in each component.

Definition 1.2.2. (a) The *atomic type* of a tuple $\bar{a}$ over a set U is

$$\operatorname{atp}(\bar{a}/U) := \big\{\, \varphi(\bar{x}, \bar{c}) \bigm| \bar{c} \subseteq U\,,\ \varphi \text{ a literal with } \mathfrak{M} \vDash \varphi(\bar{a}, \bar{c}) \,\big\}\,.$$

For a set Δ of formulae, we denote the *Δ-type* of $\bar{a}$ over U by $\operatorname{tp}_\Delta(\bar{a}/U)$. Furthermore, we define its *external type* by

$$\operatorname{etp}(\bar{a}/U) := \operatorname{atp}(\bar{a}/U) \smallsetminus \operatorname{atp}(\bar{a})\,.$$

(b) For a set Δ of formula we define the *n-ary Δ-type index* of a set A over U by

$$\operatorname{ti}^n_\Delta(A/U) := |A^n/{\approx_U}|\,,$$

where $\approx_U$ is the equivalence relation

$$\bar{a} \approx_U \bar{b} \quad :\text{iff} \quad \operatorname{tp}_\Delta(\bar{a}/U) = \operatorname{tp}_\Delta(\bar{b}/U)\,.$$

If Δ is the set of all quantifier-free formulae then we write $\operatorname{ati}^n(A/U)$ instead of $\operatorname{ti}^n_\Delta(A/U)$.

Similarly, we define the *external type index* of A over U by

$$\operatorname{eti}^n(A/U) := |A^n/{\simeq_U}|\,,$$

where

$$\bar{a} \simeq_U \bar{b} \quad :\text{iff} \quad \operatorname{etp}(\bar{a}/U) = \operatorname{etp}(\bar{b}/U)\,.$$

Definition 1.2.3. (a) Let $(U_v)_{v \in T}$ be a partition refinement of $\mathfrak{M}$. The *n-ary partition width* of $(U_v)_v$ is

$$\operatorname{pwd}_n (U_v)_{v \in T} := \sup_{v \in T} \operatorname{eti}^n (U_v / M \setminus U_v) .$$

(b) For an infinite cardinal κ we write $\operatorname{pwd} \mathfrak{M} < \kappa$ if there exists a partition refinement $(U_v)_v$ of $\mathfrak{M}$ with $\operatorname{pwd}_n (U_v)_v < \kappa$, for all $n < \omega$. If $\operatorname{pwd} \mathfrak{M} \nless \kappa$ we write $\operatorname{pwd} \mathfrak{M} \geq \kappa$. We say that $\mathfrak{M}$ has *finite partition width* if $\operatorname{pwd} \mathfrak{M} < \aleph_0$.

We will consider a structure to be tree-like if it has finite partition width.

Example. The partition refinements for linear orders and trees given in the above example have n-ary partition width 1, for every n. Hence, linear orders and trees are tree-like. Grids are a prime example of structures that are not tree-like. We will show in Lemma 2.2.4 below that every grid has a large partition width.

We can transfer bounds on the partition width from a structure $\mathfrak{M}$ to its substructures since each partition refinement of $\mathfrak{M}$ induces partition refinements of the substructures of $\mathfrak{M}$ whose width does not increase.

Lemma 1.2.4. *If* $\mathfrak{M} \subseteq \mathfrak{N}$ *and* $\operatorname{pwd} \mathfrak{N} < \kappa$ *then* $\operatorname{pwd} \mathfrak{M} < \kappa$.

Another important class of operations that, as we will show next, preserve finiteness of partition width are MSO-interpretations.

Definition 1.2.5. Let Σ and Γ be signatures. A *monadic second-order interpretation* from Σ to Γ is a sequence

$$\mathcal{I} = \langle \delta(x), (\varphi_R(\bar{x}))_{R \in \Gamma} \rangle$$

of monadic second-order formulae. Such an interpretation induces an operation mapping a Σ-structure $\mathfrak{A}$ to the Γ-structure

$$\mathcal{I}(\mathfrak{A}) := \langle \delta^{\mathfrak{A}}, (\varphi_R)^{\mathfrak{A}}_R \rangle ,$$

where the universe consists of all elements of $\mathfrak{A}$ satisfying δ and the relations are those defined by the formulae φ_R. Associated with every interpretation $\mathcal{I}$ is a *coordinate map* $\delta^{\mathfrak{A}} \to \mathcal{I}(\mathfrak{A})$ (also denoted by $\mathcal{I}$) mapping elements in $\mathfrak{A}$ to the element in $\mathcal{I}(\mathfrak{A})$ they denote.

The main property of interpretations is their compatibility with monadic second-order theories.

Lemma 1.2.6. *Let* $\mathcal{I}$ *be an* MSO-*interpretation from* Σ *to* Γ. *For every formula* $\varphi(\bar{x}) \in \mathrm{MSO}[\Gamma]$, *there exists a formula* $\varphi^{\mathcal{I}}(\bar{x}) \in \mathrm{MSO}[\Sigma]$ *such that*

$$\mathcal{I}(\mathfrak{A}) \vDash \varphi(\bar{a}) \quad \text{iff} \quad \mathfrak{A} \vDash \varphi^{\mathcal{I}}(\bar{a}) ,$$

for every Σ-*structure* $\mathfrak{A}$ *and all parameters* $\bar{a} \subseteq \delta^{\mathfrak{A}}$.

Proposition 1.2.7. *Let Σ and Γ be finite signatures and let $\mathcal{I}$ be an* MSO*-interpretation from Σ to Γ. There exist a strictly increasing function f on cardinals with $f(\aleph_0) = \aleph_0$ such that*

$$\operatorname{pwd}\mathfrak{M} < \kappa \quad \textit{implies} \quad \operatorname{pwd}\mathcal{I}(\mathfrak{M}) < f(\kappa)\,.$$

Proof. We need a variant of the external type index for monadic formulae. Let Δ be the set of all formulae of the form

$$X \subseteq Y \quad \text{and} \quad R\bar{Z}\,,$$

where X, Y, and Z_i are set variables and a R relation symbol. We define that a formula of the form $R\bar{X}$ holds in a structure if there are elements $a_i \in X_i$ such that $\bar{a} \in R$. We set

$$\begin{aligned} \operatorname{mtp}(\bar{A}/U) &:= \{\, \varphi(\bar{X},\bar{C}) \mid \varphi \in \Delta,\ \bar{C} \subseteq \wp(U),\ \mathfrak{M} \models \varphi(\bar{A},\bar{C}) \,\}\,, \\ \operatorname{emtp}(\bar{A}/U) &:= \operatorname{mtp}(\bar{A}/U) \setminus \operatorname{mtp}(\bar{A})\,, \end{aligned}$$

and $\quad \operatorname{emti}^n(X/U) := |\wp(X)^n/ \simeq_U|\,,$

where

$$\bar{A} \simeq_U \bar{B} \quad :\text{iff} \quad \operatorname{emtp}(\bar{A}/U) = \operatorname{emtp}(\bar{B}/U)\,.$$

Fix a partition refinement $(U_v)_v$ of $\mathfrak{M}$ with $\operatorname{pwd}_n(U_v)_v < \kappa$. $(U_v)_v$ induces a partition refinement $(U'_v)_v$ of $\mathcal{I}(\mathfrak{M})$ where $U'_v := \mathcal{I}(U_v)$. Let k be the quantifier rank of $\mathcal{I}$. We will compute a strictly increasing function f such that

$$\operatorname{pwd}_n(U'_v)_v \leq f\big(\operatorname{pwd}_{n+k}(U_v)_v\big) < f(\kappa)\,.$$

We have shown in [7, 5] that, for all partitions $X \cup U = M$ of the universe M, we have

$$\operatorname{eti}^n(\mathcal{I}(X)/\mathcal{I}(U)) \leq \beth_k(2^\lambda \cdot \operatorname{emti}^{n+k}(X/U))\,,$$

where λ is the number of formulae in Δ with variables $X_0, \dots, X_{n+k-1}$. (The type index on the left is computed in the structure $\mathcal{I}(\mathfrak{M})$, the one on the right in $\mathfrak{M}$.) To prove the claim it is therefore sufficient to show that

$$\operatorname{emti}^n(X/U) \leq 2^{n^2 \cdot \operatorname{eti}^n(X/U)}.$$

For $\bar{A} \in \wp(X)^n$, we set

$$e(\bar{A}/U) := \{\, \operatorname{etp}(\bar{a}/U) \mid a_i \in A_i \,\}\,.$$

The above bound follows from the following claim.

Claim. *Let X and U be disjoint sets. If $\bar{A}, \bar{B} \in \wp(X)^n$ are sets such that*

(1) $A_i \subseteq A_k$ *iff* $B_i \subseteq B_k$*, for all* $i, k < n$*,*

(2) $e(\bar{A}/U) = e(\bar{B}/U)$*,*

then we have $\mathrm{emtp}(\bar{A}/U) = \mathrm{emtp}(\bar{B}/U)$.

For a contradiction, suppose that there is some formula $\varphi(\bar{X}, \bar{Y}) \in \Delta$ where at least one variable Y_i really occurs and some parameters $\bar{C} \subseteq \wp(U)$ such that

$$\mathfrak{M} \models \varphi(\bar{A}, \bar{C}) \leftrightarrow \neg\varphi(\bar{B}, \bar{C}) .$$

By (1) and symmetry, we may assume that $\varphi := R\bar{X}\bar{Y}$ and that

$$\mathfrak{M} \models R\bar{A}\bar{C} \wedge \neg R\bar{B}\bar{C} .$$

Select elements $a_i \in A_i$ and $c_i \in C_i$ such that $\langle \bar{a}, \bar{c} \rangle \in R$, and set $\Phi := \mathrm{etp}(\bar{a}/U)$. By assumption, we have $\langle \bar{b}, \bar{c} \rangle \notin R$, for all $b_i \in B_i$. Hence, $R\bar{x}\bar{c} \in \Phi$ implies that $\Phi \in e(\bar{A}/U) \setminus e(\bar{B}/U)$. Contradiction. ☐

In particular, we obtain the following result for structures of finite partition width.

Corollary 1.2.8. *Let Σ and Γ be finite signatures and let $\mathcal{I}$ be an* MSO*-interpretation from Σ to Γ. If $\mathfrak{M}$ is a Σ-structure with finite partition width then $\mathcal{I}(\mathfrak{M})$ also has finite partition width.*

Since trees have finite partition width it follows that so does every structure interpretable in a tree. In fact, one can show that the converse holds as well.

Theorem 1.2.9 ([7, 5])**.** *Let $\mathfrak{M}$ be a structure with finite signature. $\mathfrak{M}$ has finite partition width if and only if there exist an ordinal α, a set $P \subseteq 2^{<\alpha}$, and a monadic second-order interpretation $\mathcal{I}$ with*

$$\mathfrak{M} \cong \mathcal{I}\langle 2^{<\alpha}, \preceq, P \rangle .$$

The class of structures of finite partition width admits a nice first-order model theory. This is largely due to the fact that there is a related complexity measure (the *non-standard partition width*) which is finite if and only if the partition width is finite and which furthermore is pseudo-elementary (see [7, 5] for definitions and proofs). In particular, we have the following results.

Proposition 1.2.10. *If $\mathfrak{M} \equiv_{\mathrm{FO}} \mathfrak{N}$ and $\mathfrak{M}$ has finite partition width then so does $\mathfrak{N}$.*

Theorem 1.2.11. *A structure $\mathfrak{M}$ has finite partition width if and only if there exists a sequence $\bar{w} \in \omega^\omega$ such that every finite substructure $\mathfrak{M}_0 \subseteq \mathfrak{M}$ has a partition refinement $(U_v)_v$ with $\operatorname{pwd}_n(U_v)_v \leq w_n$, for $n < \omega$.*

Theorem 1.2.12. *A set Φ of first-order formulae has a model with finite partition width if and only if there exists a sequence $\bar{w} \in \omega^\omega$ such that every finite subset $\Phi_0 \subseteq \Phi$ has a model $\mathfrak{M}$ with a partition refinement $(U_v)_v$ such that $\operatorname{pwd}_n(U_v)_v \leq w_n$, for all n.*

We conclude this section with a simple technical result which will be used in Section 2.6.

Lemma 1.2.13. *Let $\kappa := \operatorname{ti}^n_\Delta(A/U)$. There exists a set $U_0 \subseteq U$ of size $|U_0| \leq \kappa + \aleph_0$ such that, for all $\bar{a}, \bar{b} \in A^n$,*

$$\operatorname{tp}_\Delta(\bar{a}/U_0) = \operatorname{tp}_\Delta(\bar{b}/U_0) \quad \textit{implies} \quad \operatorname{tp}_\Delta(\bar{a}/U) = \operatorname{tp}_\Delta(\bar{b}/U)\,.$$

Proof. Fix a sequence $(\bar{a}^\alpha)_{\alpha<\kappa}$ of tuples $\bar{a}^\alpha \in A^n$ such that, for every $\bar{b} \in A^n$, there exists a unique index α with

$$\operatorname{tp}_\Delta(\bar{a}^\alpha/U) = \operatorname{tp}_\Delta(\bar{b}/U)\,.$$

By induction on α, we will define finite sets $C_\alpha \subseteq U$ such that, for all $\beta < \alpha$,

$$\operatorname{tp}_\Delta(\bar{a}^\alpha/C_{<\alpha}) \neq \operatorname{tp}_\Delta(\bar{a}^\beta/C_{<\alpha})\,,$$

where $C_{<\alpha} := \bigcup_{i<\alpha} C_i$. Then the set $U_0 := C_{<\kappa}$ has the desired properties.

To define C_α we consider two cases. If there is no index $\beta < \alpha$ with

$$\operatorname{tp}_\Delta(\bar{a}^\alpha/C_{<\alpha}) = \operatorname{tp}_\Delta(\bar{a}^\beta/C_{<\alpha})$$

then we can simply set $C_\alpha := \varnothing$. Otherwise, there is exactly one such index β. Since

$$\operatorname{tp}_\Delta(\bar{a}^\alpha/U) \neq \operatorname{tp}_\Delta(\bar{a}^\beta/U)$$

there are some formula $\varphi(\bar{x}, \bar{y}) \in \Delta$ and parameters $\bar{c} \subseteq U$ with

$$\mathfrak{M} \models \varphi(\bar{a}, \bar{c}) \leftrightarrow \neg\varphi(\bar{b}, \bar{c})\,.$$

We set $C_\alpha := \bar{c}$. ☐

2 Coding and indiscernibles

2.1 Dependent sequences

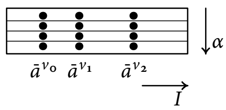
In this section we consider an indiscernible sequence $(\bar{a}^v)_{v\in I}$ and we try to find a formula $\chi(\bar{x})$ which defines the relation $\{\, \bar{a}^v \mid v \in I \,\}$. Of course, in general this is not possible. But if we allow monadic parameters there is a partial solution to this question. The combinatorial techniques used by the following lemmas are based on results by Shelah [46]. Let us start by fixing some notation for sequences.

Definition 2.1.1. Let $(\bar{a}^v)_{v\in I}$ be a sequence of α-tuples indexed by a linear order I.

(a) We denote the *order type* of $\bar{v} \in I^m$ by $\operatorname{ord}(\bar{v})$ and its *equality type* by $\operatorname{equ}(\bar{v})$. For sets $C, D \subseteq I$, we write $C < D$ if $c < d$, for all $c \in C$ and $d \in D$. Analogously, we define $\bar{u} < \bar{v}$ for tuples $\bar{u}, \bar{v} \subseteq I$.

(b) The sequence $(\bar{a}^v)_v$ is *proper* if $\bar{a}^u \cap \bar{a}^v = \varnothing$, for $u \neq v$.

(c) For $\bar{v} \in I^m$, we set



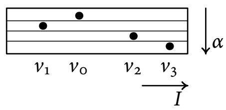
$$\bar{a}[\bar{v}] := (\bar{a}^{v_0}, \dots, \bar{a}^{v_{m-1}})\,.$$

For $J \subseteq I$ and $s \in I$ we define

$$\bar{a}[J] := (\bar{a}^v)_{v\in J} \quad \text{and} \quad \bar{a}[<s] := (\bar{a}^v)_{v<s}\,.$$

The terms $\bar{a}[>s]$, $\bar{a}[\leq s]$, and so on, are defined analogously.

(d) For $\bar{v} \in I^\alpha$, we set



$$\bar{a}\langle\bar{v}\rangle := (a_i^{v_i})_{i<\alpha}\,.$$

Before turning to the general case below let us show how to define a bijection $\bar{a}^v \mapsto \bar{b}^v$ between two sequences $(\bar{a}^v)_{v\in I}$ and $(\bar{b}^v)_{v\in I}$.

Lemma 2.1.2. *Let $(\bar{a}^v)_{v\in I}$ and $(\bar{b}^v)_{v\in I}$ be two sequences indexed by the same linear order I. If there exists a formula $\varphi(\bar{x}, \bar{y})$ (possibly with monadic parameters) and a relation $\sigma \in \{=, \neq, \leq, \geq, <, >\}$ such that*

$$\mathfrak{M} \models \varphi(\bar{c}, \bar{d}) \quad \text{iff} \quad \bar{c} = \bar{a}^u \text{ and } \bar{d} = \bar{b}^v \text{ for some } u \;\sigma\; v\,,$$

then we can construct a formula $\psi(\bar{x}, \bar{y})$ such that

$$\mathfrak{M} \models \psi(\bar{c}, \bar{d}) \quad \text{iff} \quad \bar{c} = \bar{a}^v \text{ and } \bar{d} = \bar{b}^v \text{ for some } v \in I\,.$$

Proof. If $\sigma \in \{=, \neq\}$ then we can set $\psi := \varphi$ or $\psi := \neg\varphi$. By symmetry it therefore remains to consider the case that $\sigma = \{\leq\}$. We can construct a formula ϑ such that

$$\mathfrak{M} \models \vartheta(\bar{c}, \bar{d}) \quad \text{iff} \quad \bar{c} = \bar{a}^u \text{ and } \bar{d} = \bar{a}^v \text{ for some } u \leq v\,,$$

by setting

$$\vartheta(\bar{x}, \bar{x}') := \forall \bar{y}[\varphi(\bar{x}', \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y})]\,.$$

Consequently, we obtain the desired formula ψ by

$$\psi(\bar{x}, \bar{y}) := \forall \bar{x}'[\varphi(\bar{x}', \bar{y}) \rightarrow \vartheta(\bar{x}', \bar{x})]\,. \qquad \square$$

The next lemmas provide a method to find sequences satisfying the preceding lemma.

Lemma 2.1.3. *Let Δ be a finite set of formulae, $B \subseteq M$ a set, and $(\bar{a}^u)_{u<\omega}$ an infinite sequence of tuples such that*

$$\mathrm{tp}_\Delta(\bar{a}^u/B) \neq \mathrm{tp}_\Delta(\bar{a}^v/B)\,, \quad \textit{for all } u \neq v\,.$$

There exist an infinite subset $I \subseteq \omega$, a formula $\varphi \in \Delta$, a relation $\sigma \in \{=, \neq, \leq, >\}$, a number $m < \omega$, and tuples $\bar{b}^v \in B^m$, for $v \in I$, such that

$$\mathfrak{M} \models \varphi(\bar{a}^u, \bar{b}^v) \quad \text{iff} \quad u\ \sigma\ v\,, \quad \textit{for all } u, v \in I\,.$$

Proof. We adapt the proof of Ramsey's theorem. For indices $u \neq v$, fix some formula $\varphi_{uv}(\bar{x}, \bar{y}) \in \Delta$ and a tuple $\bar{c}_{uv} \subseteq B$ with

$$\mathfrak{M} \models \varphi_{uv}(\bar{a}^u, \bar{c}_{uv}) \leftrightarrow \neg\varphi_{uv}(\bar{a}^v, \bar{c}_{uv})\,.$$

We assume that $\bar{c}_{uv} = \bar{c}_{vu}$ and $\varphi_{uv} = \varphi_{vu}$, for all $u, v < \omega$.

We define two infinite increasing sequences $u_0 < u_1 < \dots < \omega$ and $v_0 < v_1 < \dots < \omega$ of indices and a decreasing sequence $\omega = I_0 \supset I_1 \supset \dots$ of infinite sets such that, for every $i < \omega$, we have $u_i, v_i \in I_i$ and

$$\mathfrak{M} \models \varphi_{u_i v_i}(\bar{a}^{u_i}, \bar{c}_{u_i v_i}) \leftrightarrow \neg\varphi_{u_i v_i}(\bar{a}^w, \bar{c}_{u_i v_i})\,, \quad \text{for all } w \in I_{i+1}\,.$$

Note that, in particular, this implies that

$$\mathfrak{M} \models \varphi_{u_i v_i}(\bar{a}^{u_i}, \bar{c}_{u_i v_i}) \leftrightarrow \neg\varphi_{u_i v_i}(\bar{a}^{u_k}, \bar{c}_{u_i v_i})\,, \quad \text{for } i < k\,.$$

We start with $I_0 := \omega$. For the induction step, suppose that I_i has already been defined. Fix arbitrary elements $u, v \in I_i$ with $u \neq v$. By symmetry, we may assume that

$$\mathfrak{M} \models \varphi_{uv}(\bar{a}^u, \bar{c}_{uv}) \wedge \neg\varphi_{uv}(\bar{a}^v, \bar{c}_{uv})\,.$$

$$\text{Let} \quad J_0 := \{ w \in I_i \mid \mathfrak{M} \vDash \neg\varphi_{uv}(\bar{a}^w, \bar{c}_{uv}) \},$$
$$J_1 := \{ w \in I_i \mid \mathfrak{M} \vDash \varphi_{uv}(\bar{a}^w, \bar{c}_{uv}) \}.$$

If J_0 is infinite then we set $u_i := u$, $v_i := v$, and $I_{i+1} := J_0$. Otherwise, we choose $u_i := v$, $v_i := u$, and $I_{i+1} := J_1$.

Set $\bar{b}^i := \bar{c}_{u_i v_i}$. We record for every pair $i < k$ of indices which of the following relations hold

$$\mathfrak{M} \vDash \varphi_{u_i v_i}(\bar{a}^{u_i}, \bar{b}^i),$$
$$\mathfrak{M} \vDash \varphi_{u_i v_i}(\bar{a}^{u_i}, \bar{b}^k),$$
$$\mathfrak{M} \vDash \varphi_{u_i v_i}(\bar{a}^{u_k}, \bar{b}^i).$$

By Ramsey's Theorem, there exists an infinite subset $S \subseteq \omega$ such that, for all indices $i < k$ and $l < m$ in S,

- $\varphi_{u_i v_i} = \varphi_{u_k v_k}$,
- $\mathfrak{M} \vDash \varphi_{u_i v_i}(\bar{a}^{u_i}, \bar{b}^i) \leftrightarrow \varphi_{u_k v_k}(\bar{a}^{u_k}, \bar{b}^k)$,
- $\mathfrak{M} \vDash \varphi_{u_k v_k}(\bar{a}^{u_i}, \bar{b}^k) \leftrightarrow \varphi_{u_m v_m}(\bar{a}^{u_l}, \bar{b}^m)$.

Setting $\varphi := \varphi_{u_i v_i}$ it follows that, for $i < k$ in S,

$$\mathfrak{M} \vDash \varphi(\bar{a}^{v_i}, \bar{b}^i) \leftrightarrow \neg\varphi(\bar{a}^{v_k}, \bar{b}^i).$$

Consequently, we have

$$\mathfrak{M} \vDash \varphi(\bar{a}^{v_i}, \bar{b}^k) \quad \text{iff} \quad i \,\sigma\, k,$$

where $\sigma \in \{=, \neq, \leq, >\}$. ☐

Corollary 2.1.4. *Let Δ be a finite set of formulae and let $A, B \subseteq M$ be sets. If $\mathrm{ti}^n_\Delta(A/B) \geq \aleph_0$ then there exist a formula $\varphi(\bar{x}, \bar{y}) \in \Delta$, a relation $\sigma \in \{=, \neq, \leq, >\}$, a number $m < \omega$, and tuples $\bar{a}^v \in A^n$ and $\bar{b}^v \in B^m$, for $v < \omega$, such that*

$$\mathfrak{M} \vDash \varphi(\bar{a}^u, \bar{b}^v) \quad \text{iff} \quad u \,\sigma\, v.$$

For uncountable cardinals the proof is more involved.

Lemma 2.1.5. *Let κ be an infinite cardinal, Δ a set of formulae of size $|\Delta| \leq \kappa$, and $A, B \subseteq M$ sets. If $\mathrm{ti}^n_\Delta(A/B) > 2^\kappa$ then there exist a formula $\varphi(\bar{x}, \bar{y}) \in \Delta$, a number $m < \omega$, and tuples $\bar{a}^v \in A^n$ and $\bar{b}^v \in B^m$, for $v < \kappa^+$, such that*

$$\mathfrak{M} \vDash \varphi(\bar{a}^u, \bar{b}^u) \leftrightarrow \neg\varphi(\bar{a}^v, \bar{b}^u), \quad \textit{for all } u < v.$$

Proof. Let $\lambda := (2^\kappa)^+$. Fix a sequence $(\bar a^v)_{v<\lambda}$ of tuples $\bar a^v \in A^n$ such that,

$$\mathrm{tp}_\Delta(\bar a^u/B) \neq \mathrm{tp}_\Delta(\bar a^v/B)\,, \quad \text{for } u \neq v\,.$$

We construct a family of sets $S_z \subseteq \lambda$, for $z \in 2^{<\lambda^+}$, such that

- $S_{\langle\rangle} = \lambda$,
- $S_z = S_{z0} \cup S_{z1}$,
- $S_x \supseteq S_y$, for $x \leq y$,
- $S_x \cap S_y = \emptyset$, for $x \nleq y$ and $y \nleq x$, and
- if $|S_z| > 1$ then $S_{z0}, S_{z1} \neq \emptyset$.

For each z, we will choose a formula $\varphi_z(\bar x, \bar y)$ and parameters $\bar b^z \subseteq B$, and we set

$$S_z := \left\{ u < \lambda \;\middle|\; \text{for all } y \prec z \text{ we have } \mathfrak{M} \vDash \varphi_y(\bar a^u, \bar b^y) \text{ iff } y1 \leq z \right\}.$$

We define φ_z inductively. Suppose that φ_x and $\bar b^x$ have already been defined, for all $x \prec z$. Then we also know S_z. If $|S_z| \leq 1$ then we choose an arbitrary sequence $y \prec z$ and set $\varphi_z := \varphi_y$ and $\bar b^z := \bar b^y$. Otherwise, choose distinct elements $u, v \in S_z$. Since

$$\mathrm{tp}_\Delta(\bar a^u/B) \neq \mathrm{tp}_\Delta(\bar a^v/B)$$

we can find a formula $\varphi_z \in \Delta$ and parameters $\bar b^z \subseteq B$ such that

$$\mathfrak{M} \vDash \varphi_z(\bar a^u, \bar b^z) \leftrightarrow \neg\varphi_z(\bar a^v, \bar b^z)\,.$$

Having defined $(S_z)_z$ we consider the sets

$$T := \left\{ z \in 2^{<\lambda^+} \;\middle|\; |S_z| > 1 \right\} \quad \text{and} \quad F := \left\{ z \notin T \;\middle|\; y \in T \text{ for all } y \prec z \right\}.$$

Then $|S_z| \leq 1$, for all $z \in F$ and $\lambda = \bigcup_{z\in F} S_z$. Consequently, we have $|F| \geq \lambda$.

Let α be the minimal ordinal such that $T \subseteq 2^{<\alpha}$. Then $|F| \leq 2^{|\alpha|}$ implies that $\lambda \leq 2^{|\alpha|}$. Since $2^\kappa < \lambda$ it follows that $\alpha \geq \kappa^+$. Hence, there exists some $\eta \in F$ with $|\eta| \geq \kappa^+$. For $i \leq \kappa^+$, let $z_i \prec \eta$ be the prefix of η of length $|z_i| = i$, and let $c_i < 2$ be the number such that $z_i c_i \npreceq \eta$. For every i, choose some element $u_i \in S_{z_i c_i}$. Since $u_k \notin S_{z_i c_i}$, for $k > i$, it follows that

$$\mathfrak{M} \vDash \varphi_{z_i c_i}(\bar a^{u_i}, \bar b^{z_i c_i}) \leftrightarrow \neg\varphi_{z_i c_i}(\bar a^{u_k}, \bar b^{z_i c_i})\,, \quad \text{for } i < k\,.$$

By the Pigeon Hole Principle, there exists a subset $I \subseteq \kappa^+$ such that $\varphi_{z_i c_i} = \varphi_{z_k c_k}$, for all $i, k \in I$. Hence, $(\bar a^{u_i})_{i\in I}$ and $(\bar b^{z_i c_i})_{i \in I}$ are the desired sequences. □

Corollary 2.1.6. *Let κ be an infinite cardinal, Δ a set of formulae of size $|\Delta| \leq \kappa$, and $A, B \subseteq M$ sets. If $\mathrm{ti}^n_\Delta(A/B) > 2^{2^\kappa}$ then there exist a formula $\varphi(\bar{x}, \bar{y}) \in \Delta$, a relation $\sigma \in \{=, \neq, \leq, >\}$, a number $m < \omega$, and tuples $\bar{a}^\nu \in A^n$ and $\bar{b}^\nu \in B^m$, for $\nu < \kappa^+$, such that*

$$\mathfrak{M} \vDash \varphi(\bar{a}^u, \bar{b}^\nu) \quad \text{iff} \quad u \;\sigma\; \nu\,.$$

Proof. By Lemma 2.1.5, there exist a formula φ and sequences $(\bar{a}^i)_{i<(2^\kappa)^+}$ and $(\bar{b}^i)_{i<(2^\kappa)^+}$ such that

$$\mathfrak{M} \vDash \varphi(\bar{a}^i, \bar{b}^i) \leftrightarrow \neg\varphi(\bar{a}^k, \bar{b}^i)\,, \quad \text{for } i < k\,.$$

By the Erdős-Rado Theorem, we have $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$. Hence, we can can find a subsequence $I \subseteq (2^\kappa)^+$ of size $|I| \geq \kappa^+$ such that

$$\mathfrak{M} \vDash \varphi(\bar{a}^i, \bar{b}^k) \leftrightarrow \varphi(\bar{a}^j, \bar{b}^l)\,, \quad \begin{array}{l}\text{for all indices } i, j, k, l \in I \text{ with} \\ \mathrm{ord}(ik) = \mathrm{ord}(jl)\,.\end{array}$$

It follows that there is some relation $\sigma \in \{=, \neq, \leq, >\}$ such that, for all $i, k \in I$,

$$\mathfrak{M} \vDash \varphi(\bar{a}^k, \bar{b}^i) \quad \text{iff} \quad k \;\sigma\; i\,. \qquad \square$$

Let us turn to the general case of Lemma 2.1.2 where we may have more than two sequences. We recall some notions from model theory. A sequence $(\bar{a}^i)_{i\in I}$ of α-tuples $\bar{a}^i$ is *indiscernible* over a set U if

$$\mathrm{tp}(\bar{a}[\bar{\imath}]/U) = \mathrm{tp}(\bar{a}[\bar{\jmath}]/U)\,, \quad \begin{array}{l}\text{for all increasing tuples } \bar{\imath}, \bar{\jmath} \subseteq I \text{ of the} \\ \text{same length.}\end{array}$$

If this equation holds for arbitrarily ordered tuples $\bar{\imath}, \bar{\jmath} \subseteq I$ the sequence is *totally indiscernible.* We adopt the usual convention of working in a sufficiently saturated monster model $\mathbb{M}$ into which we can embed every model $\mathfrak{M}$ under consideration. All elements and sets are tacitly assumed to be contained in $\mathbb{M}$. By an *U-automorphism,* we mean an automorphism π of $\mathbb{M}$ with $\pi|_U = \mathrm{id}_U$. We will frequently use the following standard facts from model theory concerning indiscernible sequences.

Lemma 2.1.7. *Let $(\bar{a}^\nu)_{\nu\in I}$ be an infinite indiscernible sequence over U. For every strictly increasing partial map $\beta : I \to I$, there exists a U-automorphism π such that*

$$\pi(\bar{a}^\nu) = \bar{a}^{\beta(\nu)}\,, \quad \textit{for all } \nu \in \mathrm{dom}\,\beta\,.$$

Lemma 2.1.8. *Let $(\bar{a}^\nu)_{\nu\in I}$ be an indiscernible sequence over U. For every order embedding $\alpha : I \to J$ there exists an indiscernible sequence $(\bar{b}^\nu)_{\nu\in J}$ over U such that $\bar{b}^{\alpha(\nu)} = \bar{a}^\nu$, for $\nu \in I$.*

To generalise Lemma 2.1.2 we look at the fine structure of an indiscernible sequence. In [46] Shelah defines an equivalence relation on the indices of a certain sequence $(\bar{a}^\nu)_{\nu\in I}$ of α-tuples (actually enumerations of models) by calling two indices $i, k < \alpha$ equivalent if the bijection $a^\nu_i \mapsto a^\nu_k$, $\nu \in I$, is MSO-definable. Shelah's main result concering this equivalence relation is a characterisation via indiscernibility. Inspired by this work we consider the case of arbitrary indiscernible sequences. Taking the characterisation in terms of indiscernibility as the definition we show that this equivalence relation gives rise to definable bijections $a^\nu_i \mapsto a^\nu_k$, $\nu \in I$. The main ideas of the proof of this fact in Theorem 2.1.19 below are already contained in [46]. Our contribution consists in streamlining the presentation, showing that the result holds without the special assumptions of Shelah, and obtaining more precise information about the formulae defining the bijections.

Definition 2.1.9. (a) Let $\varphi(\bar{x}^0, \dots, \bar{x}^{k-1})$ be a formula where each $\bar{x}^i$ is an α-tuple of variables. A sequence $(\bar{a}^\nu)_{\nu\in I}$ of α-tuples is *Δ-indiscernible* if, for all indices $\bar{u}^i, \bar{v}^i \in I^\alpha$, $i < k$, with $\operatorname{ord}(\bar{u}^0 \dots \bar{u}^{k-1}) = \operatorname{ord}(\bar{v}^0 \dots \bar{v}^{k-1})$, we have

$$\mathfrak{M} \models \varphi\big(\bar{a}\langle\bar{u}^0\rangle, \dots, \bar{a}\langle u^{k-1}\rangle\big) \leftrightarrow \varphi\big(\bar{a}\langle\bar{v}^0\rangle, \dots, \bar{a}\langle v^{k-1}\rangle\big)\,.$$

(b) Let Δ be a set of such formulae. $(\bar{a}^\nu)_{\nu\in I}$ is *Δ-indiscernible* if it is φ-indiscernible, for every $\varphi \in \Delta$. If Δ is the set of all formulae over a set U of parameters we say that $(\bar{a}^\nu)_{\nu\in I}$ is indiscernible over U.

Example. A sequence $(\bar{a}^i)_i$ of 4-tuples satisfying

$$\mathfrak{M} \models \varphi(a^i_0, a^k_1, a^l_2, a^m_3) \quad \text{iff} \quad i = k \text{ or } (i < k \text{ and } l = m)$$

is φ-indiscernible.

The relation $\{\, \bar{a}^\nu \mid \nu \in I \,\}$ is usually not definable. Instead, we define relations $\{\, \bar{a}^\nu|_p \mid \nu \in I \,\}$ for certain subsets $p \subseteq \alpha$. The main part of this section consists in the proof that the sets p where this is possible form a partition of α.

Definition 2.1.10. (a) A *partition* of a set X is a set $P \subseteq \wp(X)$ such that $X = \bigcup P$ and $p \cap q = \emptyset$, for distinct $p, q \in P$.

(b) Every partition P on X induces the equivalence relation

$$x \approx_P y \quad :\text{iff} \quad \text{there is some } p \in P \text{ with } x, y \in p\,.$$

(c) The *refinement order* on partitions P and Q of X is defined by

$$P \sqsubseteq Q \quad :\text{iff} \quad \approx_P \subseteq \approx_Q\,,$$

and, for a family F of partitions of X, we define their *common refinement* by

$$\textstyle\bigsqcap F := X/{\approx} \quad \text{where} \quad {\approx} := \bigcap_{P\in F} \approx_P\,.$$

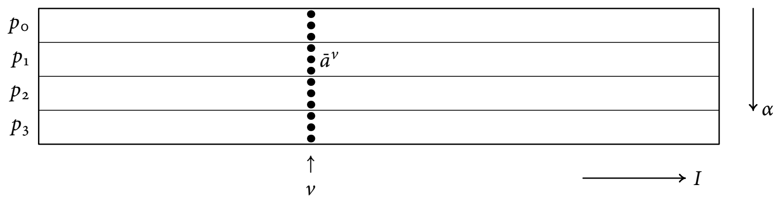
Definition 2.1.11. Let $(\bar{a}^\nu)_{\nu \in I}$ be a sequence of α-tuples and $\varphi(\bar{x}^0, \dots, \bar{x}^k)$ a formula where each $\bar{x}^i$ is an α-tuple of variables. A *Δ-partition* of $(\bar{a}^\nu)_{\nu \in I}$ is a partition P of α such that

$$\mathfrak{M} \vDash \varphi\big(\bar{a}\langle\bar{u}^0\rangle, \dots, \bar{a}\langle\bar{u}^k\rangle\big) \leftrightarrow \varphi\big(\bar{a}\langle\bar{v}^0\rangle, \dots, \bar{a}\langle\bar{v}^k\rangle\big)\,,$$

for all indices $\bar{u}^i, \bar{v}^i \in I^\alpha$, $i \leq k$, such that

$$\operatorname{ord}(\bar{u}^0|_p \dots \bar{u}^k|_p) = \operatorname{ord}(\bar{v}^0|_p \dots \bar{v}^k|_p)\,, \quad \text{for every } p \in P\,.$$

Let Δ be a set of formulae. A *Δ-partition* is a partition P that is a φ-partition, for every $\varphi \in \Delta$.



Equivalently, P is a Δ-partition of $(\bar{a}^i)_i$ if, for every $p \in P$, the 'band' $(\bar{a}^i|_p)_i$ is indiscernible over its complement $(\bar{a}^i|_{\alpha \setminus p})_i$.

Example. Let $(\bar{a}^i)_i$ be an indiscernible sequence of 4-tuples and suppose that $\varphi(x_0 x_1 x_2 x_3)$ is a formula such that

$$\mathfrak{M} \vDash \varphi(a_0^i, a_1^k, a_2^l, a_3^m) \quad \text{iff} \quad i = k \text{ or } (i < k \text{ and } l = m)\,.$$

There are two φ-partitions of $[4]$. The trivial partition with just one class and the partition with classes $\{0, 1\}$ and $\{2, 3\}$.

We will show that there is a unique minimal Δ-partition. We start by pointing out that there exists at least one Δ-partition. Then we show that the class of these partitions is closed under intersections.

Lemma 2.1.12. *If $(\bar{a}^\nu)_{\nu \in I}$ is a Δ-indiscernible sequence of α-tuples then $\{\alpha\}$ is a Δ-partition.*

Lemma 2.1.13. *If $(P_i)_{i<\kappa}$ is a decreasing sequence of Δ-partitions then $\bigsqcap_{i<\kappa} P_i$ is a Δ-partition.*

Proof. If κ is finite then we have $\bigsqcap_{i<\kappa} P_i = P_{\kappa-1}$, which is a Δ-partition. For infinite κ the claim follows from the fact that every formula $\varphi \in \Delta$ contains only finitely many variables. □

Lemma 2.1.14. *Let $(\bar{a}^\nu)_{\nu \in I}$ be an infinite sequence of α-tuples. If P and Q are Δ-partitions then so is $P \sqcap Q$.*

Proof. It is sufficient to prove the claim for $\Delta = \{\varphi\}$. Since φ contains only finitely many variables we may assume w.l.o.g. that α is finite and that

$$P = \{p_0, \dots, p_{n-1}\} \quad \text{and} \quad Q = \{q_0, \dots, q_{m-1}\}\,.$$

For $i < m$, let $q'_i := \alpha \setminus q_i$. Since

$$P \sqcap Q = P \sqcap \{q_0, q'_0\} \sqcap \dots \sqcap \{q_{m-1}, q'_{m-1}\}$$

it is sufficient to prove the claim for $Q = \{q, q'\}$.

Let us introduce some shorthand. For $\bar u_i \in I^{p_i \cap q}$ and $\bar v_i \in I^{p_i \cap q'}$, we set

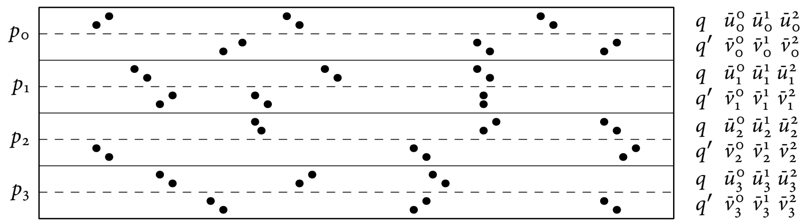
$$\bar a[\bar u_0, \dots, \bar u_{n-1}, \bar v_0, \dots, \bar v_{n-1}] := \bar a\langle \bar x\rangle\,,$$

where x_i is

- the l-th element of $\bar u_k$, if i is the l-th element of $p_k \cap q$,
- the l-th element of $\bar v_k$, if i is the l-th element of $p_k \cap q'$.

Suppose that $\varphi = \varphi(\bar x^0, \dots, \bar x^k)$. For $\bar u^l_i \in I^{p_i \cap q}$ and $\bar v^l_i \in I^{p_i \cap q'}$, we define

$$\begin{aligned}\varphi\big[\bar u^0_0 \dots \bar u^k_0,\ \dots,\ \bar u^0_{n-1} \dots \bar u^k_{n-1},\ \bar v^0_0 \dots \bar v^k_0,\ \dots,\ \bar v^0_{n-1} \dots \bar v^k_{n-1}\big] :=\\ \varphi\big(\bar a[\bar u^0_0, \dots, \bar u^0_{n-1}, \bar v^0_0, \dots, \bar v^0_{n-1}], \dots, \bar a[\bar u^k_0, \dots, \bar u^k_{n-1}, \bar v^k_0, \dots, \bar v^k_{n-1}]\big)\,.\end{aligned}$$



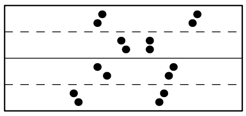
We have to show that

$$\mathfrak M \models \varphi[\bar u_0, \dots, \bar u_{n-1}, \bar v_0, \dots, \bar v_{n-1}] \leftrightarrow \varphi[\bar s_0, \dots, \bar s_{n-1}, \bar t_0, \dots, \bar t_{n-1}]\,,$$

whenever $\operatorname{ord}(\bar u_i) = \operatorname{ord}(\bar s_i)$ and $\operatorname{ord}(\bar v_i) = \operatorname{ord}(\bar t_i)$. If we prove the following special case then the general one will follow by symmetry (w.r.t. permutations of P and Q) and induction.

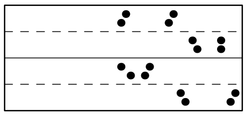
Claim. *If* $\operatorname{ord}(\bar u_0) = \operatorname{ord}(\bar w_0)$ *then*

$$\begin{aligned}\mathfrak M \models\ & \varphi[\bar u_0, \bar u_1, \dots, \bar u_{n-1}, \bar v_0, \dots, \bar v_{n-1}]\\ \leftrightarrow\ & \varphi[\bar w_0, \bar u_1, \dots, \bar u_{n-1}, \bar v_0, \dots, \bar v_{n-1}]\,.\end{aligned}$$



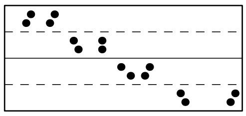
Let $\bar{u}_* := \bar{u}_1 \dots \bar{u}_{n-1}$ and $\bar{v}_* := \bar{v}_1 \dots \bar{v}_{n-1}$. Since I is infinite we can find indices $\bar{s}_\circ, \bar{t}_\circ, \bar{s}_*, \bar{t}_* \subseteq I$ such that

$$\operatorname{ord}(\bar{s}_\circ \bar{s}_*) = \operatorname{ord}(\bar{u}_\circ \bar{u}_*)\,, \quad \operatorname{ord}(\bar{t}_\circ \bar{t}_*) = \operatorname{ord}(\bar{v}_\circ \bar{v}_*)\,, \quad \bar{s}_\circ, \bar{s}_* < \bar{t}_\circ, \bar{t}_*\,.$$



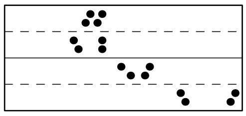
Since Q is a φ-partition we have

$$\mathfrak{M} \vDash \varphi[\bar{u}_\circ, \bar{u}_*, \bar{v}_\circ, \bar{v}_*] \leftrightarrow \varphi[\bar{s}_\circ, \bar{s}_*, \bar{t}_\circ, \bar{t}_*]\,.$$



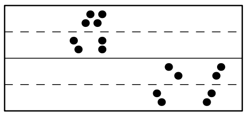
Fix indices $\bar{s}'_\circ, \bar{t}'_\circ$ such that

$$\operatorname{ord}(\bar{s}'_\circ \bar{t}'_\circ) = \operatorname{ord}(\bar{s}_\circ \bar{t}_\circ) \quad \text{and} \quad \bar{s}'_\circ, \bar{t}'_\circ < \bar{s}_\circ, \bar{s}_*\,.$$



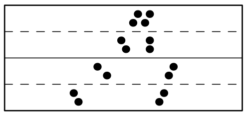
Since P is a φ-partition we have

$$\mathfrak{M} \vDash \varphi[\bar{s}_\circ, \bar{s}_*, \bar{t}_\circ, \bar{t}_*] \leftrightarrow \varphi[\bar{s}'_\circ, \bar{s}_*, \bar{t}'_\circ, \bar{t}_*]\,.$$



Choose $\bar{s}''_\circ$ such that $\operatorname{ord}(\bar{s}''_\circ \bar{t}'_\circ) = \operatorname{ord}(\bar{w}_\circ \bar{v}_\circ)$. Since $\operatorname{ord}(\bar{s}''_\circ \bar{s}_*) = \operatorname{ord}(\bar{s}'_\circ \bar{s}_*)$ and Q is a φ-partition it follows that

$$\mathfrak{M} \vDash \varphi[\bar{s}'_\circ, \bar{s}_*, \bar{t}'_\circ, \bar{t}_*] \leftrightarrow \varphi[\bar{s}''_\circ, \bar{s}_*, \bar{t}'_\circ, \bar{t}_*]\,.$$



Finally, let $\bar{s}'_*, \bar{t}'_* \subseteq I$ be indices such that

$$\begin{aligned}\operatorname{ord}(\bar{s}''_\circ \bar{s}'_*) &= \operatorname{ord}(\bar{s}''_\circ \bar{s}_*)\,,\\ \operatorname{ord}(\bar{t}'_\circ \bar{t}'_*) &= \operatorname{ord}(\bar{t}'_\circ \bar{t}_*)\,,\\ \operatorname{ord}(\bar{s}'_* \bar{t}'_*) &= \operatorname{ord}(\bar{u}_* \bar{v}_*)\,.\end{aligned}$$

As Q is a φ-partition we have

$$\mathfrak{M} \vDash \varphi[\bar{s}''_\circ, \bar{s}_*, \bar{t}'_\circ, \bar{t}_*] \leftrightarrow \varphi[\bar{s}''_\circ, \bar{s}'_*, \bar{t}'_\circ, \bar{t}'_*]\,.$$

Furthermore, $\operatorname{ord}(\bar{s}''_\circ \bar{t}'_\circ) = \operatorname{ord}(\bar{w}_\circ \bar{v}_\circ)$ and $\operatorname{ord}(\bar{s}'_* \bar{t}'_*) = \operatorname{ord}(\bar{u}_* \bar{v}_*)$ implies that

$$\mathfrak{M} \vDash \varphi[\bar{s}''_\circ, \bar{s}'_*, \bar{t}'_\circ, \bar{t}'_*] \leftrightarrow \varphi[\bar{w}_\circ, \bar{u}_*, \bar{v}_\circ, \bar{v}_*]\,,$$

because P is a φ-partition. □

Combining the preceding lemmas we obtain the following result.

Theorem 2.1.15. *For every infinite Δ-indiscernible sequence $(\bar{a}^\nu)_{\nu\in I}$, there exists a unique minimal Δ-partition P.*

Definition 2.1.16. Let $(\bar{a}^\nu)_{\nu \in I}$ be an infinite Δ-indiscernible sequence of α-tuples and let P be the minimal Δ-partition of α corresponding to $(\bar{a}^\nu)_\nu$.

(a) The elements of P are called *Δ-classes.*

(b) We set

$$\asymp_\Delta := \approx_P .$$

Two indices i and k are *Δ-dependent* if $i \asymp_\Delta k$. Otherwise, they are *Δ-independent.*

(c) If Δ is the set of all first-order formulae over U we also also speak of *U-partitions, U-classes, U-independent* indices, etc. and we write $\asymp_U$ instead of $\asymp_\Delta$.

Remark. Note that, if $i < \alpha$ is an index such that no variable x^l_i appears in Δ then $\{i\}$ is a Δ-class. Hence, if Δ is finite then every Δ-class is finite.

Remark. Let $(\bar{a}^\nu)_{\nu \in I}$ be an infinite indiscernible sequence over U. For every U-class p, the sequence $(\bar{a}^\nu|_p)_{\nu \in I}$ is indiscernible over $U \cup \bar{a}|_{\alpha \smallsetminus p}[I]$.

It particular, this means that we can use U-automorphisms to shift each U-class independently.

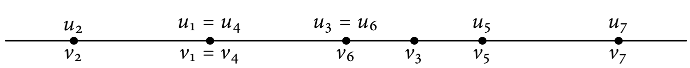
Lemma 2.1.17. *Let $(\bar{a}^\nu)_{\nu \in I}$ be an infinite indiscernible sequence over U and let P be its minimal U-partition. For every family $(\beta_p)_{p \in P}$ of strictly increasing maps $\beta_p : I \to I$, there exists a U-automorphism π such that*

$$\pi(\bar{a}^\nu|_p) = \bar{a}^{\beta_p(\nu)}|_p\,, \quad \textit{for all } p \in P \textit{ and } \nu \in \operatorname{dom}\beta_p\,.$$

An argument we will frequently employ below with worth singling out. Suppose we are given a sequence $x_0, \dots, x_n$ where x_0 has some property P while x_n does not. Then there is some index i with $x_i \in P$ and $x_{i+1} \notin P$. For instance, if $x_0 = u_0 \dots u_{m-1}$ and $x_n = v_0 \dots v_{m-1}$ are tuples then we can use the sequence $x_i := v_0 \dots v_{i-1} u_i \dots u_{m-1}$ to conclude that there are tuples $\bar{u}' \in P$ and $\bar{v}' \notin P$ that differ in exactly one component. A more involved example that appears in the proof of the next theorem is the following one. For an ordered index set I, indices $\bar{u}, \bar{v} \in I^n$, and a number $m < n$, we define $\bar{u} \rightharpoondown_m \bar{v}$ iff there exists some $k < n$ such that

- $u_k \neq v_k$ and $u_i = v_i$, for $i \neq k$, and
- either there are exactly m indices $i \neq k$ with $u_i = u_k$ and there is no i with

$$u_k < u_i \leq v_k \quad \text{or} \quad v_k \leq u_i < u_k\,,$$

$\bar{u} \rightharpoondown_1 \bar{v}$ 

or there are exactly m indices $i \neq k$ with $v_i = v_k$ and there is no i with

$$v_k < v_i \leq u_k \quad \text{or} \quad u_k \leq v_i < v_k \,.$$

Let $\rightharpoondown_{<m} := \rightharpoondown_0 \cup \dots \cup \rightharpoondown_{m-1}$.

Lemma 2.1.18. *If I is densely ordered then any two tuples $\bar{u}, \bar{v} \in I^n$ are connected by a $\rightharpoondown_{<n}$-path.*

Proof. For a contradiction, suppose that $\bar{u}$ and $\bar{v}$ are not connected. As explained above we may assume that $\bar{u}$ and $\bar{v}$ differ in exactly one component. Say $\bar{u} = x\bar{z}$ and $\bar{v} = y\bar{z}$. Since the definition of $\rightharpoondown_m$ is invariant under permutations of the tuples we may assume that $\bar{z}$ is increasing and

$$z_0 \leq \dots \leq z_{k-1} \leq x \leq z_k \leq \dots \leq z_{l-1} \leq y \leq z_l \leq \dots \leq z_{n-2} \,.$$

We choose k and l such that $x < z_k$ and $z_{l-1} < y$. We derive a contradiction by induction on $l - k$. If $k = l$ then we have

$$x\bar{z} \rightharpoondown_{<n} y\bar{z} \,.$$

Contradiction. Suppose that $k < l$. We claim that

$$x\bar{z} \rightharpoondown_{<n} \cdots \rightharpoondown_{<n} z_k\bar{z} \,.$$

Hence, the result follows by induction hypothesis. If $z_{k-1} = x < z_k$ then we can take any element $z_{k-1} < x' < z_k$ and it follows that

$$x\bar{z} \rightharpoondown_{<n} x'\bar{z} \rightharpoondown_{<n} z_k\bar{z} \,.$$

If $z_{k-1} < x < z_k$ then we immediately have

$$x\bar{z} \rightharpoondown_{<n} z_k\bar{z} \,. \qquad \square$$

After these preparations we can finally prove that, for every Δ-class p, we can define the relation $\{\, \bar{a}^v|_p \mid v \in I \,\}$ with the help of monadic parameters. In the constructions this will allow us below to replace sequences $(\bar{a}^v)_v$ of tuples by sequences $(a^v_0)_v$ of singletons.

Theorem 2.1.19. *Suppose that $(\bar{a}^v)_{v \in I}$ is an infinite φ-indiscernible sequence of α-tuples where φ has r free variables. For each φ-class p and every finite subset $q \subseteq p$, there exists a formula $\chi_q(\bar{x}; \bar{y}, \bar{z}, \bar{Z})$ with the following property.*

If $\bar{s}, \bar{t} \in I^r$ are strictly increasing r-tuples with $\bar{s} < \bar{t}$ and

$$A_i := \{\, a^v_i \mid v \in I \,,\ \bar{s} < v < \bar{t} \,\} \,, \quad \text{for } i \in p \,,$$

then we have

$$\mathfrak{M} \vDash \chi_q(\bar{c}; \bar{a}[\bar{s}], \bar{a}[\bar{t}], \bar{A}) \quad \text{iff} \quad \bar{c} = \bar{a}^v|_q \,, \quad \text{for some } v \in I \text{ with } \bar{s} < v < \bar{t} \,.$$

Proof. The proof is based on [46, Fact II.1.5]. We prove the claim by induction on $n := |q|$. For $q = \{i\}$, we can set

$$\chi_q(x) := A_i x .$$

Furthermore, if q and q' are sets such that $q \cap q' \neq \varnothing$ and χ_q and $\chi_{q'}$ exist, then we can define

$$\chi_{q \cup q'}(\bar{x}\bar{y}\bar{z}) := \chi_q(\bar{x}\bar{y}) \wedge \chi_{q'}(\bar{y}\bar{z}) ,$$

where the variables $\bar{x}$ correspond to the elements of $q \smallsetminus q'$, $\bar{y}$ to $q \cap q'$, and $\bar{z}$ to $q' \smallsetminus q$.

Consequently, there exists a partition $p = q_0 \cup \cdots \cup q_{n-1}$ such that each q_i is a maximal subset of p with the property that χ_{q_i} exists. We have to show that $n = 1$ and $q_0 = p$. Let $\bar{b}_* := \bar{a}^\nu|_{\alpha \smallsetminus p}$, for an arbitrary index ν. For $\bar{\nu} \in I^n$, we define

$$\varphi[\bar{\nu}] := \varphi(\bar{a}^{\nu_0}|_{q_0}, \dots, \bar{a}^{\nu_{n-1}}|_{q_{n-1}}, \bar{b}_*) .$$

We will show that

$$\mathfrak{M} \vDash \varphi[\bar{u}] \leftrightarrow \varphi[\bar{\nu}] , \quad \text{for all } \bar{u}, \bar{\nu} \in I^n .$$

It follows that each q_i is a φ-class which implies that $q_i = p$.

By Lemma 2.1.18 and the remarks preceding it, it is sufficient to prove that

$$\bar{u} \rightharpoondown_m \bar{\nu} \quad \text{implies} \quad \mathfrak{M} \vDash \varphi[\bar{u}] \leftrightarrow \varphi[\bar{\nu}] .$$

We prove this claim by induction on m. Let k be the index witnessing the fact that $\bar{u} \rightharpoondown_m \bar{\nu}$. By symmetry, we may assume that $\bar{u}$ is increasing, that $u_k < \nu_k$, and that $u_k \in \{ u_i \mid i \neq k \}$. Hence, we have

$$u_0 \leq \cdots \leq u_{k-m-1} < u_{k-m} = \cdots = u_k < \nu_k < u_{k+1} \leq \cdots \leq u_{n-1} .$$

Define

$$\bar{s} := u_0 \dots u_{k-m-1} , \quad u := u_k , \quad \nu := \nu_k , \quad \bar{t} := u_{k+1} \dots u_{n-1} ,$$

and set $\bar{b}_- := \bar{a}^{u_0}|_{q_0} \dots \bar{a}^{u_{k-m-1}}|_{q_{k-m-1}}$ and $\bar{b}_+ := \bar{a}^{u_{k+1}}|_{q_{k+1}} \dots \bar{a}^{u_{n-1}}|_{q_{n-1}}$.

For $m = 0$, the claim follows immediately by indiscernibility of $(\bar{a}^\nu)_\nu$. Suppose that $m = 1$ and that

$$\mathfrak{M} \vDash \varphi[\bar{s}, u, u, \bar{t}] \wedge \neg\varphi[\bar{s}, u, \nu, \bar{t}] .$$

If $\mathfrak{M} \vDash \neg\varphi[\bar{s}, \nu, u, \bar{t}]$ then we have

$$\mathfrak{M} \vDash \varphi[\bar{s}, x, y, \bar{t}] \quad \text{iff} \quad x = y ,$$

and we can define

$$\chi_{q_{k-1} \cup q_k}(\bar{x}, \bar{y}) := \chi_{q_{k-1}}(\bar{x}) \wedge \chi_{q_k}(\bar{y}) \wedge \varphi(\bar{b}_-, \bar{x}, \bar{y}, \bar{b}_+, \bar{b}_*)\,,$$

in contradiction to our choice of q_k.

Thus, we have $\mathfrak{M} \vDash \varphi[\bar{s}, v, u, \bar{t}]$. This implies that

$$\mathfrak{M} \vDash \varphi[\bar{s}, x, y, \bar{t}] \quad \text{iff} \quad x \geq y\,.$$

As in Lemma 2.1.2, we obtain a formula

$$\vartheta(\bar{x}, \bar{x}') := \forall \bar{y}[\chi_{q_k}(\bar{y}) \wedge \varphi(\bar{b}_-, \bar{x}, \bar{y}, \bar{b}_+, \bar{b}_*) \to \varphi(\bar{b}_-, \bar{x}', \bar{y}, \bar{b}_+, \bar{b}_*)]$$

such that

$$\mathfrak{M} \vDash \vartheta(\bar{a}^x|_{q_{k-1}}, \bar{a}^y|_{q_{k-1}}) \quad \text{iff} \quad x \leq y\,,$$

and we can set

$$\begin{aligned}\chi_{q_{k-1} \cup q_k}(\bar{x}, \bar{y}) := \chi_{q_{k-1}}(\bar{x}) \wedge \chi_{q_k}(\bar{y}) & \\ \wedge \forall \bar{x}'[\chi_{q_{k-1}}(\bar{x}') \wedge \varphi(\bar{b}_-, \bar{x}', \bar{y}, \bar{b}_+, \bar{b}_+) &\to \vartheta(\bar{x}', \bar{x})]\,.\end{aligned}$$

Contradiction.

It remains to consider the case that $m > 1$. Again, assume that

$$\mathfrak{M} \vDash \varphi[\bar{s}, u \dots u, u, \bar{t}] \wedge \neg\varphi[\bar{s}, u \dots u, v, \bar{t}]\,.$$

By indiscernibility, the former implies that

$$\mathfrak{M} \vDash \varphi[\bar{s}, w \dots w, \bar{t}]\,, \qquad \text{for all } w \in I \text{ with } \bar{s} < w < \bar{t}\,.$$

On the other hand, if $\bar{w} \in I^{m+1}$ is a tuple such that $\bar{s} < \bar{w} < \bar{t}$ and $|\operatorname{rng} w| > 1$ then $\bar{s}\bar{w}\bar{t} \rightharpoondown_{<m} \bar{s}u \dots uv\bar{t}$. Hence, by induction hypothesis, we have

$$\mathfrak{M} \vDash \neg\varphi[\bar{s}, \bar{w}, \bar{t}]\,, \qquad \text{for all such } \bar{w}\,.$$

Consequently, we have

$$\mathfrak{M} \vDash \varphi[\bar{s}, \bar{w}, \bar{t}] \quad \text{iff} \quad w_0 = \dots = w_m\,,$$

and we can define

$$\begin{aligned}\chi_{q_{k-m} \cup \dots \cup q_k}(\bar{x}_0, \dots, \bar{x}_{k-m}) := \chi_{q_{k-m}}(\bar{x}_0) \wedge \dots \wedge \chi_{q_k}(\bar{x}_{k-m}) & \\ \wedge\, \varphi(\bar{b}_-, \bar{x}_0, \dots, \bar{x}_{k-m}, \bar{b}_+, \bar{b}_*)&\,,\end{aligned}$$

in contradiction to our choice of q_k. $\square$

2.2 Pairing functions and coding

In [1] Baldwin and Shelah argue that the monadic theories of structures are hopelessly complicated if they *admit coding*, i.e., if they contain a first-order definable pairing function. Then they proceed by classifying the remaining structures by their first-order theories. Baldwin and Shelah show that, if the first-order theory is stable then structures that do not admit coding have a tree-like decomposition with countable components. The unstable case is considered in [46] but the resulting theory remains fragmentary. In the first part of the thesis we complete the picture by proving that every structure that does not admit coding has a partition refinement of bounded (though infinite) width. This also gives an alternative proof of the already known results on stable structures.

In this and the next section we collect conditions that imply the definability of a pairing function. Special emphasis is placed on indiscernible sequences. We start in this section by presenting the needed definitions and results from [1], together with some simple consequences. The next section contains mostly new results.

Definition 2.2.1. A structure $\mathfrak{M}$ *admits coding* if there exist an elementary extension $\mathfrak{N} \succeq \mathfrak{M}$, unary predicates $\bar{P}$, and infinite sets $A, B, C \subseteq N$ such that in the structure $(\mathfrak{N}, \bar{P})$ there exists a first-order definable bijection $A \times B \to C$.

An alternative characterisation of coding is based on the existence of two equivalence relations.

Lemma 2.2.2. *Suppose that $\varphi(x, y)$ and $\psi(x, y)$ are formulae (with monadic parameters) and $(c^{uv})_{u,v<\omega}$ are elements such that*

$$\begin{aligned} \mathfrak{M} &\vDash \varphi(c^{uv}, c^{st}) \quad \text{iff} \quad u = s\,, \\ \mathfrak{M} &\vDash \psi(c^{uv}, c^{st}) \quad \text{iff} \quad v = t\,. \end{aligned}$$

Then $\mathfrak{M}$ admits coding.

Proof. The formula $\chi(x, y, z) := \varphi(x, z) \wedge \psi(y, z)$ defines the bijection

$$\{ c^{u0} \mid u < \omega \} \times \{ c^{0v} \mid v < \omega \} \to \{ c^{uv} \mid u, v < \omega \}$$

sending the pair (c^{u0}, c^{0v}) to c^{uv}. □

It is not difficult to show that structures admitting coding have large partition width. A weak version of the converse will be established in Theorem 2.6.3.

Lemma 2.2.3 ([7, 5]). *If $\mathfrak{M}$ admits coding then* $\operatorname{pwd} \mathfrak{M} \geq \aleph_0$.

This lemma is a special case of the following result. Let us call a function $f : A \times B \to C$ *cancellative* if $f(a,b) = f(a',b)$ implies $a = a'$ and $f(a,b) = f(a,b')$ implies $b = b'$.

Lemma 2.2.4. *Let M be a set and $f : A \times B \to C$ (the graph of) a cancellative function where $A, B, C \subseteq M$ are sets of size $|A|, |B| > \kappa$. Then we have*

$$\mathrm{pwd}_2(U_v)_v \geq \kappa\,, \quad \textit{for every partition refinement } (U_v)_v \textit{ of } (M, f)\,.$$

Proof. For a contradiction suppose that there exists a partition refinement $(U_v)_{v \in T}$ of (M, f) such that

$$\mathrm{eti}^2_0(U_v/M \setminus U_v) < \kappa\,, \quad \text{for all } v \in T\,.$$

For $v \in T$, set $A_v := A \cap U_v$, $B_v := B \cap U_v$, and $C_v := C \cap U_v$.

We claim that there exists a vertex $v \in T$ such that $|A_v| \geq \kappa$ and $|A \setminus A_v| \geq \kappa$. Suppose otherwise. Then $S := \{\, v \in T \mid |A_v| \geq \kappa \,\}$ forms a chain. Let $(v_i)_{i<\alpha}$ be an increasing enumeration of S and let $d_i \in [2]$ be the number such that $v_{i+1} = v_i d_i$, for $i < \alpha$. Since $|A_v| = |A_{v0}| + |A_{v1}|$ it follows that α is a limit ordinal. Let

$$X_i := A_{v_i(1-d_i)}\,, \quad Z := \bigcap_{i<\alpha} A_{v_i}\,, \quad \text{and} \quad I := \{\, i < \alpha \mid X_i \neq \varnothing \,\}\,.$$

Then $A = Z \cup \bigcup_{i<\alpha} X_i$. Since $|A| > \kappa$ and $|Z| < \kappa$ it follows that

$$\sum_{i \in I} |X_i| = \sum_{i<\alpha} |X_i| = |A \setminus Z| > \kappa\,.$$

Since $|X_i| < \kappa$ and κ^+ is regular we have $|I| > \kappa$. Consequently, there is some $\beta < \alpha$ such that $|I \cap \beta| \geq \kappa$. Setting $v := v_\beta$ it follows that

$$|A_v| > \kappa \quad \text{and} \quad |A \setminus A_v| = \sum_{i<\beta} |X_i| \geq |I \cap \beta| \geq \kappa\,.$$

Contradiction.

Having found a vertex $v \in T$ with $|A_v| \geq \kappa$ and $|A \setminus A_v| \geq \kappa$ we distinguish two cases. We have $|B_v| \geq \kappa$ or $|B \setminus B_v| \geq \kappa$. By symmetry, we may assume the latter.

First, we prove that there are less than κ elements $b \in B \setminus B_v$ such that $f(a,b) = c$, for some $a \in A_v$ and $c \in C_v$. Otherwise, we could find elements $a, a' \in A_v$, $c, c' \in C_v$, and $b, b' \in B \setminus B_v$ with

$$b \neq b'\,, \quad f(a,b) = c\,, \quad f(a',b') = c'\,,$$

$$\text{and} \quad \mathrm{etp}_0(ac/M \setminus U_v) = \mathrm{etp}_0(a'c'/M \setminus U_v)\,.$$

But then $f(a,b) = c$ implies $f(a',b) = c'$. Hence, $f(a',b) = f(a',b')$. Since f is cancellative it follows that $b = b'$. Contradiction.

Consequently, there is some $b \in B \setminus B_\nu$ such that $f(a,b) \in C \setminus C_\nu$, for all $a \in A_\nu$. Since $|A_\nu| \geq \kappa$ we can find $a, a' \in A_\nu$ with $\operatorname{etp}_0(a/M \setminus U_\nu) = \operatorname{etp}_0(a'/M \setminus U_\nu)$. Hence, we have $f(a,b) = c$ iff $f(a',b) = c$, for all $c \in C \setminus C_\nu$. Contradiction. □

Corollary 2.2.5. *If $\mathfrak{M}$ admits coding then for every cardinal κ, there exists an elementary extension $\mathfrak{N} \geq \mathfrak{M}$ with* $\operatorname{pwd}\mathfrak{N} \geq \kappa$.

Proof. Since $\mathfrak{M}$ admits coding there exists an FO-interpretation $\mathcal{I}$ such that, for every cardinal κ, we can find an elementary extension $\mathfrak{N}_\kappa$ of $\mathfrak{M}$ and unary parameters $\bar{P}_\kappa$ such that $\mathcal{I}(\mathfrak{N}_\kappa, \bar{P}_\kappa) = (N_\kappa, f_\kappa)$ where $f_\kappa : A \times B \to C$ is a bijective function with $|A|, |B| = \kappa$.

By Proposition 1.2.7, there exists a strictly increasing function g such that

$$\operatorname{pwd}(\mathfrak{N}_\kappa, \bar{P}_\kappa) < \kappa \quad \text{implies} \quad \operatorname{pwd}(N_\kappa, f_\kappa) < g(\kappa)\,.$$

Given a cardinal κ set $\lambda := g(\kappa)$ and let μ be the minimal cardinal such that

$$\operatorname{pwd}(\mathfrak{N}_\lambda, \bar{P}_\lambda) < \mu\,.$$

Note that this implies that μ is also the minimal cardinal with $\operatorname{pwd}\mathfrak{N}_\lambda < \mu$. By the preceding lemma, we have

$$g(\kappa) = \lambda \leq \operatorname{pwd}(\mathfrak{N}_\lambda, f_\lambda) < g(\mu)\,.$$

Since g is strictly increasing it follows that $\kappa < \mu$. By choice of μ this implies that $\operatorname{pwd}\mathfrak{N}_\lambda \geq \kappa$. □

A first simple criterion for coding is the independence property.

Definition 2.2.6. Let T be a first-order theory. A formula $\varphi(\bar{x}, \bar{y})$ has the *independence property* (w.r.t. T) if there exists a model $\mathfrak{M}$ of T containing sequences $(\bar{a}_X)_{X \subseteq \omega}$ and $(\bar{b}_i)_{i<\omega}$ such that

$$\mathfrak{M} \models \varphi(\bar{a}_X, \bar{b}_i) \quad \text{iff} \quad i \in X\,.$$

We say that a structure $\mathfrak{M}$ has the independence property if there exists a formula φ that has the independence property w.r.t. $\operatorname{Th}(\mathfrak{M})$. If $\bar{a}_X$ and $\bar{b}_i$ are singletons we say that $\mathfrak{M}$ has the *independence property on singletons.*

Lemma 2.2.7. *If $\mathfrak{M}$ has the independence property on singletons then it admits coding.*

Proof. Fix sequences $(a_X)_{X\subseteq\omega}$ and $(b_i)_{i\in\omega}$ and a formula $\varphi(x,y)$ such that

$$\mathbb{M} \vDash \varphi(a_X, b_i) \quad \text{iff} \quad i \in X\,.$$

Fix disjoint infinite sets $U, V \subseteq \{\, b_i \mid i < \omega \,\}$ and define $f : U \times V \to M$ by $f(b_i, b_k) := a_{\{i,k\}}$. Then we have

$$f(x,y) = z \quad \text{iff} \quad \mathbb{M} \vDash \varphi(z,x) \wedge \varphi(z,y)\,,$$

for $x \in U$, $y \in V$, and $z \in f(U,V)$. □

In [1] it is shown that the independence property and the independence property on singletons coincide if we allow unary predicates.

Lemma 2.2.8 (Baldwin, Shelah). *Suppose that $\mathfrak{M}$ has the independence property. There exists an elementary extension $\mathfrak{N} \succeq \mathfrak{M}$ and unary predicates $\bar{P}$ such that $(\mathfrak{N}, \bar{P})$ has the independence property on singletons.*

Consequently, the independence property implies coding.

Corollary 2.2.9. *If $\mathfrak{M}$ has the independence property then it admits coding.*

In the next section we study the following question. Given an indiscernible sequence $(\bar{a}^\nu)_{\nu\in I}$ and an arbitrary element c what is their relationship? Is the sequence also indiscernible over c or can one distinguish intervals of I with the help of c? (We use the term 'interval' for arbitrary convex subsets. We do not require the existence of a supremum or infimum.) As an example we give a characterisation of the independence property in these terms, which is basically due to Shelah (see [47] and [48].)

Definition 2.2.10. Let $\varphi(\bar{x})$ be a formula and $(\bar{a}^\nu)_{\nu\in I}$ a sequence. We define

$$[\![\varphi(\bar{a}^\nu)]\!]_{\nu\in I} := \{\, \nu \in I \mid \mathbb{M} \vDash \varphi(\bar{a}^\nu) \,\}\,.$$

Lemma 2.2.11. *A formula $\varphi(\bar{x},\bar{y})$ has the independence property if and only if there exists an indiscernible sequence $(\bar{a}^\nu)_{\nu\in I}$ and a tuple $\bar{c}$ such that the set $[\![\varphi(\bar{c},\bar{a}^\nu)]\!]_{\nu\in I}$ cannot be written as union of finitely many intervals.*

Proof. $(\Rightarrow)$ Let $(\bar{a}_i)_{i<\omega}$ and $(\bar{b}_X)_{X\subseteq\omega}$ be sequences such that

$$\mathbb{M} \vDash \varphi(\bar{b}_X, \bar{a}_i) \quad \text{iff} \quad i \in X\,.$$

By compactness, we may assume that $(\bar{a}_i)_{i<\omega}$ is indiscernible. Take the set $X := \{\, 2i \mid i < \omega \,\}$ of even numbers and set $\bar{c} := \bar{b}_X$. Then $[\![\varphi(\bar{c},\bar{a}_i)]\!]_i$ has the desired property.

$(\Leftarrow)$ Fix a strictly increasing or a strictly decreasing subsequence $(u_i)_{i<\omega}$ of I such that every interval (u_i, u_{i+1}) contains elements of both $[\![\varphi(\bar{c},\bar{a}^\nu)]\!]_\nu$

and $[\![\neg\varphi(\bar{c},\bar{a}^\nu)]\!]_\nu$. Let $J := \{ u_i \mid i < \omega \}$ and set $\bar{b}_i := \bar{a}^{u_i}$. For every set $X \subseteq \omega$, we can fix a strictly increasing function $\alpha_X : J \to I$ such that

$$\alpha_X(u_i) \in [\![\varphi(\bar{c},\bar{a}^\nu)]\!]_\nu \quad \text{iff} \quad i \in X\,.$$

Let π_X be an automorphism such that $\pi(\bar{a}^{\alpha_X(\nu)}) = \bar{a}^\nu$, for $\nu \in J$, and set $\bar{c}_X := \pi(\bar{c})$. Then it follows that

$$\mathbb{M} \models \varphi(\bar{c}_X, \bar{b}_i) \quad \text{iff} \quad i \in X\,.$$

Consequently, φ has the independence property. □

Corollary 2.2.12. *Let $(\bar{a}^\nu)_{\nu\in I}$ be an infinite indiscernible sequence and $\bar{c}$ a tuple such that the sets*

$$[\![\varphi(\bar{c},\bar{a}^\nu)]\!]_{\nu\in I} \quad \textit{and} \quad [\![\neg\varphi(\bar{c},\bar{a}^\nu)]\!]_{\nu\in I}$$

are both infinite. If $(\bar{a}^\nu)_\nu$ is totally indiscernible then $\mathfrak{M}$ admits coding.

Proof. By taking a suitable subsequence we may assume that I is countable. We can choose a bijection $\alpha : \mathbb{Q} \to I$ such that the sets

$$[\![\varphi(\bar{c},\bar{a}^{\alpha(\nu)})]\!]_{\nu\in\mathbb{Q}} \quad \text{and} \quad [\![\neg\varphi(\bar{c},\bar{a}^{\alpha(\nu)})]\!]_{\nu\in\mathbb{Q}}$$

are dense in $\mathbb{Q}$. If $(\bar{a}^\nu)_{\nu\in I}$ is totally indiscernible then so is the rearranged sequence $(\bar{a}^{\alpha(\nu)})_{\nu\in\mathbb{Q}}$. By the preceding lemma it follows that φ has the independence property. □

2.3 Indiscernibles and coding

In order to develop a structure theory for structures that do not admit coding we investigate indiscernible sequences. In the following we derive a sequence of lemmas containing more and more strict conditions on definable intervals of indiscernible sequence. We will prove that the U-classes of such an indiscernible sequence are not affected if we add a new element c to U, i.e., every U-class is also a $(U \cup \{c\})$-class. The main result of this section states that, if the structure in question does not admit coding, then we can extend each indiscernible sequence $(\bar{a}^\nu)_{\nu\in I}$ to cover every given set, i.e., we can find an indiscernible sequence $(\bar{b}^\nu)_{\nu\in J}$ with $\bar{b}^\nu \supseteq \bar{a}^\nu$, for $\nu \in I$, such that $\bar{b}[J]$ contains the given set. As a consequence it follows that every structure without coding has a basically linear structure.

Note that the result obviously fails for arbitrary structures. For instance, if $(\bar{a}_X)_{X\subseteq\omega}$ and $(\bar{b}_i)_{i<\omega}$ witness the independence property, then we cannot extend $(\bar{b}_i)_i$ to include the element $\bar{a}_{\{1,5\}}$.

Let us start with a simple example that illuminates the general structure of the more involved arguments below. Given two indiscernible sequences $(a^v)_{v\in I}$ and $(b^v)_{v\in I}$ with certain additional properties, we construct a family $(c^{uv})_{u,v\in I}$ and a definable bijection $(a_u, b_v) \mapsto c^{uv}$.

Lemma 2.3.1. *Let $(a^v)_{v\in\mathbb{Z}}$ and $(b^v)_{v\in\mathbb{Z}}$ be sequences such that $(a^v)_v$ is indiscernible over $U \cup b[\mathbb{Z}]$ and $(b^v)_v$ is indiscernible over $U \cup a[\mathbb{Z}]$. If there exist an element c, formulae $\varphi(x, y)$ and $\psi(x, y)$ over U, and relations $\rho, \sigma \in \{=, \neq, \leq, \geq, <, >\}$ such that*

$$\begin{aligned} &\mathfrak{M} \vDash \varphi(a^v, c) \quad \text{iff} \quad v \,\rho\, 0 \\ \textit{and} \quad &\mathfrak{M} \vDash \psi(b^v, c) \quad \text{iff} \quad v \,\sigma\, 0\,, \end{aligned}$$

then $\mathfrak{M}$ admits coding.

Proof. Let π_{st} be an U-automorphism such that

$$\pi_{st}(a^v) = a^{v+s} \quad \text{and} \quad \pi_{st}(b^v) = b^{v+t},$$

and set $c^{st} := \pi_{st}(c)$. It follows that

$$\begin{aligned} \mathfrak{M} \vDash \varphi(a^v, c^{st}) \quad &\text{iff} \quad \mathfrak{M} \vDash \varphi(a^{v-s}, c) \\ &\text{iff} \quad v - s \,\rho\, 0 \quad \text{iff} \quad v \,\rho\, s\,, \end{aligned}$$

and similarly

$$\mathfrak{M} \vDash \psi(b^v, c^{st}) \quad \text{iff} \quad v \,\sigma\, t\,.$$

Let $A := \{\, a^v \mid v \in I \,\}$ and $B := \{\, b^v \mid v \in I \,\}$. We can construct formulae $\chi(x, y)$ and $\vartheta(x, y)$ such that

$$\begin{aligned} \mathfrak{M} \vDash \chi(c^{uv}, c^{st}) \quad &\text{iff} \quad u = s\,, \\ \mathfrak{M} \vDash \vartheta(c^{uv}, c^{st}) \quad &\text{iff} \quad v = t\,, \end{aligned}$$

by setting

$$\begin{aligned} \chi(x, y) &:= (\forall z.Az)[\varphi(z, x) \leftrightarrow \varphi(z, y)]\,, \\ \vartheta(x, y) &:= (\forall z.Bz)[\psi(z, x) \leftrightarrow \psi(z, y)]\,. \end{aligned}$$

By Lemma 2.2.2 it follows that $\mathfrak{M}$ admits coding. □

The following criterion for coding appears in [46]. For the readers convenience, we repeat its proof.

Lemma 2.3.2 (Shelah). *Let $(\bar{a}^v)_{v\in I}$ be an infinite indiscernible sequence over U. Suppose that there exists a U-class p, an element $c \in \mathbb{M}$, a formula ψ over U, and indices $s < t$ such that*

- $\mathfrak{M} \vDash \psi(c, \bar{a}^s|_p, \bar{a}^t|_p)$,
- $\mathfrak{M} \vDash \neg\psi(c, \bar{a}^s|_p, \bar{a}^v|_p)$ *for infinitely many* $v > t$,
- $\mathfrak{M} \vDash \neg\psi(c, \bar{a}^v|_p, \bar{a}^t|_p)$ *for infinitely many* $v < s$.

Then $\mathfrak{M}$ *admits coding.*

Proof. Replacing the sequence $(\bar{a}^v)_v$ by $(\bar{a}^v|_p)_v$ we may assume that $\bar{a}^v|_p = \bar{a}^v$. Hence, we can omit p. Furthermore, by considering a suitable subsequence we may assume that

$$\mathfrak{M} \vDash \neg\psi(c, \bar{a}^s, \bar{a}^v), \quad \text{for all } v > t,$$
$$\mathfrak{M} \vDash \neg\psi(c, \bar{a}^v, \bar{a}^t), \quad \text{for all } v < s.$$

Finally, we may assume by compactness that $I = \mathbb{R}$. By Theorem 2.1.19, there exists a formula $\chi(\bar{x})$ with monadic parameters such that

$$\mathfrak{M} \vDash \chi(\bar{b}) \quad \text{iff} \quad \bar{b} = \bar{a}^v \text{ for some } v \in I.$$

Replacing ψ by the formula

$$\psi'(x, y, z) := \exists\bar{y}'\exists\bar{z}'\big[\chi(y\bar{y}') \wedge \chi(z\bar{z}') \wedge \psi(x, y\bar{y}', z\bar{z}')\big]$$

we can assume that $\bar{a}^v = a^v$ is a singleton.

For each pair $u < v$, fix an order isomorphism $\alpha_{uv} : I \to I$ with $\alpha(s) = u$ and $\alpha(t) = v$ and let π_{uv} be a U-automorphism such that $\pi_{uv}(\bar{a}^x) = \bar{a}^{\alpha(x)}$, for $x \in I$. We set $c^{uv} := \pi_{uv}(c)$. To simplify notation, we set $\psi_{st}[u, v] := \psi(c^{st}, a^u, a^v)$. By Ramsey's theorem and compactness, we may assume that

$$\mathfrak{M} \vDash \psi_{st}[u, s] \leftrightarrow \psi_{st}[u', s]$$
$$\mathfrak{M} \vDash \psi_{st}[u, t] \leftrightarrow \psi_{st}[u', t]$$
$$\mathfrak{M} \vDash \psi_{st}[s, u] \leftrightarrow \psi_{st}[s, u']$$
$$\mathfrak{M} \vDash \psi_{st}[t, u] \leftrightarrow \psi_{st}[t, u']$$
$$\mathfrak{M} \vDash \psi_{st}[u, v] \leftrightarrow \psi_{st}[u', v']$$

for all s, t, u, v, u', v' with $s < t$ such that s, t, u, v and s, t, u', v' have the same order type.

(a) First, we consider the case that there exists some infinite subset $J \subseteq I$ and a formula $\eta(x, y; a^r)$ such that

$$\mathfrak{M} \vDash \eta(a^u, a^v; a^r) \quad \text{iff} \quad u < v \qquad \text{for all } u, v \in J.$$

For fixed $s, t \in J$ we partition $J = J_0 \cup \dots \cup J_4$ where

$$J_0 := \{\, v \in J \mid v < s \,\}, \quad J_1 := \{s\}, \quad J_2 := \{\, v \in J \mid s < v < t \,\},$$
$$J_3 := \{t\}, \quad J_4 := \{\, v \in J \mid v > t \,\}.$$

The sets $A^i := \{ a^\nu \mid \nu \in J_i \}$, $i < 5$, are definable from a^s and a^t using the formula η and the parameters a^r and $A := \{ a^\nu \mid \nu \in J \}$. Let $\rho^i(x; a^s, a^t)$ be the formula defining them. We define

$$\begin{aligned}\vartheta(x,y,z) := Ay \wedge Az \wedge \eta(y,z;a^r) \\ \wedge \bigwedge_{i,k<5} \forall y' \forall z' \big[\rho^i(y';y,z) \wedge \rho^k(z';y,z) \to \vartheta_{i,k}(x,y',z')\big]\end{aligned}$$

where

$$\vartheta_{i,k} := \begin{cases} \psi(x,y',z') & \text{if } \mathfrak{M} \vDash \psi_{st}[u,v] \text{ for all } u \in J_i,\ v \in J_k\,, \\ \neg\psi(x,y',z') & \text{if } \mathfrak{M} \vDash \neg\psi_{st}[u,v] \text{ for all } u \in J_i,\ v \in J_k\,. \end{cases}$$

Again, we write $\vartheta_{st}[u,v] := \vartheta(c^{st}, a^u, a^v)$.

Then $\mathfrak{M} \vDash \vartheta_{st}[s,t]$. For all $s, t \in J$ with $s < t$, we record whether,

$$\mathfrak{M} \vDash \vartheta_{st}[s_0,t_0] \wedge \vartheta_{st}[s_1,t_1] \quad \text{implies} \quad s_0 = s_1 \text{ and } t_0 = t_1\,,$$

for $s_0, s_1, t_0, t_1 \in J$.

If there exists an infinite subset $J' \subseteq J$ such that this is the case for all $s, t \in J'$, then, by taking every other element of J', we obtain an infinite subset $J'' \subseteq J'$ such that $\vartheta(x,y,z)$ defines the function $c^{st} \mapsto (a^s, a^t)$ for $s \in J''_0$ and $t \in J''_1$ where $J''_0 \cup J''_1 = J''$ is a partition with $J''_0 < J''_1$ and $|J''_0|, |J''_1| \geq \aleph_0$.

Otherwise, there exists an infinite subset $J' \subseteq J$ such that, for all $s, t \in J'$, the above condition does not hold. Fix $s, t \in J'$ and let s_0, s_1, t_0, t_1 be counterexamples. By symmetry, we may assume that $t_0 < t_1$. That is, we have $s_0 < t_0 < t_1$ and $s_1 < t_1$. For arbitrary indices $u_0, v_0 \in J$ with $s_1 < u_0 < v_0$ we can find $u_1, v_1 \in J$ with $\max\{s_1, t_0\} < u_1 < v_1$ such that u_0, v_0, t_1 and u_1, v_1, t_1 have the same order type.

$\mathfrak{M} \vDash \vartheta_{st}[s_1, t_1]$ implies that

$$\mathfrak{M} \vDash \psi_{st}[u_1,v_1] \leftrightarrow \psi_{st}[u_0,v_0]\,,$$

and $\mathfrak{M} \vDash \vartheta_{st}[s_0,t_0]$ implies, for $t_0 < u_2 < v_2$, that

$$\mathfrak{M} \vDash \psi_{st}[u_1,v_1] \leftrightarrow \psi_{st}[u_2,v_2]\,.$$

Thus, the truth value of $\psi_{st}[u,v]$ is the same for all $v > u > s_1$. This is a property of c^{st} and s_1 which fails for any $s' < s_1$ since $\mathfrak{M} \vDash \psi_{st}[s_1,t_1]$ but $\mathfrak{M} \vDash \neg\psi_{st}[s_1,v]$ for $v > t_1$. Hence, a^{s_1} and, therefore, a^{t_1} are definable from c^{st}. It follows that we can define the bijection $c^{st} \mapsto (a^s, a^t)$ for $s \in J'_0$ and $t \in J'_1$ where $J'_0 \cup J'_1 = J'$ is a partition as above.

(B) Now, suppose that no such set $J \subseteq I$ exists. We consider five cases.

(1) $\mathfrak{M} \vDash \psi_{st}[v,t]$ for some/all v strictly between s and t. By indiscernibility, we can assume that there are infinitely many elements less than t. By

Ramsey's Theorem, there is some infinite set $L \subseteq J' := \{ v \in I \mid v < t \}$ such that $(c^{vt}a^v)_{v\in L}$ is $\psi(x, y, a^t)$-indiscernible. Since

$$\mathfrak{M} \vDash \psi(c^{st}, a^v, a^t) \wedge \neg\psi(c^{vt}, a^s, a^t) \qquad \text{for } s, v \in L\,,\ s < v\,,$$

we have, for $s, v \in L$,

$$\mathfrak{M} \vDash \psi(c^{st}, a^v, a^t) \quad \text{iff} \quad s \leq v\,.$$

That is, the sequences $(c^{st})_{s\in L}$ and $(a^s)_{s\in L}$ are of $\psi(x, y, a^t)$-type $\leq$. Hence, by Lemma 2.1.2, there exists a formula ρ such that

$$\mathfrak{M} \vDash \rho(\bar{b}; a^t) \quad \text{iff} \quad \bar{b} = c^{st}a^s \text{ for some } s \in L\,,$$

and we can order $(a^s)_{s\in L}$ by the formula

$$\eta_0(x, y; a^t) := \exists z[\rho(z, x; a^t) \wedge \psi(z, y, a^t)]\,.$$

Since $|L| \geq \aleph_0$ this contradicts our assumption.

(2) Similarly, if (1) does not hold and $\mathfrak{M} \vDash \psi_{st}[v, t]$ for some/all $v > t$, then we can choose s such that there are infinitely many elements above s and we find some infinite set $L \subseteq \{ v \in I \mid v > s \}$ such that $(c^{sv}a^v)_{v\in L}$ is $\psi(x, a^s, y)$-indiscernible. Since

$$\mathfrak{M} \vDash \psi(c^{st}, a^s, a^v) \wedge \neg\psi(c^{sv}, a^s, a^t) \qquad \text{for } t, v \in L\,,\ t < v\,,$$

we can order $(a^v)_{v\in L}$ and obtain a contradiction as above.

(3) If $\mathfrak{M} \vDash \psi_{st}[s, v]$ for some $s < v < t$ then we obtain a contradiction analogously to (1).

(4) If (3) does not hold and $\mathfrak{M} \vDash \psi_{st}[s, v]$ for $v < s$, then we can argue as in (2).

(5) Let $J \subseteq I$ be a dense and co-dense subset of I. As the above cases can not occur, if $s, t \in I$, $s < t$, then s is the unique index $v \neq t$ with $\mathfrak{M} \vDash \psi_{st}[v, t]$ and t is the unique $v \neq s$ with $\mathfrak{M} \vDash \psi_{st}[s, v]$. We claim that (s, t) is the only pair of indices in J with these properties. Hence, we can use c^{st} to code the pair (a^s, a^t) by the formula

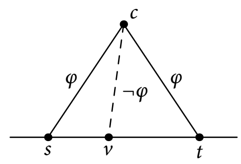
$$\begin{aligned}\vartheta(x, y, z) := {} & (\forall y'.Ay')(\psi(x, y', z) \leftrightarrow y' = y) \\ & \wedge (\forall z'.Az')(\psi(x, y, z') \leftrightarrow z' = z)\,,\end{aligned}$$

where $A := \{ a^v \mid v \in J \}$.

To prove our claim suppose that J contains another pair (s', t') of such indices. Since between any two elements of J there is another one of I, we can find some $s'' \in I$ such that $s'' \neq s'$ and the order types of s, t, s', t' and s, t, s'', t' are the same. By indiscernibility, it follows that $\mathfrak{M} \vDash \psi_{st}[s', t']$ implies $\mathfrak{M} \vDash \psi_{st}[s'', t']$ in contradiction to the uniqueness of s'. $\square$

In the preceding lemma we have considered the case that the truth value of φ changes if we move the index v outside the interval $[s,t]$. The next lemma states a dual version of this result where we consider instead indices $v \in (s,t)$.

Lemma 2.3.3. *Let $(\bar{a}^v)_{v \in I}$ be an infinite indiscernible sequence over U. If there exist an element c, a U-class p, a formula φ, and indices $s < t$ such that*



$$\begin{aligned} &\mathfrak{M} \vDash \varphi(c, \bar{a}^s|_p) \wedge \varphi(c, \bar{a}^t|_p)\,, \\ &\mathfrak{M} \vDash \neg\varphi(c, \bar{a}^v|_p)\,, \quad \textit{for infinitely many } s < v < t\,, \end{aligned}$$

then $\mathfrak{M}$ admits coding.

Proof. W.l.o.g. assume that $\bar{a}^v|_p = \bar{a}^v$. By Ramsey's theorem and compactness, we may assume that $I = \mathbb{R}$ and

$$\begin{aligned} &\mathfrak{M} \vDash \neg\varphi(c, \bar{a}^v)\,, && \text{for all } s < v < t\,, \\ &\mathfrak{M} \vDash \varphi(c, \bar{a}^u) \leftrightarrow \varphi(c, \bar{a}^v)\,, && \text{for all } u, v < s\,, \\ &\mathfrak{M} \vDash \varphi(c, \bar{a}^u) \leftrightarrow \varphi(c, \bar{a}^v)\,, && \text{for all } u, v > t\,. \end{aligned}$$

For $u < v$, fix an order isomorphism $\alpha_{uv} : I \to I$ with $\alpha(s) = u$ and $\alpha(t) = v$ and let π_{uv} be an U-automorphism such that $\pi_{uv}(\bar{a}^x) = \bar{a}^{\alpha(x)}$. We set $c^{uv} := \pi_{uv}(c)$. Fix a partition $I = I_0 \cup I_1$ into infinite sets I_0 and I_1 with $I_0 < I_1$, $s \in I_0$ and $t \in I_1$.

First, consider the case that $\mathfrak{M} \vDash \varphi(c, \bar{a}^v)$, for all $v < s$. We can define the order of $(\bar{a}^v)_{v \in I_0}$ by

$$\vartheta(\bar{x}, \bar{y}) := (\forall z.Cz)[\varphi(z, \bar{x}) \to \varphi(z, \bar{y})]\,,$$

where $C := \{\, c^{uv} \mid u \in I_0,\ v \in I_1 \,\}$. Let $\chi(\bar{x})$ be a formula with monadic parameters such that

$$\mathfrak{M} \vDash \chi(\bar{b}) \quad \text{implies} \quad \bar{b} = \bar{a}^v\,, \text{ for some } v \in I_0\,.$$

For the formula

$$\psi(z, \bar{x}) := \varphi(z, \bar{x}) \wedge \forall \bar{y}[\chi(\bar{y}) \wedge \vartheta(\bar{x}, \bar{y}) \to \neg\varphi(z, \bar{y})]$$

we have

$$[\![\psi(c^{st}, \bar{a}^v)]\!]_{v \in I} = [\![\varphi(c^{st}, \bar{a}^v)]\!]_{v \in I} \smallsetminus (-\infty, s)\,, \quad \text{for } s \in I_0 \text{ and } t \in I_1\,.$$

Similarly, if $\mathfrak{M} \vDash \varphi(c^{st}, \bar{a}^v)$, for all $v > t$, then we can construct a formula ψ such that

$$[\![\psi(c^{st}, \bar{a}^v)]\!]_{v \in I} = [\![\varphi(c^{st}, \bar{a}^v)]\!]_{v \in I} \smallsetminus (t, \infty)\,.$$

Consequently, we can assume that

$$[\![\varphi(c^{st}, \bar{a}^v)]\!]_v = \{s, t\}\,, \quad \text{for all } s \in I_0 \text{ and } t \in I_1\,.$$

For all $s, u \in I_0$ and $t, v \in I_1$, it follows that

$$\mathfrak{M} \models \varphi(c^{st}, \bar{a}^u) \wedge \varphi(c^{st}, \bar{a}^v) \quad \text{iff} \quad u = s \text{ and } v = t\,.$$

Let $\chi(\bar{x})$ be a formula with monadic parameters such that

$$\mathfrak{M} \models \chi(\bar{b}) \quad \text{implies} \quad \bar{b} = \bar{a}^v\,, \text{ for some } v \in I_0\,.$$

It follows that the formula

$$\psi(x, y, z) := \exists \bar{x}' \exists \bar{y}' [\chi(x\bar{x}') \wedge \chi(y\bar{y}') \wedge \varphi(z, x\bar{x}') \wedge \varphi(z, y\bar{y}')]\,.$$

defines the bijection $(a_0^u, a_0^v) \mapsto c^{uv}$, for $u \in I_0$ and $v \in I_1$. ☐

For sequences $(\bar{a}^v)_v$ with a single U-class, it follows that, in the absence of coding, the structure of sets of the form $[\![\varphi(c, \bar{a}^v)]\!]_v$ is quite simple.

Corollary 2.3.4. *Suppose that $\mathfrak{M}$ does not admit coding and let $(\bar{a}^v)_{v \in I}$ be an indiscernible sequence over U where the order I has no minimal and no maximal element.*

For every U-class p, each element c, and all formulae $\varphi(x, \bar{y})$ over U, one of the following cases holds:

- $\big|[\![\varphi(c, \bar{a}^v|_p)]\!]_v\big| \leq 1$
- $\big|[\![\neg\varphi(c, \bar{a}^v|_p)]\!]_v\big| \leq 1$
- $[\![\varphi(c, \bar{a}^v|_p)]\!]_v$ *is an initial segment of I.*
- $[\![\varphi(c, \bar{a}^v|_p)]\!]_v$ *is a final segment of I.*

Proof. We simplify notation by setting $[\![\varphi]\!] := [\![\varphi(c, \bar{a}^v|_p)]\!]_v$ and similarly for $[\![\neg\varphi]\!]$. Suppose that $[\![\varphi]\!]$ and $[\![\neg\varphi]\!]$ both contain at least two elements. We consider three cases.

(a) Suppose that, for every $v \in I$ there are elements $u, u' \in [\![\varphi]\!]$ with $u < v < u'$. We fix indices $s, t \in [\![\neg\varphi]\!]$ with $s < t$. The formula

$$\psi(z, \bar{x}, \bar{y}) := \neg\varphi(z, \bar{x}) \wedge \neg\varphi(z, \bar{y})$$

and the indices $s < t$ satisfy the conditions of Lemma 2.3.2. Hence, $\mathfrak{M}$ admits coding. A contradiction.

(b) If, for every $v \in I$, there are elements $u, u' \in [\![\neg\varphi]\!]$ with $u < v < u'$ then we obtain a contradiction as in (a) by exchanging φ and $\neg\varphi$.

(c) It follows that there are indices $s \leq t$ such that either

$$(-\infty, s) \subseteq [\![\varphi]\!] \quad \text{and} \quad (t, \infty) \subseteq [\![\neg\varphi]\!]\,,$$

$$\text{or} \qquad (-\infty, s) \subseteq [\![\neg\varphi]\!] \quad \text{and} \quad (t, \infty) \subseteq [\![\varphi]\!] \,.$$

By symmetry, we may assume the former. If $s = t$ then we are done.

For a contradiction, suppose that there are elements $s \leq u < v \leq t$ with $u \in [\![\neg\varphi]\!]$ and $v \in [\![\varphi]\!]$. By indiscernibility and compactness, we may assume that I is dense. If $(u, v) \cap [\![\varphi]\!]$ is infinite then $\neg\varphi$ and the pair $u < t$ satisfy the conditions of Lemma 2.3.3. Otherwise, $(u, v) \cap [\![\neg\varphi]\!]$ is infinite and φ and the pair $s < v$ satisfy these conditions. In both cases it follows that $\mathfrak{M}$ admits coding. Contradiction. □

Remark. If the order I in the corollary is (Dedekind) complete then we can rephrase the statement as follows: there exists an index $s \in I$ and a relation $\sigma \in \{\varnothing, I \times I, =, \neq, \leq, \geq, <, >\}$ such that

$$\mathfrak{M} \vDash \varphi(c, \bar{a}^v|_p) \quad \text{iff} \quad v \,\sigma\, s \,.$$

In the remainder of this section we generalise this result. We start by considering formulae $\varphi(c, \bar{a}[\bar{v}])$ talking about several elements of the sequence. Then we generalise the results to the case of several U-classes.

Lemma 2.3.5. *Suppose that $\mathfrak{M}$ does not admit coding. Let $(\bar{a}^v)_{v \in I}$ be an indiscernible sequence over U and p a U-class. For every element $c \in \mathbb{M}$, there exists a linear order $J \supseteq I$, an element $s \in J$, and an indiscernible sequence $(\bar{b}^v)_{v \in J}$ over U such that $\bar{b}^v = \bar{a}^v|_p$, for $v \in I$, and*

$$\mathfrak{M} \vDash \varphi(c, \bar{b}[\bar{u}]) \leftrightarrow \varphi(c, \bar{b}[\bar{v}]) \,,$$

for every formula φ over U and all indices $\bar{u}, \bar{v} \subseteq J$ with $\mathrm{ord}(s\bar{u}) = \mathrm{ord}(s\bar{v})$.

Proof. Replacing $\bar{a}^v$ by $\bar{a}^v|_p$ we may assume that p is the only U-class. Let J be a (Dedekind) complete dense order extending I and let $(\bar{b}^v)_{v \in J}$ be an indiscernible sequence extending $(\bar{a}^v)_{v \in I}$.

If $(\bar{b}^v)_v$ is indiscernible over $U \cup \{c\}$ then there is nothing to do. Otherwise, there are a formula φ and tuples $\bar{u}, \bar{v} \subseteq J$ with $\mathrm{ord}(\bar{u}) = \mathrm{ord}(\bar{v})$ such that

$$\mathfrak{M} \vDash \varphi(c, \bar{b}[\bar{u}]) \wedge \neg\varphi(c, \bar{b}[\bar{v}]) \,.$$

We can choose $\bar{u}$ and $\bar{v}$ such that there is exactly one index i with $u_i \neq v_i$. Hence, we may assume w.l.o.g. that $\bar{u} = u\bar{r}^0\bar{r}^1$ and $\bar{v} = v\bar{r}^0\bar{r}^1$ where

$$r^0_0 < \cdots < r^0_{m-1} < u < v < r^1_0 < \cdots < r^1_{l-1} \,.$$

Fix the interval $J_0 := (r^0_{m-1}, r^1_0) \subseteq J$. The sequence $(\bar{b}^v)_{v \in J_0}$ is indiscernible over $U \cup \bar{b}[\bar{r}^0\bar{r}^1]$. We can apply Corollary 2.3.4 to the element c and the sequence $(\bar{b}^v)_{v \in J_0}$ to find an index $s \in J_0$ and a relation $\sigma \in \{=, \neq, <, \leq\}$ such that

$$\mathfrak{M} \vDash \varphi(c, \bar{b}^x, \bar{b}[\bar{r}^0 \bar{r}^1]) \quad \text{iff} \quad x \,\sigma\, s\,, \quad \text{for all } x \in J_0\,.$$

We claim that s is the desired index.

Suppose otherwise. Then there is some formula ψ and indices $\bar{u}, \bar{v} \subseteq J$ with $\operatorname{ord}(s\bar{u}) = \operatorname{ord}(s\bar{v})$ such that

$$\mathfrak{M} \vDash \psi(c, \bar{b}[\bar{u}]) \wedge \neg\psi(c, \bar{b}[\bar{v}])\,.$$

Again we may assume that $\bar{u} = u\bar{r}^2\bar{r}^3$ and $\bar{v} = v\bar{r}^2\bar{r}^3$ with $r^2_0 < \cdots < r^2_{m'-1} < u < v < r^3_0 < \cdots < r^3_{l'-1}$. Let $J_1 := (r^2_{m'-1}, r^3_0) \subseteq J$. As above there is some index $t \in J_1$ and some $\rho \in \{=, \neq, <, \leq\}$ such that

$$\mathfrak{M} \vDash \psi(c, \bar{b}^x, \bar{b}[\bar{r}^2 \bar{r}^3]) \quad \text{iff} \quad x \,\rho\, t\,.$$

$\operatorname{ord}(su) = \operatorname{ord}(sv)$ and $u \leq t \leq v$ implies that $t \neq s$. Hence, there exist infinite convex subsets $I_0 \subseteq J_0$ and $I_1 \subseteq J_1$ with $s \in I_0$ and $t \in I_1$ such that $I_0 \cap I_1 = \varnothing$, $\bar{r}^0\bar{r}^1 \cap I_1 = \varnothing$, and $\bar{r}^2\bar{r}^3 \cap I_0 = \varnothing$. Furthermore, there are formulae $\varphi'(x, \bar{y})$ and $\psi'(x, \bar{y})$ with monadic parameters such that

$$\begin{aligned}
&\mathfrak{M} \vDash \varphi'(c, \bar{b}^x) \quad \text{iff} \quad x = s\,, \qquad \text{for all } x \in I_0\,,\\
&\mathfrak{M} \vDash \psi'(c, \bar{b}^x) \quad \text{iff} \quad x = t\,, \qquad \text{for all } x \in I_1\,.
\end{aligned}$$

For $u \in I_0$ and $v \in I_1$, fix order isomorphisms $\alpha_u : I_0 \to I_0$ and $\beta_v : I_1 \to I_1$ with $\alpha_u(s) = u$ and $\beta_v(t) = v$. Let π_{uv} be a U-automorphism such that

$$\begin{aligned}
\pi_{uv}(\bar{b}^x) &= \bar{b}^{\alpha_u(x)} && \text{for } x \in I_0\,,\\
\pi_{uv}(\bar{b}^x) &= \bar{b}^{\beta_v(x)} && \text{for } x \in I_1\,,\\
\pi_{uv}(\bar{b}^x) &= \bar{b}^x && \text{for } x \in J \setminus (I_0 \cup I_1)\,,
\end{aligned}$$

and set $c^{uv} := \pi_{uv}(c)$. For $u, s \in I_0$ and $v, t \in I_1$, it follows that

$$\mathfrak{M} \vDash \varphi'(c^{uv}, \bar{b}^s) \wedge \psi'(c^{uv}, \bar{b}^t) \quad \text{iff} \quad u = s \text{ and } v = t\,.$$

Contradiction. □

Lemma 2.3.6. *Suppose that $\mathfrak{M}$ does not admit coding. Let $(\bar{a}^v)_{v \in I}$ be an indiscernible sequence over U, $c \in \mathbb{M}$ an element, $\varphi(z, \bar{x}^0, \ldots, \bar{x}^{m-1})$ a formula over U, and p a U-class. Set*

$$\varphi[c, \bar{v}] := \varphi(c, \bar{a}^{v_0}|_p, \ldots, \bar{a}^{v_{m-1}}|_p)\,.$$

If there are indices $\bar{u}, \bar{v} \in I^m$ such that

$$\mathfrak{M} \vDash \varphi[c, \bar{u}] \wedge \neg\varphi[c, \bar{v}]$$

then there either exists a formula $\vartheta(\bar{x},\bar{y})$ *(with monadic parameters) such that*

$$\mathfrak{M} \vDash \vartheta(\bar{a}^x|_p, \bar{a}^y|_p) \quad \text{iff} \quad x \leq y\,,$$

or there exist an index $s \in I$ *such that* $\operatorname{equ}(s\bar{x}) = \operatorname{equ}(s\bar{y})$ *implies*

$$\mathfrak{M} \vDash \varphi[c,\bar{x}] \leftrightarrow \varphi[c,\bar{y}]\,.$$

Proof. By Lemma 2.3.5, there is an index s such that the truth value of $\varphi[c,\bar{x}]$ only depends on $\operatorname{ord}(s\bar{x})$. Suppose that there are indices $\bar{u},\bar{v} \in I^m$ with $\operatorname{equ}(s\bar{u}) = \operatorname{equ}(s\bar{v})$ and

$$\mathfrak{M} \vDash \varphi[c,\bar{u}] \wedge \neg\varphi[c,\bar{v}]\,.$$

We construct a formula ϑ that defines the ordering of I. By adding unused variables to φ we may assume that $s \in \bar{u}$. Furthermore, by changing φ we may assume that $u_i \neq u_k$ and $v_i \neq v_k$, for $i \neq k$. Let k be the minimal index such that

$$\mathfrak{M} \vDash \neg\varphi[c, v_0 \dots v_k u_{k+1} \dots u_{m-1}]\,.$$

Since

$$\begin{aligned}\mathfrak{M} \vDash\; &\varphi[c, v_0 \dots v_{k-1} u_k u_{k+1} \dots u_{m-1}] \\ &\wedge \neg\varphi[c, v_0 \dots v_{k-1} v_k u_{k+1} \dots u_{m-1}]\end{aligned}$$

we may assume that there is some index k such that $u_i = v_i$, for $i \neq k$. W.l.o.g. assume that $k = 0$ and that $u_0 < v_0$. Since $\operatorname{ord}(\bar{u}) \neq \operatorname{ord}(\bar{v})$ there must be at least one index $k > 0$ with $u_0 < u_k < v_0$. By a similar argument as above we may assume that there is exactly one such index. Hence, we may assume that

$$\bar{u} = ut\bar{r}_0\bar{r}_1 \quad \text{and} \quad \bar{v} = vt\bar{r}_0\bar{r}_1 \quad \text{where } \bar{r}_0 < u < t < v < \bar{r}_1\,.$$

We consider two cases.

(a) Suppose that $t \neq s$. Then $\operatorname{equ}(s\bar{u}) = \operatorname{equ}(s\bar{v})$ implies that $s \in \bar{r}_0\bar{r}_1$. Since $\operatorname{ord}(vu\bar{r}_0\bar{r}_1) = \operatorname{ord}(vt\bar{r}_0\bar{r}_1)$ it follows that

$$\mathfrak{M} \vDash \varphi[c, uv\bar{r}_0\bar{r}_1] \wedge \neg\varphi[c, vu\bar{r}_0\bar{r}_1]\,.$$

Fix a linear order $J \supseteq I$ and a strictly increasing function $\alpha : I \to J$ such that $\alpha(\bar{r}_0) < I < \alpha(\bar{r}_1)$. Let $(\bar{b}^v)_{v\in J}$ be an indiscernible sequence extending $(\bar{a}^v)_{v\in I}$ and fix a U-automorphism π such that $\pi(\bar{a}^x) = \bar{b}^{\alpha(x)}$. We set $d := \pi(c)$. For $x, y \in I$ with $x \neq y$ it follows that

$$\mathfrak{M} \vDash \varphi[d, xy\alpha(\bar{r}_0)\alpha(\bar{r}_1)] \quad \text{iff} \quad x < y\,.$$

Hence, we can define

$$\vartheta(\bar{x}, \bar{y}) := \bar{x} = \bar{y} \vee \varphi(d, \bar{x}, \bar{y}, \bar{b}[\alpha(\bar{r}_0\bar{r}_1)]) .$$

(b) It remains to consider the case that $t = s$. Then we have

$$\mathfrak{M} \vDash \varphi[c, us\bar{r}_0\bar{r}_1] \wedge \neg\varphi[c, vs\bar{r}_0\bar{r}_1] .$$

Fix a linear order $J \supseteq I$, tuples $\bar{w}_0, \bar{w}_1 \subseteq J$ with $\bar{w}_0 < I < \bar{w}_1$, and an indiscernible sequence $(\bar{b}^v)_{v\in J}$ extending $(\bar{a}^v)_{v\in J}$. For each $t \in I$, let $\alpha_t : I \to J$ be an order embedding such that $\alpha_t(s\bar{r}_0\bar{r}_1) = t\bar{w}_0\bar{w}_1$ and choose a U-automorphism π_t with $\pi(\bar{a}^x) = \bar{b}^{\alpha(x)}$. Setting $c^t := \pi_t(c)$ it follows, for $x \neq t$, that

$$\mathfrak{M} \vDash \varphi[c^t, xt\bar{w}_0\bar{w}_1] \quad \text{iff} \quad x < t .$$

By Theorem 2.1.19, there is a formula χ (with monadic parameters) such that

$$\mathfrak{M} \vDash \chi(c, \bar{a}) \quad \text{iff} \quad \bar{a} = \bar{a}^x \text{ and } c = c^x \text{ , for some } x \in I .$$

If we define

$$\vartheta(\bar{x}, \bar{y}) := \bar{x} = \bar{y} \vee \exists z\big(\chi(z, \bar{y}) \wedge \varphi(z, \bar{x}, \bar{y}, \bar{b}[\bar{w}_0\bar{w}_1])\big)$$

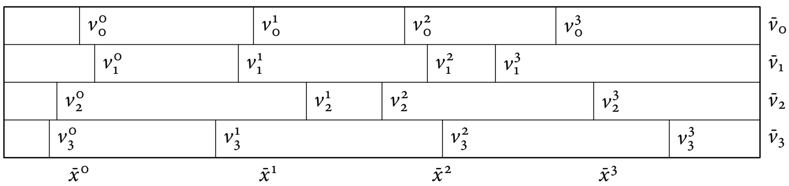
it follows that

$$\mathfrak{M} \vDash \vartheta(\bar{a}^x, \bar{a}^y) \quad \text{iff} \quad x \leq y . \qquad \square$$

Next we consider the case that there are several U-classes. The following lemma roughly states that, when adding an element c to U, the partition into U-classes does not change.

Lemma 2.3.7. *Let $(\bar{a}^v)_{v\in I}$ be an infinite indiscernible sequence over U, $c \in \mathbb{M}$ be an element, $\varphi(z, \bar{x}^0, \dots, \bar{x}^{m-1})$ a formula over U, and let $p_0, \dots, p_{k-1}$ be the U-classes corresponding to the variables in $\bar{x}^0, \dots, \bar{x}^{m-1}$. For indices $\bar{v}_0, \dots, \bar{v}_{k-1} \in I^m$, we set*

$$\varphi[c, \bar{v}_0, \dots, \bar{v}_{k-1}] := \\ \qquad \varphi\big(c, \bar{a}^{v_0^0}|_{p_0} \dots \bar{a}^{v_{k-1}^0}|_{p_{k-1}}, \dots, \bar{a}^{v_0^{m-1}}|_{p_0} \dots \bar{a}^{v_{k-1}^{m-1}}|_{p_{k-1}}\big) .$$



If there are indices $\bar{u}_0, \bar{u}_1, \bar{v}_0, \dots, \bar{v}_{k-1} \in I^m$ *such that* $\operatorname{ord}(\bar{u}_i) = \operatorname{ord}(\bar{v}_i)$, *for* $i < 2$, *and*

$$\begin{aligned}
&\mathfrak{M} \vDash \varphi[c, \bar{v}_0, \bar{v}_1, \bar{v}_2, \dots, \bar{v}_{k-1}] \\
&\mathfrak{M} \vDash \neg\varphi[c, \bar{u}_0, \bar{v}_1, \bar{v}_2, \dots, \bar{v}_{k-1}] \\
&\mathfrak{M} \vDash \neg\varphi[c, \bar{v}_0, \bar{u}_1, \bar{v}_2, \dots, \bar{v}_{k-1}]
\end{aligned}$$

then $\mathfrak{M}$ *admits coding.*

Proof. For a contradiction, suppose that $\mathfrak{M}$ does not admit coding. Since $(\bar{a}^v|_{p_0 \cup p_1})_v$ is indiscernible over $U \cup \bar{a}|_{p_2 \cup \dots \cup p_{k-1}}[I]$ we may w.l.o.g. assume that $k = 2$. Further, note that the sequence $(\bar{a}^v|_{p_0})_v$ is indiscernible over $U \cup \bar{a}|_{p_1}[I]$.

For fixed $\bar{y} \in I^m$, there are two cases. The truth value of $\varphi[c, \bar{x}, \bar{y}]$ might only depend on $\operatorname{ord}(\bar{x})$. Otherwise, we may assume, by Lemma 2.3.5, that there exists a unique index $s(\bar{y})$ such that the truth value of $\varphi[c, \bar{x}, \bar{y}]$ only depends on $\operatorname{ord}(\bar{x}s(\bar{y}))$. Similarly, if, for $\bar{x} \in I^m$, $\varphi[c, \bar{x}, \bar{y}]$ depends on more that just $\operatorname{ord}(\bar{y})$ then there exists a unique index $t(\bar{x})$ such that the truth value of $\varphi[c, \bar{x}, \bar{y}]$ only depends on $\operatorname{ord}(\bar{y}t(\bar{x}))$.

By compactness, we may assume that $I = \mathbb{R}$. For every pair of order automorphisms $\alpha, \beta : I \to I$, fix a U-automorphism $\pi_{\alpha\beta}$ such that

$$\begin{aligned}
\pi_{\alpha\beta}(\bar{a}^v|_{p_0}) &= \bar{a}^{\alpha(v)}|_{p_0}\,, \\
\pi_{\alpha\beta}(\bar{a}^v|_{p_1}) &= \bar{a}^{\beta(v)}|_{p_1}\,.
\end{aligned}$$

First, we prove that we have $s(\bar{y}) = s(\bar{y}')$, for all $\bar{y}, \bar{y}' \in I^m$ such that $s(\bar{y})$ and $s(\bar{y}')$ are defined. For a contradiction, suppose that $s(\bar{y}) < s(\bar{y}')$. For $u < v$ in I let $\alpha_{uv} : I \to I$ be an order isomorphism such that $\alpha_{uv}(s(\bar{y})) = u$ and $\alpha_{uv}(s(\bar{y}')) = v$, and set $c^{uv} := \pi_{\alpha_{uv},\mathrm{id}}(c)$. We construct formulae $\psi(z, \bar{x})$ and $\psi'(z, \bar{x})$ (with monadic parameters) such that

$$\begin{aligned}
&\mathfrak{M} \vDash \psi(c, \bar{a}^v|_{p_0}) \quad \text{iff} \quad v = s(\bar{y})\,, \\
\text{and} \quad &\mathfrak{M} \vDash \psi'(c, \bar{a}^v|_{p_0}) \quad \text{iff} \quad v = s(\bar{y}')\,.
\end{aligned}$$

Let χ_{p_0} be the formula from Theorem 2.1.19 defining the relation $\{\, \bar{a}^v|_{p_0} \mid v \in I \,\}$. If the linear ordering on the sequence $(\bar{a}^v|_{p_0})_{v \in I}$ is definable by a formula over $U \cup \{c\} \cup \bar{a}|_{p_1 \cup \dots \cup p_{m-1}}[I]$ then we can define $\psi(z, \bar{x})$ by

$$\begin{aligned}
\chi_{p_0}(\bar{x}) \wedge \forall \bar{u}^0 \cdots \forall \bar{u}^{m-1} \forall \bar{v}^0 \cdots \forall \bar{v}^{m-1} \\
\Big[\bigwedge_{i<m} (\chi_{p_0}(\bar{u}^i) \wedge \chi_{p_0}(\bar{v}^i)) \\
\wedge \operatorname{ord}(\bar{x}\bar{u}^0 \dots \bar{u}^{m-1}) = \operatorname{ord}(\bar{x}\bar{v}^0 \dots \bar{v}^{m-1}) \\
\to (\varphi'(z, \bar{u}^0, \dots, \bar{u}^{m-1}) \leftrightarrow \varphi'(z, \bar{v}^0, \dots, \bar{v}^{m-1})) \Big]
\end{aligned}$$

where $\varphi'(z, \bar{x}^0, \dots, \bar{x}^{m-1})$ is an abbreviation for

$$\varphi\big(z,\ \bar{x}^0,\ \bar{a}^{y_0}|_{p_1}, \dots,\ \bar{x}^{m-1},\ \bar{a}^{y_{m-1}}|_{p_1}\big)\,.$$

If the ordering is not definable then it follows by Lemma 2.3.6 that the truth value of $\varphi[c, \bar{u}, \bar{y}]$ only depends on $\operatorname{equ}(s\bar{u})$. In this case we can replace the condition $\operatorname{ord}(\bar{x}\bar{u}^0 \dots) = \operatorname{ord}(\bar{x}\bar{v}^0 \dots)$ in the above formula by the formula

$$\operatorname{equ}(\bar{x}\bar{u}^0 \dots \bar{u}^{m-1}) = \operatorname{equ}(\bar{x}\bar{v} \dots \bar{v}^{m-1})\,.$$

The formula $\psi'(z, \bar{x})$ is defined analogously. It follows that

$$\mathfrak{M} \models \psi(c^{uv}, \bar{a}^x|_{p_0}) \wedge \psi'(c^{uv}, \bar{a}^y|_{p_0}) \quad \text{iff} \quad x = u \text{ and } y = v\,.$$

Fixing disjoint intervals $I_0, I_1 \subseteq I$ with $I_0 < I_1$ we obtain a definable bijection $\bar{a}|_{p_0}[I_0] \times \bar{a}|_{p_0}[I_1] \to \{\, c^{uv} \mid u \in I_0, v \in I_1 \,\}$. Contradiction.

In the same way it follows that $t(\bar{x}) = t(\bar{x}')$ if these values are defined. By assumption, there are indices $\bar{x} := \bar{v}_0$ and $\bar{y} := \bar{v}_1$ such that $s(\bar{y})$ and $t(\bar{x})$ are defined. Let us denote these values by s and t. As above we can construct formulae $\vartheta_0(z, \bar{x})$ and $\vartheta_1(z, \bar{y})$ such that

$$\mathfrak{M} \models \vartheta_0(c, \bar{a}^x|_{p_0}) \quad \text{iff} \quad x = s\,,$$
$$\text{and} \quad \mathfrak{M} \models \vartheta_1(c, \bar{a}^y|_{p_1}) \quad \text{iff} \quad y = t\,.$$

For $u, v \in I$, Let $\alpha_u, \beta_v : I \to I$ be order isomorphisms such that $\alpha_u(s) = u$ and $\beta_v(t) = v$, and set $c^{uv} := \pi_{\alpha_u, \beta_v}$. It follows that

$$\mathfrak{M} \models \vartheta_0(c^{uv}, \bar{a}^x|_{p_0}) \wedge \vartheta_1(c^{uv}, \bar{a}^y|_{p_1}) \quad \text{iff} \quad x = u \text{ and } y = v\,.$$

Consequently, $\mathfrak{M}$ admits coding. ☐

Lemma 2.3.8. *Suppose that $\mathfrak{M}$ does not admit coding. Let $(\bar{a}^v)_{v \in I}$ be an indiscernible sequence over U. For every element c such that $(\bar{a}^v)_v$ is not indiscernible over $U \cup \{c\}$, there exist a linear order $J \supseteq I$, an indiscernible sequence $(\bar{b}^v)_{v \in J}$ with $\bar{b}^v = \bar{a}^v$, for $v \in I$, and a unique index $s \in J$ such that*

$$\mathfrak{M} \models \varphi(c, \bar{b}[\bar{u}]) \leftrightarrow \varphi(c, \bar{b}[\bar{v}])\,,$$

for all formulae φ over U and all tuples $\bar{u}, \bar{v} \subseteq J$ with $\operatorname{ord}(s\bar{u}) = \operatorname{ord}(s\bar{v})$.

Proof. Let $\alpha := |\bar{a}^v|$. By Lemma 2.3.7, there is a U-class p such that the sequence $(\bar{a}^v|_{\alpha \setminus p})_v$ is indiscernible over $U \cup \bar{a}|_p[I] \cup \{c\}$. Furthermore, by Lemma 2.3.5 there exists a linear order $J \supseteq I$, an indiscernible sequence $(\bar{b}^v)_{v \in J}$ with $\bar{b}^v = \bar{a}^v$, for $v \in I$, and an index $s \in J$ such that

$$\mathfrak{M} \models \varphi(c, \bar{b}|_p[\bar{u}]) \leftrightarrow \varphi(c, \bar{b}|_p[\bar{v}])\,,$$

for all formulae φ over $U \cup \bar{b}|_{\alpha \setminus p}[J]$ and all indices $\bar{u}, \bar{v} \subseteq J$ with $\operatorname{ord}(s\bar{u}) = \operatorname{ord}(s\bar{v})$. It follows that

$$\mathfrak{M} \vDash \varphi(c, \bar{b}[\bar{u}]) \leftrightarrow \varphi(c, \bar{b}[\bar{v}]),$$

for all formulae φ over U and all indices $\bar{u}, \bar{v} \subseteq J$ with $\operatorname{ord}(s\bar{u}) = \operatorname{ord}(s\bar{v})$. $\square$

It follows that we can generalise Corollary 2.3.4 to sequences with several U-classes.

Corollary 2.3.9. *Suppose that $\mathfrak{M}$ does not admit coding and let $(\bar{a}^v)_{v \in I}$ be an indiscernible sequence over U where the order I has no minimal and no maximal element.*

For each element c and all formulae $\varphi(x, \bar{y})$ over U, one of the following cases holds:

- $\big|[\![\varphi(c, \bar{a}^v)]\!]_v\big| \leq 1$
- $\big|[\![\neg\varphi(c, \bar{a}^v)]\!]_v\big| \leq 1$
- $[\![\varphi(c, \bar{a}^v)]\!]_v$ *is an initial segment of I.*
- $[\![\varphi(c, \bar{a}^v)]\!]_v$ *is a final segment of I.*

Combining the preceding lemmas we finally obtain the main result of this section. The next theorem states that we can extend each indiscernible sequence to cover every given element.

Theorem 2.3.10. *Suppose that $\mathfrak{M}$ does not admit coding. Let $(\bar{a}^v)_{v \in I}$ be an indiscernible sequence over U. For every element c, there exist a linear order $J \supseteq I$ and an indiscernible sequence $(\bar{b}^v c^v)_{v \in J}$ over U such that $\bar{b}^v = \bar{a}^v$, for $v \in I$, and $c = c^v$, for some $v \in J$.*

Proof. W.l.o.g. assume that I is infinite and (Dedekind) complete. If $(\bar{a}^v)_v$ is indiscernible over c then we can set $c^v := c$, for all v. Otherwise, it follows by Lemma 2.3.8 that there exist a linear order $J \supseteq I$, an indiscernible sequence $(\bar{b}^v)_{v \in J}$ with $\bar{b}^v = \bar{a}^v$, for $v \in I$, and a unique index $s \in J$ such that

$$\mathfrak{M} \vDash \varphi(c, \bar{b}[\bar{u}]) \leftrightarrow \varphi(c, \bar{b}[\bar{v}]),$$

for all formulae φ over U and all tuples $\bar{u}, \bar{v} \subseteq J$ such that $\operatorname{ord}(s\bar{u}) = \operatorname{ord}(s\bar{v})$.

For $u \in J$, let $\alpha_u : J \to J$ be an order isomorphism with $\alpha_u(s) = u$. Choose U-automorphisms π_u with $\pi_u(\bar{b}^v) = \bar{b}^{\alpha_u(v)}$ and set $c^u := \pi_u(c)$.

Let Φ be the set of all formulae $\varphi(\bar{x}, \bar{y})$ such that, for some infinite subset $J_0 \subseteq J$, we have

$$\mathfrak{M} \vDash \varphi(\bar{b}[\bar{u}], c[\bar{u}]), \quad \text{for all increasing sequences } \bar{u} \subseteq J_0.$$

For every formula φ we have $\varphi \in \Phi$ or $\neg\varphi \in \Phi$, by Ramsey's theorem. Furthermore, Φ is closed under entailment. Let $\Psi \subseteq \Phi$ be a maximal consistent subset of Φ. If there were a formula φ with $\varphi \notin \Psi$ and $\neg\varphi \notin \Psi$ then $\Psi \cup \{\varphi\}$ and $\Psi \cup \{\neg\varphi\}$ were inconsistent. Hence, we would have $\Psi \vDash \neg\varphi$ and $\Psi \vDash \varphi$. This implies that $\Psi \vDash \varphi \wedge \neg\varphi$ and Ψ is inconsistent. Contradiction.

It follows that Ψ is a complete type. Let $(\hat{b}^v \hat{c}^v)_{v \in J}$ be a sequence realising Ψ. Since $\mathrm{tp}(\hat{c}^s/U \cup (\hat{b}^v)_v) = \mathrm{tp}(c/U \cup (\bar{b}^v)_v)$ there exists an U-isomorphism π with $\pi(\hat{c}^s) = c$ and $\pi(\hat{b}^v) = \bar{b}^v$, for all $v \in J$. It follows that the sequence $(\bar{b}^v \pi(\hat{c}^v))_v$ is the desired indiscernible sequence. □

By induction it follows that we can extend each indiscernible sequence to cover every given set of elements.

Corollary 2.3.11. *Suppose that $\mathfrak{M}$ does not admit coding. Let $(\bar{a}^v)_{v \in I}$ be an indiscernible sequence of α-tuples over U. For every set $C \subseteq \mathbb{M}$, there exist a linear order $J \supseteq I$ and an indiscernible sequence $(\bar{b}^v)_{v \in J}$ of β-tuples over U with $\beta \geq \alpha$ such that $C \subseteq \bar{b}[J]$ and $\bar{a}^v = \bar{b}^v|_\alpha$, for $v \in I$.*

We conclude this section by an investigation of the U-partition of a sequence of the form $(\bar{a}^v|_N)_v$, for an arbitrary set $N \subseteq \alpha$. We start by generalising Lemma 2.3.7.

Lemma 2.3.12. *Suppose that $(\bar{a}^v c^v)_{v \in I}$ is an infinite indiscernible sequence over U and let P be the minimal U-partition for the sequence $(\bar{a}^v)_{v \in I}$. Let $\varphi(\bar{z}, \bar{x}^0, \dots, \bar{x}^{m-1})$ be a formula over U and let $p_0, \dots, p_{k-1} \in P$ be the U-classes corresponding to the variables in $\bar{x}^0, \dots, \bar{x}^{m-1}$. For $\bar{t}, \bar{v}_0, \dots, \bar{v}_{k-1} \in I^m$, we set*

$$\begin{aligned}&\varphi[\bar{t}, \bar{v}_0, \dots, \bar{v}_{k-1}] :=\\ &\quad \varphi\big(c[\bar{t}], \bar{a}^{v_0^0}|_{p_0} \dots \bar{a}^{v_{k-1}^0}|_{p_{k-1}}, \dots, \bar{a}^{v_0^{m-1}}|_{p_0} \dots \bar{a}^{v_{k-1}^{m-1}}|_{p_{k-1}}\big).\end{aligned}$$

If there are indices $\bar{u}_0, \bar{u}_1, \bar{v}_0, \dots, \bar{v}_{k-1}, \bar{t} \in I^m$ such that $\mathrm{ord}(\bar{u}_i) = \mathrm{ord}(\bar{v}_i)$, for $i < 2$, and

$$\begin{aligned}&\mathfrak{M} \vDash \varphi[\bar{t}, \bar{v}_0, \bar{v}_1, \bar{v}_2, \dots, \bar{v}_{k-1}]\\ &\mathfrak{M} \vDash \neg\varphi[\bar{t}, \bar{u}_0, \bar{v}_1, \bar{v}_2, \dots, \bar{v}_{k-1}]\\ &\mathfrak{M} \vDash \neg\varphi[\bar{t}, \bar{v}_0, \bar{u}_1, \bar{v}_2, \dots, \bar{v}_{k-1}]\end{aligned}$$

then $\mathfrak{M}$ admits coding.

Proof. Replacing U by $U \cup \bar{a}|_{p_2 \cup \dots \cup p_{k-1}}[I]$ we may assume that $k = 2$. By assumption there are tuples $\bar{u}_0, \bar{u}_0', \bar{u}_1, \bar{u}_1' \in I^m$ such that

$$\mathrm{ord}(\bar{u}_i) = \mathrm{ord}(\bar{u}_i') = \mathrm{ord}(\bar{v}_i)$$

and

$$\begin{aligned}
\mathfrak{M} &\models \varphi[\bar{t}, \bar{u}_0, \bar{v}_1] \wedge \neg\varphi[\bar{t}, \bar{u}'_0, \bar{v}_1]\,, \\
\mathfrak{M} &\models \varphi[\bar{t}, \bar{v}_0, \bar{u}_1] \wedge \neg\varphi[\bar{t}, \bar{v}_0, \bar{u}'_1]\,.
\end{aligned}$$

As usual we may assume that $\bar{u}_i$ and $\bar{u}'_i$ differ only in one component. Thus, suppose that $\bar{u}_i = u_i \bar{r}_i$ and $\bar{u}'_i = u'_i \bar{r}_i$. Furthermore, we may assume that

$$|[u_i, u'_i] \cap \bar{t}| \leq 1$$

since, if $u_i \leq t_k < t_l \leq u'_i$ then we can replace either u_i or u'_i by some index between t_k and t_l. Hence, suppose that there are indices k and l such that

$$[u_0, u'_0] \cap \bar{t} \subseteq \{t_k\} \quad \text{and} \quad [u_1, u'_1] \cap \bar{t} \subseteq \{t_l\}\,.$$

Let α be an order isomorphism with $\alpha(t_l) = t_k$. W.l.o.g. suppose that $k = 0$ and let $\bar{t} = t_0 \bar{t}'$. It follows that

$$\mathfrak{M} \models \varphi[t_0 \alpha(\bar{t}'), \alpha(\bar{v}_0), \alpha(\bar{u}_0)] \wedge \neg\varphi[t_0 \alpha(\bar{t}'), \alpha(\bar{v}_0), \alpha(\bar{u}'_0)]\,.$$

Fix indices s_-, s_+ such that

$$s_- < u_0 u'_0 \alpha(u_1) \alpha(u'_1) < s_+ \quad \text{and} \quad (s_-, s_+) \cap \bar{t} = \{t_0\}\,,$$

and set $J := (s_-, s_+)$. The subsequence $(\bar{a}^\nu)_{\nu \in J}$ is indiscernible over the set $V := U \cup \bar{a}[I \smallsetminus J]$. Defining

$$\psi(z, \bar{x}_0 \bar{y}_0, \bar{x}_1 \bar{y}_1) := \varphi(z, c[\bar{t}'], \bar{x}_0, \bar{x}_1) \wedge \varphi(z, c[\alpha(\bar{t}')], \bar{y}_0, \bar{y}_1)$$

we obtain a formula over V such that

$$\begin{aligned}
\mathfrak{M} &\models \psi[c^{t_0}, \bar{u}_0 \alpha(\bar{v}_0), \bar{v}_1 \alpha(\bar{u}_1)]\,, \\
\mathfrak{M} &\models \neg\psi[c^{t_0}, \bar{u}'_0 \alpha(\bar{v}_0), \bar{v}_1 \alpha(\bar{u}_1)]\,, \\
\mathfrak{M} &\models \neg\psi[c^{t_0}, \bar{u}_0 \alpha(\bar{v}_0), \bar{v}_1 \alpha(\bar{u}'_1)]\,.
\end{aligned}$$

By Lemma 2.3.7 it follows that $\mathfrak{M}$ admits coding. □

It follows that the Δ-dependence of two indices i and k is a 'local' property since it only depends on the sequence $(a^\nu_i a^\nu_k)_\nu$, not on all of $(\bar{a}^\nu)_\nu$.

Theorem 2.3.13. *Suppose that $\mathfrak{M}$ does not admit coding. Let $(\bar{a}^\nu)_{\nu \in I}$ be an indiscernible sequence over U with $|\bar{a}^\nu| = \alpha$, and let $N \subseteq \alpha$. If P is the U-partition of $(\bar{a}^\nu)_\nu$ then the U-partition of $(\bar{a}^\nu|_N)_\nu$ is $\{\, p \cap N \mid p \in P \,\}$.*

Proof. It is sufficient to consider the case that $N = \alpha \setminus \{n\}$. Then the general case will follow by induction. Let P be the U-partition of $(\bar{a}^v|_N)_v$. Consider a formula $\varphi(z^0\bar{x}^0, \dots, z^{m-1}\bar{x}^{m-1})$ over U where the variables z^i correspond to a^v_n while $\bar{x}^i$ correspond to $\bar{a}^v|_N$. Let $p_0, \dots, p_{k-1} \subseteq N$ be the U-classes appearing in the variables $\bar{x}^i$. By Lemma 2.3.12, it follows that, for every $\bar{t} \in I^m$, there exists some class p_l such that the truth value of φ only depends on the class p_l, i.e.,

$$\begin{aligned}\mathfrak{M} \vDash \varphi\big(&a_n^{t_0}\bar{a}^{u^0_0}|_{p_0}\dots\bar{a}^{u^0_{k-1}}|_{p_{k-1}}, \dots, a_n^{t_{m-1}}\bar{a}^{u^{m-1}_0}|_{p_0}\dots\bar{a}^{u^{m-1}_{k-1}}|_{p_{k-1}}\big)\\ \leftrightarrow \varphi\big(&a_n^{t_0}\bar{a}^{v^0_0}|_{p_0}\dots\bar{a}^{v^0_{k-1}}|_{p_{k-1}}, \dots, a_n^{t_{m-1}}\bar{a}^{v^{m-1}_0}|_{p_0}\dots\bar{a}^{v^{m-1}_{k-1}}|_{p_{k-1}}\big),\end{aligned}$$

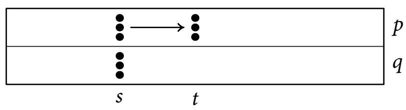
for all indices $\bar{u}^i, \bar{v}^i \in I^m$ with $\operatorname{ord}(\bar{u}^i) = \operatorname{ord}(\bar{v}^i)$, for $i < k$, and $\bar{u}^l = \bar{v}^l$. By indiscernibility, this index is the same for all $\bar{t}$. It follows that the U-class of n is either $\{n\}$ or $p_l \cup \{n\}$, while the other U-classes are p_j, $j \neq l$. ☐

2.4 The commutation order

We have seen that the relation $\asymp_U$ partitioning an indiscernible sequence into its U-classes is well-behaved for structures that do not admit coding. In this section we introduce a refinement of $\asymp_U$.

Definition 2.4.1. Suppose that $(\bar{a}^v)_{v \in I}$ is an indiscernible sequence of α-tuples over U. For sets $p, q \subseteq \alpha$ of indices, we define $p \trianglelefteq_U q$ iff, for some/all $s < t$ in I, we have

$$\operatorname{tp}\big(\bar{a}^s|_p\bar{a}^s|_q / U \cup \bar{a}[<s] \cup \bar{a}[>t]\big) \neq \operatorname{tp}\big(\bar{a}^t|_p\bar{a}^s|_q / U \cup \bar{a}[<s] \cup \bar{a}[>t]\big).$$



For single indices $i, k < \alpha$, we write $i \trianglelefteq_U k$ instead of $\{i\} \trianglelefteq_U \{k\}$.

Our aim in this section is to show that the relation $\trianglelefteq_U$ linearly preorders every U-class. We start by showing that the U-classes are exactly the connected components of this relation. Recall that $\asymp_U$ is the equivalence relation associated with the U-classes.

Lemma 2.4.2. *Let $(\bar{a}^v)_v$ be an indiscernible sequence of α-tuples over U. For $i, k < \alpha$, we have*

$$i \asymp_U k \quad \text{iff} \quad i \trianglelefteq_U k \text{ or } k \trianglelefteq_U i\,.$$

Proof. ($\Leftarrow$) follows immediately from the definition of $\asymp_U$.

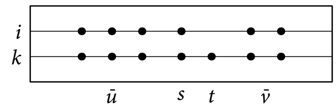
($\Rightarrow$) Suppose that $i \ntrianglelefteq_U k$ and $k \ntrianglelefteq_U i$. We have to show that $i \not\asymp_U k$, i.e.,

$$\mathrm{tp}(a_i[\bar{u}]a_k[\bar{v}]/U) = \mathrm{tp}(a_i[\bar{s}]a_k[\bar{t}]/U)\,,$$

for all $\bar{u}, \bar{v}, \bar{s}, \bar{t} \subseteq I$ with $\mathrm{ord}(\bar{u}) = \mathrm{ord}(\bar{s})$ and $\mathrm{ord}(\bar{v}) = \mathrm{ord}(\bar{t})$. As usual we only need to consider the case that $\bar{u}$ and $\bar{v}$ differ at only one component. Hence, consider indices

$$u_0 < \dots < u_{m-1} < s < t < v_0 < \dots < v_{n-1}\,.$$

It is sufficient to show that



$$\begin{aligned}\mathrm{tp}(a_i[\bar{u}s\bar{v}]a_k[\bar{u}t\bar{v}]/U) &= \mathrm{tp}(a_i[\bar{u}s\bar{v}]a_k[\bar{u}s\bar{v}]/U)\\ &= \mathrm{tp}(a_i[\bar{u}t\bar{v}]a_k[\bar{u}s\bar{v}]/U)\,.\end{aligned}$$

For the first equation, note that $i \ntrianglelefteq_U k$ implies

$$\mathrm{tp}(a_i^t a_k^s/U \cup a_i[\bar{u}\bar{v}] \cup a_k[\bar{u}\bar{v}]) = \mathrm{tp}(a_i^s a_k^s/U \cup a_i[\bar{u}\bar{v}] \cup a_k[\bar{u}\bar{v}])\,.$$

Similarly, $k \ntrianglelefteq_U i$ implies that

$$\mathrm{tp}(a_i^s a_k^t/U \cup a_i[\bar{u}\bar{v}] \cup a_k[\bar{u}\bar{v}]) = \mathrm{tp}(a_i^s a_k^s/U \cup a_i[\bar{u}\bar{v}] \cup a_k[\bar{u}\bar{v}])\,,$$

as desired. ☐

Lemma 2.4.3. *Let $(\bar{a}^v)_v$ be an indiscernible sequence over U.*

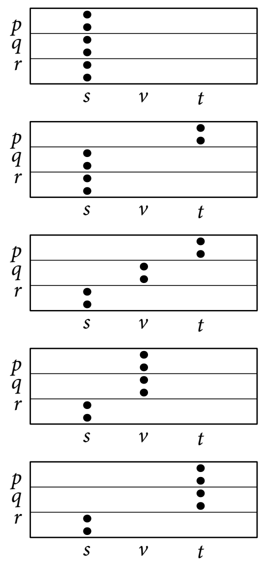
(a) *$p \trianglelefteq_U q$ implies that $p_+ \trianglelefteq_U q_+$, for all $p_+ \supseteq p$ and $q_+ \supseteq q$.*

(b) *If $p \ntrianglelefteq_U q \cup r$ and $q \ntrianglelefteq_U r$ then $p \cup q \ntrianglelefteq_U r$.*

(c) *If $p \cup q \ntrianglelefteq_U r$ and $p \ntrianglelefteq_U q$ then $p \ntrianglelefteq_U q \cup r$.*

Proof. (a) follows immediately from the definition.

(b) For $s < v < t$, we have



$$\begin{aligned}&\mathrm{tp}\big(\bar{a}^s|_p\bar{a}^s|_q\bar{a}^s|_r/U \cup \bar{a}[<s] \cup \bar{a}[>t]\big)\\ &= \mathrm{tp}\big(\bar{a}^t|_p\bar{a}^s|_q\bar{a}^s|_r/U \cup \bar{a}[<s] \cup \bar{a}[>t]\big) && (p \ntrianglelefteq_U q \cup r)\\ &= \mathrm{tp}\big(\bar{a}^t|_p\bar{a}^v|_q\bar{a}^s|_r/U \cup \bar{a}[<s] \cup \bar{a}[>t]\big) && (q \ntrianglelefteq_U r)\\ &= \mathrm{tp}\big(\bar{a}^v|_p\bar{a}^v|_q\bar{a}^s|_r/U \cup \bar{a}[<s] \cup \bar{a}[>t]\big) && (p \ntrianglelefteq_U q)\\ &= \mathrm{tp}\big(\bar{a}^t|_p\bar{a}^t|_q\bar{a}^s|_r/U \cup \bar{a}[<s] \cup \bar{a}[>t]\big)\,,\end{aligned}$$

as desired.

(c) For $s < v < t$, we have

$$\begin{aligned}
&\operatorname{tp}\big(\bar a^s|_p \bar a^s|_q \bar a^s|_r / U \cup \bar a[<s] \cup \bar a[>t]\big) \\
&= \operatorname{tp}\big(\bar a^v|_p \bar a^v|_q \bar a^s|_r / U \cup \bar a[<s] \cup \bar a[>t]\big) && (p \cup q \ntrianglelefteq_U r) \\
&= \operatorname{tp}\big(\bar a^t|_p \bar a^v|_q \bar a^s|_r / U \cup \bar a[<s] \cup \bar a[>t]\big) && (p \ntrianglelefteq_U q) \\
&= \operatorname{tp}\big(\bar a^t|_p \bar a^s|_q \bar a^s|_r / U \cup \bar a[<s] \cup \bar a[>t]\big)\,, && (q \ntrianglelefteq_U r)
\end{aligned}$$

as desired. □

Lemma 2.4.4. *Suppose that $\mathfrak{M}$ does not admit coding and let $(\bar a^v)_{v \in I}$ be an indiscernible sequence of α-tuples over U. Let $p, q \subseteq \alpha$ and $i \in \alpha$. If $p \ntrianglelefteq_U q$ then $p \cup \{i\} \ntrianglelefteq_U q$ or $p \ntrianglelefteq_U q \cup \{i\}$.*

Proof. W.l.o.g. assume that I is dense. Fix $s < t$ in I. Since $p \ntrianglelefteq_U q$ we have

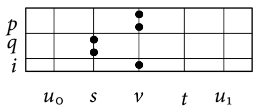
$$\operatorname{tp}(\bar a^t|_p \bar a^s|_q / U \cup \bar a[<s] \cup \bar a[>t]) = \operatorname{tp}(\bar a^s|_p \bar a^s|_q / U \cup \bar a[<s] \cup \bar a[>t])\,.$$

Hence, there exists an element $c \in \mathbb{M}$ such that

$$\begin{aligned}
&\operatorname{tp}(\bar a^t|_p \bar a^s|_q c / U \cup \bar a[<s] \cup \bar a[>t]) \\
&= \operatorname{tp}(\bar a^s|_p \bar a^s|_q a^s_i / U \cup \bar a[<s] \cup \bar a[>t])\,.
\end{aligned}$$

For a contradiction, suppose that $p \cup \{i\} \trianglelefteq_U q$ and $p \trianglelefteq_U q \cup \{i\}$. Then there are formulae $\varphi(\bar x, \bar y, z)$ and $\psi(\bar x, \bar y, z)$ over $U \cup \bar a[<s] \cup \bar a[>t]$ such that, for $s < v \leq t$,

$$\begin{aligned}
&\mathfrak{M} \vDash \varphi(\bar a^s|_p, \bar a^s|_q, a^s_i)\,, && \mathfrak{M} \vDash \psi(\bar a^s|_p, \bar a^s|_q, a^s_i)\,, \\
&\mathfrak{M} \nvDash \varphi(\bar a^v|_p, \bar a^s|_q, a^v_i)\,, && \mathfrak{M} \nvDash \psi(\bar a^v|_p, \bar a^s|_q, a^s_i)\,.
\end{aligned}$$



Let u_0 be the maximal index $u < s$ such that an element of $\bar a^u$ appears in φ or ψ, and let u_1 be the minimal index $u_1 > t$ appearing in φ or ψ. Then

$$\mathfrak{M} \nvDash \varphi(\bar a^t|_p, \bar a^s|_q, a^t_i) \quad \text{implies} \quad \mathfrak{M} \nvDash \varphi(\bar a^s|_p, \bar a^v|_q, a^s_i) \quad \text{for } u_0 < v < s\,.$$

Setting $\chi := \varphi \wedge \psi$ it follows that

$$\begin{aligned}
&\mathfrak{M} \vDash \chi(\bar a^s|_p, \bar a^s|_q, a^s_i)\,, \\
&\mathfrak{M} \nvDash \chi(\bar a^v|_p, \bar a^s|_q, a^s_i)\,, && \text{for } s < v < u_1\,, \\
&\mathfrak{M} \nvDash \chi(\bar a^s|_p, \bar a^v|_q, a^s_i)\,, && \text{for } u_0 < v < s\,.
\end{aligned}$$

By choice of c this implies that

$$\begin{aligned}
&\mathfrak{M} \vDash \chi(\bar a^t|_p, \bar a^s|_q, c)\,, \\
&\mathfrak{M} \nvDash \chi(\bar a^v|_p, \bar a^s|_q, c)\,, && \text{for } t < v < u_1\,, \\
&\mathfrak{M} \nvDash \chi(\bar a^t|_p, \bar a^v|_q, c)\,, && \text{for } u_0 < v < s\,.
\end{aligned}$$

Therefore, we can use Lemma 2.3.2 to conclude that $\mathfrak{M}$ admits coding. Contradiction. □

Corollary 2.4.5. *Suppose that $\mathfrak{M}$ does not admit coding and let $(\bar{a}^\nu)_\nu$ be an indiscernible sequence of α-tuples over U.*

(a) *$p \trianglelefteq_U i \trianglelefteq_U q$ implies $p \trianglelefteq_U q$, for $p, q \subseteq \alpha$ and $i \in \alpha$.*

(b) *$\trianglelefteq_U$ linearly preorders every U-class.*

Proof. (a) Suppose that $p \ntrianglelefteq_U q$. Then we have $p \cup \{i\} \ntrianglelefteq_U q$ or $p \ntrianglelefteq_U q \cup \{i\}$, by Lemma 2.4.4. In the former case, it follows by monotonicity that $i \ntrianglelefteq_U q$ while in the latter case we have $p \ntrianglelefteq_U i$.

(b) $\trianglelefteq_U$ is clearly reflexive. In (a) we have shown that it is transitive. Hence, $\trianglelefteq_U$ is a preorder. To show that it is linear on each U-class note that $i \asymp_U k$ implies $i \trianglelefteq_U k$ or $k \trianglelefteq_U i$. □

Corollary 2.4.6. *Suppose that $\mathfrak{M}$ does not admit coding and let $(\bar{a}^\nu)_\nu$ be an indiscernible sequence over U.*

(a) *$p \trianglelefteq_U q$ if and only if $i \trianglelefteq_U q$, for some $i \in p$.*

(b) *$i \trianglelefteq_U q$ if and only if $i \trianglelefteq_U k$, for some $k \in q$.*

(c) *$p \trianglelefteq_U q$ if and only if $i \trianglelefteq_U k$, for some $i \in p$ and $k \in q$.*

Proof. (a) By monotonicity it follows that $p \ntrianglelefteq_U q$ implies $i \ntrianglelefteq_U q$ for all $i \in p$. We prove the converse by induction on $|p|$. Suppose that $p \cup \{i\} \trianglelefteq_U q$. If $p \trianglelefteq_U q$ then the claim follows by induction hypothesis. Hence, we may assume that $p \ntrianglelefteq_U q$. Since $p \cup \{i\} \trianglelefteq_U q$ it follows by Lemma 2.4.4 that $p \ntrianglelefteq_U q \cup \{i\}$. If $i \ntrianglelefteq_U q$ then we would have $p \cup \{i\} \ntrianglelefteq_U q$, by Lemma 2.4.3 (b). Consequently, we have $i \trianglelefteq_U q$.

(b) The proof is analogous to (a). By monotonicity, $i \ntrianglelefteq_U q$ implies $i \ntrianglelefteq_U k$ for all $k \in q$. We prove the converse by induction on $|q|$. Suppose that $i \trianglelefteq_U q \cup \{k\}$. If $i \trianglelefteq_U q$ then the claim follows by induction hypothesis. Hence, we may assume that $i \ntrianglelefteq_U q$. By Lemma 2.4.4, it follows that $\{i, k\} \ntrianglelefteq_U q$. If $i \ntrianglelefteq_U k$ then we would have $i \ntrianglelefteq_U q \cup \{k\}$, by Lemma 2.4.3 (c). Consequently, we have $i \trianglelefteq_U k$.

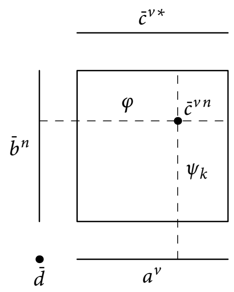
(c) follows immediately from (a) and (b). □

Since $\trianglelefteq_U$ is a preorder on each $\asymp_U$-class it follows that we can divide each U-class into the classes of this preorder which we call *strong U-classes.*

Definition 2.4.7. Suppose that $(\bar{a}^\nu)_{\nu \in I}$ is an indiscernible sequence of α-tuples over U. A *strong U-class* is an equivalence class for the relation

$$\{\langle i, k\rangle \in \alpha \times \alpha \mid i \trianglelefteq_U k \text{ and } k \trianglelefteq_U i\}.$$

We have shown above that every U-class is partitioned into one or several strong U-classes that are linearly ordered by $\trianglelefteq_U$. Sets of the form $\bar{a}^\nu|_p$, for a U-class p, will be the building blocks of the partition refinement we will construct in Section 2.6. To compute the width of the resulting partition refinement we have to bound the type index $\mathrm{ti}^n_\Delta(\bar{a}^\nu|_p/U \cup \bar{a}[\neq\nu])$ of such sets. This will be done in the next theorem. Let us start with two technical lemmas that are needed in its proof.



Lemma 2.4.8. *Suppose that there are formulae φ, ψ_k, and ψ^*_k, monadic parameters $\bar{P}$, and sequences $(a^\nu)_{\nu\in I}$, $(\bar{b}^n)_{n\in N}$, $(\bar{c}^{\nu n})_{\nu\in I, n\in N}$, $(\bar{c}^{\nu *})_{\nu\in I}$, and $\bar{d}$ satisfying the following conditions:*

- *The sequence $(a^\nu \bar{c}^{\nu *}(\bar{c}^{\nu n})_n)_{\nu\in I}$ is indiscernible over $(\bar{b}^n)_n \cup \bar{d}$.*
- *I and N are infinite.*
- *There is some $\sigma \in \{=, \neq, <, >, \leq, \geq\}$ such that*
$$\mathfrak{M} \vDash \varphi(\bar{b}^i, \bar{c}^{\nu k}, \bar{c}^{\nu *}, \bar{d}) \quad \text{iff} \quad i \,\sigma\, k\,.$$
- *There are relations $\rho_k \in \{=, \leq, \geq\}$ such that*
$$\mathfrak{M} \vDash \psi_k(c^{ui}_k, a^\nu, \bar{d}) \quad \text{iff} \quad u \,\rho_k\, \nu\,.$$
- $\mathfrak{M} \vDash \psi^*_k(c^{u*}_k, a^\nu, \bar{d}, \bar{P}) \quad \text{iff} \quad u = \nu\,.$

Then $\mathfrak{M}$ admits coding.

Proof. Set
$$\begin{aligned} A &:= \{\, a^\nu \mid \nu \in I \,\}\,, & C^*_k &:= \{\, c^{\nu *}_k \mid \nu \in I \,\}\,,\\ B_k &:= \{\, b^n_k \mid n \in N \,\}\,, & C^\nu_k &:= \{\, c^{\nu n}_k \mid n \in N \,\}\,, \end{aligned}$$
and $C_k := \bigcup_\nu C^\nu_k$. The formula
$$\vartheta^*(x, \bar{z}) := Ax \wedge \bigwedge_k [C^*_k z_k \wedge \psi^*_k(z_k, x, \bar{d}, \bar{P})]$$
satisfies
$$\mathfrak{M} \vDash \vartheta^*(a, \bar{c}) \quad \text{iff} \quad a = a^\nu \text{ and } \bar{c} = \bar{c}^{\nu *}, \text{ for some } \nu \in I\,.$$
We also construct a formula $\hat{\psi}_k$ such that
$$\mathfrak{M} \vDash \hat{\psi}_k(a, c) \quad \text{iff} \quad a = a^\nu \text{ and } c = c^{\nu n}_k \text{ for some } \nu \in I \text{ and } n \in N\,.$$
If ρ_k equals $=$ then we can simply set
$$\hat{\psi}_k(x, z) := Ax \wedge C_k z \wedge \psi_k(z, x, \bar{d})\,.$$
Suppose that $\rho_k \in \{\leq, \geq\}$. Defining
$$\chi(x, x') := Ax \wedge Ax' \wedge \forall z[Qz \wedge \psi_l(z, x, \bar{d}) \to \psi_l(z, x', \bar{d})]\,,$$

where $Q := \{ c_l^{\nu 0} \mid \nu \in I \}$, we obtain a formula such that

$$\mathfrak{M} \vDash \chi(a, a') \quad \text{iff} \quad a = a^u \text{ and } a' = a^\nu \text{ for some } u \; \rho_k \; \nu \,.$$

Hence, we can set

$$\hat{\psi}_k(x, z) := Ax \wedge C_k z \wedge \forall x'[Ax \to [\chi(x', x) \leftrightarrow \psi_k(z, x', \bar{d})]] \,.$$

Let $N^+ := \mathbb{Z} + N + \mathbb{Z}$ be the extension of the ordering N by two copies of $\mathbb{Z}$. By compactness, we can find extensions $(\bar{b}^n)_{n \in N^+}$ and $(\bar{c}^{\nu n})_{\nu \in I, n \in N^+}$ of $(\bar{b}^n)_n$ and $(\bar{c}^{\nu n})_{\nu, n}$ that behave in the same way with respect to the formulae ψ_k and φ. W.l.o.g. assume that $|\bar{b}^n|$ and $|\bar{c}^{\nu n}|$ are minimal. Then $(\bar{b}^n \bar{c}^{\nu n})_n$ forms a single φ-class and, by Theorem 2.1.19, there exists a formula

$$\eta\big(\bar{y}, \bar{z}, \bar{c}^{\nu *}, \bar{b}[\bar{m}], \bar{c}^\nu[\bar{m}], \bar{B}, \bar{C}^\nu\big)$$

with parameters $\bar{B}$, $\bar{C}^\nu$, $\bar{c}^{\nu *}$, $\bar{b}^{m_0}, \dots, \bar{b}^{m_l}$, $\bar{c}^{\nu m_0}, \dots, \bar{c}^{\nu m_l}$, for $\bar{m} \subseteq N^+ \setminus N$, such that

$$\mathfrak{M} \vDash \eta\big(\bar{b}, \bar{c}, \bar{c}^{\nu *}, \bar{b}[\bar{m}], \bar{c}^\nu[\bar{m}], \bar{B}, \bar{C}^\nu\big)$$

iff $\quad \bar{b} = \bar{b}^n$ and $\bar{c} = \bar{c}^{\nu n}$, for some $n \in N$.

Set $P_k^m := \{ c_k^{\nu m} \mid \nu \in I \}$ and

$$\zeta_0(x, \bar{z}^*, \bar{u}) := Ax \wedge \vartheta^*(x, \bar{z}^*) \wedge \bigwedge_{k,i} [P_k^{m_i} u_k^i \wedge \hat{\psi}_k(x, u_k^i)] \,.$$

Then we have

$$\mathfrak{M} \vDash \zeta_0(a, \bar{c}^*, \bar{e}) \quad \text{iff} \quad a = a^\nu,\ \bar{c}^* = \bar{c}^{\nu *}, \text{ and } \bar{e} = \bar{c}^\nu[\bar{m}], \text{ for some } \nu \in I \,.$$

Let $\hat{\eta}(x, \bar{y}, \bar{z}, \bar{c}^{\nu *}, \bar{b}[\bar{m}], \bar{c}^\nu[\bar{m}], \bar{B}, \bar{C})$ be the formula obtained from η by replacing the parameter C_k^ν by the formula $\hat{\psi}_k$ and set

$$\zeta(x, \bar{y}, \bar{z}, \bar{z}^*, \bar{u}) := \zeta_0(x, \bar{z}^*, \bar{u}) \wedge \hat{\eta}(x, \bar{y}, \bar{z}, \bar{z}^*, \bar{b}[\bar{m}], \bar{u}, \bar{B}, \bar{C}) \,.$$

Then it follows that

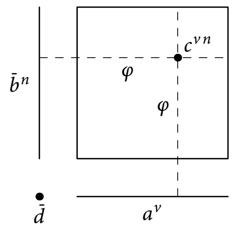
$$\mathfrak{M} \vDash \zeta(a, \bar{b}, \bar{c}, \bar{c}^*, \bar{e}) \quad \text{iff} \quad a = a^\nu,\ \bar{b} = \bar{b}^n,\ \bar{c} = \bar{c}^{\nu n},\ \bar{c}^* = \bar{c}^{\nu *}, \text{ and } \bar{e} = \bar{c}^\nu[\bar{m}], \text{ for some } \nu \in I \text{ and } n \in N \,.$$

Consequently, we have

$$\mathfrak{M} \vDash \exists \bar{y}' \exists \bar{z}' \exists \bar{z}^* \exists \bar{u} \zeta(a, b\bar{y}', c\bar{z}', \bar{z}^*, \bar{u})$$

iff $\quad a = a^\nu,\ b = b_0^n$, and $c = c_0^{\nu n}$, for some $\nu \in I$ and $n \in N$,

and $\mathfrak{M}$ admits coding. $\square$



Lemma 2.4.9. *Suppose that there are sequences $\bar{d}$, $(a^v)_{v \in I}$, $(\bar{b}^n)_{n<\omega}$, and $(c^{vn})_{v \in I, n<\omega}$ and a formula φ satisfying the following conditions:*

- $(a^v(c^{vn})_n)_{v \in I}$ *is indiscernible over* $(\bar{b}^n)_n \cup \bar{d}$.
- *I is dense and it has no least element and no greatest one.*
- *There is some $\rho \in \{=, \leq, \geq\}$ such that*

$$\mathfrak{M} \vDash \varphi(a^u, \bar{b}^n, c^{vn}, \bar{d}) \quad \text{iff} \quad u \,\rho\, v\,.$$

- *There are relations $\sigma_0 \in \{=, \leq, \geq\}$ and $\sigma_-, \sigma_+ \in \{\varnothing, I \times I, =, \neq, <, >, \leq, \geq\}$ such that*

$$\begin{aligned} \mathfrak{M} &\vDash \varphi(a^v, \bar{b}^k, c^{vn}, \bar{d}) && \text{iff} && k \,\sigma_0\, n\,, \\ \mathfrak{M} &\vDash \varphi(a^u, \bar{b}^k, c^{vn}, \bar{d}) && \text{iff} && k \,\sigma_-\, n\,, && \textit{for } u < v\,, \\ \mathfrak{M} &\vDash \varphi(a^u, \bar{b}^k, c^{vn}, \bar{d}) && \text{iff} && k \,\sigma_+\, n\,, && \textit{for } u > v\,. \end{aligned}$$

Then $\mathfrak{M}$ admits coding.

Proof. We start by constructing a formula ψ such that

$$\mathfrak{M} \vDash \psi(a^v, \bar{b}^n, c^{vn}) \quad \text{and} \quad \mathfrak{M} \nvDash \psi(a^u, \bar{b}^k, c^{vn}) \quad \text{for } u \neq v\,.$$

Let $A := \{\, a^v \mid v \in I \,\}$, $C^0 := \{\, C^{v0} \mid v \in I \,\}$, and $C := \{\, c^{vn} \mid v \in I,\ n < \omega \,\}$. If ρ equals $=$ then we can set

$$\psi(x, \bar{y}, z) := \forall x'(Ax' \rightarrow (\varphi(x', \bar{y}, z, \bar{d}) \leftrightarrow x' = x))\,.$$

Clearly, we have $\mathfrak{M} \vDash \psi(a^v, \bar{b}^n, \bar{c}^{vn})$ and, by indiscernibility, it follows that $\mathfrak{M} \nvDash \psi(a^u, \bar{b}^k, \bar{c}^{vn})$, for all $u \neq v$.

For $\rho \in \{\leq, \geq\}$, we define

$$\chi(x, x') := Ax \wedge Ax' \wedge \forall z[C^0 z \wedge \varphi(x', \bar{b}^0, z, \bar{d}) \rightarrow \varphi(x, \bar{b}^0, z, \bar{d})]\,.$$

This formula satisfies

$$\mathfrak{M} \vDash \vartheta(a, a') \quad \text{iff} \quad a = a^u \text{ and } a' = a^v \text{ for some } u \,\rho\, v\,.$$

Hence, we can obtain the desired formula ψ by setting

$$\psi(x, \bar{y}, z) := \forall x'[Ax' \rightarrow (\varphi(x', \bar{y}, z, \bar{d}) \leftrightarrow \vartheta(x', x))]\,.$$

Again, by indiscernibility, we have $\mathfrak{M} \nvDash \psi(a^u, \bar{b}^k, \bar{c}^{vn})$, for all $u \neq v$.

If we can show that the constructed formula ψ satisfies

$$\mathfrak{M} \nvDash \psi(a^v, \bar{b}^k, c^{vn}) \quad \text{for all } k \neq n\,,$$

then it follows that

$$\mathfrak{M} \vDash \psi(a^u, \bar{b}^k, c^{vn}) \quad \text{iff} \quad u = v \text{ and } k = n\,,$$

and $\mathfrak{M}$ admits coding. Hence, suppose that

$$\mathfrak{M} \vDash \psi(a^v, \bar{b}^k, c^{vn}) \quad \text{for some } k < n\,.$$

Then $\sigma_0 = \leq$. Fix some $s \in I$. W.l.o.g. assume that $|\bar{b}^n|$ is minimal. Then we can use Theorem 2.1.19 to find a formula $\eta(\bar{y}, z)$ (with monadic parameters) such that

$$\mathfrak{M} \vDash \eta(\bar{b}, c) \quad \text{iff} \quad \bar{b} = \bar{b}^n \text{ and } c = c^{sn},\ \text{for some } n\,.$$

Defining

$$\vartheta(\bar{y}, \bar{y}') \coloneqq \exists z \eta(\bar{y}, z) \wedge \exists z \eta(\bar{y}', z) \wedge \forall z(\eta(\bar{y}', z) \to \eta(\bar{y}, z))$$

we obtain a formula such that

$$\mathfrak{M} \vDash \vartheta(\bar{b}, \bar{b}') \quad \text{iff} \quad \bar{b} = \bar{b}^k \text{ and } \bar{b}' = \bar{b}^n,\ \text{for some } k \leq n\,.$$

If we define

$$\begin{aligned} \zeta_0(x, \bar{y}, z) &\coloneqq Ax \wedge Cz \wedge \exists z' \eta(\bar{y}, z') \wedge \psi(x, \bar{y}, z)\,, \\ \zeta(x, \bar{y}, z) &\coloneqq \zeta_0(x, \bar{y}, z) \wedge \forall \bar{y}'[\zeta_0(x, \bar{y}', z) \to \vartheta(\bar{y}', \bar{y})]\,, \end{aligned}$$

then we have

$$\begin{aligned} \mathfrak{M} \vDash \zeta(a, \bar{b}, c) \quad \text{iff} \quad & a = a^v,\ \bar{b} = \bar{b}^n, \text{ and } c = c^{vn}, \\ & \text{for some } v \in I \text{ and } n < \omega\,. \end{aligned}$$

Again, $\mathfrak{M}$ admits coding.

The remaining case that $\mathfrak{M} \vDash \psi(a^v, \bar{b}^k, c^{vn})$, for some $k > n$, is handled symmetrically. □

Theorem 2.4.10. *Suppose that $(\bar{a}^v)_{v \in I}$ is a proper infinite indiscernible sequence over U and let Δ be a set of formulae (over $\varnothing$) such that $2^{|\Delta|} \leq \kappa$ where $\kappa \coloneqq |\Sigma| + \aleph_0$ is the number of first-order formulae over the signature Σ. If there exist a U-class p, an index $v \in I$, and a number $n < \omega$ such that*

$$\mathrm{ti}^n_\Delta(\bar{a}^v|_p / U \cup \bar{a}[\neq v]) > \kappa$$

then $\mathfrak{M}$ admits coding.

Proof. By compactness, we may assume that I is dense without endpoints. Fix n-tuples $\bar{c}^{\nu i} \subseteq \bar{a}^{\nu}|_p$, for $i < \kappa^+$ such that

$$\mathrm{tp}_\Delta(\bar{c}^{\nu i}/U \cup \bar{a}[\neq\nu]) \neq \mathrm{tp}_\Delta(\bar{c}^{\nu k}/U \cup \bar{a}[\neq\nu])\,, \quad \text{for } i \neq k\,.$$

Choose some element $d^\nu \in \bar{a}^\nu|_p$ and indices $s < \nu < t$ in I. To simplify notation we set $W := U \cup \bar{a}[<s] \cup \bar{a}[>t]$. By indiscernibility, we have

$$\mathrm{tp}_\Delta(\bar{c}^{\nu i}/W) \neq \mathrm{tp}_\Delta(\bar{c}^{\nu k}/W)\,, \quad \text{for } i \neq k\,.$$

For every $s < u < t$, let $\alpha_u : I \to I$ be an order isomorphism such that $\alpha_u(\nu) = u$ and $\alpha_u(x) = x$, for $x < s$ or $x > t$. Let π_u be a U-automorphism such that $\pi_u(\bar{a}^x) = \bar{a}^{\alpha_u(x)}$, for all $x \in I$. For $s < u < t$, set $\bar{c}^{ui} := \pi_u(\bar{c}^{\nu i})$ and $d^u := \pi_u(d^\nu)$.

By Lemma 2.4.2, all indices in the U-class p are related via $\trianglelefteq_U$. Hence, we can find, for every $i < \kappa^+$ and all $k < n$, a formula $\psi^i_k(x, y, \bar{z})$, a tuple $\bar{e}^i_k \subseteq W$, and a relation $\rho^i_k \in \{=, \leq, \geq\}$ such that

$$\mathfrak{M} \vDash \psi^i_k(c^{ui}_k, d^\nu, \bar{e}^i_k) \quad \text{iff} \quad u\ \rho^i_k\ \nu\,.$$

By choice of κ there exists a subset $J \subseteq \kappa^+$ of size $|J| = \kappa^+$ such that $\psi^i_k = \psi^l_k$ and $\rho^i_k = \rho^l_k$, for all $i, l \in J$. We denote this formula by ψ_k and the corresponding relation by ρ_k.

We can use Lemma 2.1.5 to find an infinite subset $J_0 \subseteq J$, a formula $\varphi \in \Delta$, and parameters $\bar{b}^i \in W^m$, for $i \in J_0$, such that

$$\mathfrak{M} \vDash \varphi(\bar{b}^i, \bar{c}^{\nu i}) \leftrightarrow \neg\varphi(\bar{b}^i, \bar{c}^{\nu k})\,, \quad \text{for } i < k \text{ in } J_0\,.$$

By Ramsey's theorem, there exists an infinite subset $J_1 \subseteq J_0$ and a relation $\sigma \in \{=, \neq, \leq, >\}$ such that

$$\mathfrak{M} \vDash \varphi(\bar{b}^i, \bar{c}^{\nu k}) \quad \text{iff} \quad i\ \sigma\ k\,,$$

for $i, k \in J_1$. There is a φ-class $H \subseteq [m + n]$ of the sequence $(\bar{b}^i\bar{c}^{\nu i})_i$ containing indices j, l with $j < m$ and $m \leq l < m + n$. If we replace in $\bar{b}^i$ every component b^i_l with $l \in [m] \setminus H$ by b^0_l and we replace in $\bar{c}^{\nu i}$ every component $c^{\nu i}_l$ with $m + l \in [m + n] \setminus H$ by $c^{\nu 0}_l$ then we obtain two sequences that still satisfy

$$\mathfrak{M} \vDash \varphi(\bar{b}^i, \bar{c}^{\nu k}) \quad \text{iff} \quad i\ \sigma\ k\,.$$

Therefore, we may assume that there are sequences $(\bar{b}^i)_{i \in J_1}$ and $(\bar{c}^{\nu i})_{i \in J_1}$ and tuples $\bar{b}_* \subseteq W$ and $\bar{c}^\nu_* \subseteq \bar{a}^\nu|_p$ such that

$$\mathfrak{M} \vDash \varphi(\bar{b}^i, \bar{b}_*, \bar{c}^{\nu k}, \bar{c}^\nu_*) \quad \text{iff} \quad i\ \sigma\ k$$

and the sequence $(\bar{b}^i \bar{c}^{\nu i})_i$ has a single φ-class. To show that $\mathfrak{M}$ admits coding we distinguish two cases.

First assume that, for every k, we can choose ψ_k and $\bar{e}^i_k$ such that $\bar{e}^i_k = \bar{e}^l_k$, for all $i, l < \omega$. Then the sequences $(\bar{b}^i)_{i \in J_1}$, $(\bar{c}^{\nu i})_{\nu \in I, i \in J_1}$, $(\bar{c}^\nu_*)_{\nu \in I}$, and $(d^\nu)_{\nu \in I}$, and the tuple $\bar{b}_* \bar{e}^i_0 \dots \bar{e}^i_{n-1}$ satisfy the conditions of Lemma 2.4.8. Consequently, $\mathfrak{M}$ admits coding.

It remains to consider the case that there is some k such that we cannot choose the $\bar{e}^i_k$ to be equal. Then we can find an infinite subset $J_2 \subseteq J_1$ and a relation $\rho \in \{=, \neq, <, >, \leq, \geq\}$ such that, for all $i, l \in J_2$, we have

$$\mathfrak{M} \vDash \psi_k(c^{\nu i}_k, d^\nu, \bar{e}^l_k) \quad \text{iff} \quad i \, \rho \, l \,.$$

The sequences $(c^{\nu i}_k)_{\nu \in I, i \in J_2}$, $(\bar{c}^\nu_*)_{\nu \in I}$, $(\bar{e}^i_k)_{i \in J_2}$, and $(d^\nu)_{\nu \in I}$ satisfy the conditions of Lemma 2.4.9. Hence, $\mathfrak{M}$ admits coding. ☐

2.5 Finite satisfiability

One way to extend the notion of a non-forking type to arbitrary theories consists in considering finitely satisfiable types. Of course, many properties of forking – like symmetry and locality – are lost in this transition. Fortunately, sufficiently many basic properties remain to make the notion useful. Except for a few minor lemmas and changes of presentation all of the definitions and results in this section are taken from [46, 47, 48]. We include the proofs for convenience.

Definition 2.5.1. (a) A type p is *finitely satisfiable* in a set A if, for every finite subset $p_0 \subseteq p$, there exists a tuple $\bar{a} \subseteq A$ satisfying p_0.

(b) Let $\mathfrak{u}$ be an ultrafilter over A^α and let $U \subseteq M$ be a set of parameters. The *average type* of $\mathfrak{u}$ over U is

$$\mathrm{Av}(\mathfrak{u}/U) := \{\, \varphi(\bar{x}, \bar{c}) \mid \bar{c} \subseteq U,\ [\![\varphi(\bar{a}, \bar{c})]\!]_{\bar{a} \in A^\alpha} \in \mathfrak{u} \,\}\,.$$

Example. (a) Suppose that $\mathfrak{M} = (M, E)$ is a structure where E is an equivalence relation with infinitely many classes all of which are infinite. Let $U \subseteq V \subseteq M$ be sets and $a \in M \setminus V$ an element with E-class $[a]$. The type $\mathrm{tp}(a/V)$ is finitely satisfiable in U if and only if

- $[a] \cap V = \emptyset$ and U/E is infinite, or
- $[a] \cap V \neq \emptyset$ and $[a] \cap U$ is infinite.

(b) Let $\mathfrak{M} = (M, <)$ be a dense linear order, $U \subseteq V \subseteq M$ sets, and $a \in M \setminus V$. The type $\mathrm{tp}(a/V)$ is finitely satisfiable in U if and only if, for all $v, v' \in V$ with $v < a < v'$, there is some $u \in U$ with $v < u < v'$.

The connection between average types and types that are finitely satisfiable is given by the following lemma.

Lemma 2.5.2. (a) $U \subseteq V$ *implies* $\mathrm{Av}(\mathfrak{u}/U) \subseteq \mathrm{Av}(\mathfrak{u}/V)$.

(b) *Let* $\mathfrak{u}$ *be an ultrafilter over* A^α *and* $U \subseteq M$ *a set of parameters. Then* $\mathrm{Av}(\mathfrak{u}/U)$ *is a complete* α*-type over* U *which is finitely satisfiable in* A.

(c) *For every partial* α*-type* p *over* U *which is finitely satisfiable in* A*, there exists some ultrafilter* $\mathfrak{u}$ *over* A^α *such that* $p \subseteq \mathrm{Av}(\mathfrak{u}/U)$.

The next two lemmas summarise the basic properties of finitely satisfiable types that hold without any stability assumption.

Definition 2.5.3. Let p be a type, A a set, and $\Delta \subseteq \mathrm{FO}$. We say that p *Δ-splits* over A if there are tuples $\bar b_0, \bar b_1$ with $\mathrm{tp}_\Delta(\bar b_0/A) = \mathrm{tp}_\Delta(\bar b_1/A)$ and a formula $\varphi(\bar x; \bar y) \in \Delta$ such that $p \vDash \varphi(\bar x; \bar b_0)$ but $p \vDash \neg\varphi(\bar x; \bar b_1)$.

Lemma 2.5.4. (a) *Every* α*-type* p *over* B *which is finitely satisfiable in* A *can be extended to a complete type* $q \in S^\alpha(B)$ *which is also finitely satisfiable in* A.

(b) *If* $\mathrm{tp}_\Delta(C_0/A \cup B)$ *is finitely satisfiable in* A *and* $\mathrm{tp}_\Delta(C_1/A \cup B \cup C_0)$ *is finitely satisfiable in* $A \cup C_0$ *then* $\mathrm{tp}_\Delta(C_0 \cup C_1/A \cup B)$ *is finitely satisfiable in* A.

(c) *If* p *is finitely satisfiable in* A *then* p *does not* Δ*-split over* A.

Proof. (a) Clearly, if $(p_i)_i$ is an increasing sequence of types that are finitely satisfiable in A, then $\bigcup_i p_i$ is also finitely satisfiable in A. Therefore, it is sufficient to prove that, if $\varphi \in \mathrm{FO}$ and $\bar b \subseteq B$, either $p_0 := p \cup \{\varphi(\bar x, \bar b)\}$ or $p_1 := p \cup \{\neg\varphi(\bar x, \bar b)\}$ is finitely satisfiable in A.

For a contradiction, suppose otherwise. Then there are finite sets $q_0 \subseteq p_0$ and $q_1 \subseteq p_1$ that are not satisfiable in A. $q := (q_0 \cup q_1) \cap p$ is a finite subset of p and, hence, realised by some tuple $\bar a \in A^m$. If $\mathfrak{M} \vDash \varphi(\bar a, \bar b)$ then $\bar a$ realises q_0 and, otherwise, it realises q_1. Contradiction.

(b) Let $\varphi(\bar x, \bar y; \bar a, \bar b) \in \mathrm{tp}_\Delta(C_0 \cup C_1/A \cup B)$ and let $\bar c_0 \subseteq C_0$ and $\bar c_1 \subseteq C_1$ be the tuples corresponding to the variables $\bar x$ and $\bar y$. Then

$$\varphi(\bar c_0, \bar y; \bar a, \bar b) \in \mathrm{tp}_\Delta(C_1/A \cup B \cup C_0)$$

and there is some $\bar d_1 \subseteq A$ with $\mathfrak{M} \vDash \varphi(\bar c_0, \bar d_1; \bar a, \bar b)$. Hence,

$$\varphi(\bar x, \bar d_1; \bar a, \bar b) \in \mathrm{tp}_\Delta(C_0/A \cup B)$$

and there exists some $\bar d_0 \subseteq A$ with $\mathfrak{M} \vDash \varphi(\bar d_0, \bar d_1; \bar a, \bar b)$.

(c) Let $\bar b_0, \bar b_1$ be tuples with $\mathrm{tp}_\Delta(\bar b_0/A) = \mathrm{tp}_\Delta(\bar b_1/A)$ and suppose that $p \subseteq \mathrm{Av}(\mathfrak{u}/B)$ where $\bar b_0, \bar b_1 \subseteq B$. If $\varphi(\bar x; \bar b_0) \in p$ then

$$S := \{\, \bar a \subseteq A \mid \mathfrak{M} \vDash \varphi(\bar a; \bar b_0) \,\} \in \mathfrak{u}\,.$$

By assumption, any $\bar a \subseteq A$ satisfies $\mathfrak{M} \vDash \varphi(\bar a; \bar b_0) \leftrightarrow \varphi(\bar a; \bar b_1)$. Hence,

$$S = \{\, \bar a \subseteq A \mid \mathfrak{M} \vDash \varphi(\bar a; \bar b_1) \,\}$$

which implies that $\varphi(\bar x; \bar b_1) \in \mathrm{Av}(\mathfrak{u}/B)$. Therefore, $\neg\varphi(\bar x; \bar b_1) \notin p$ and p does not Δ-split over A. □

According to the preceding lemma the extension and transitivity properties of non-forking types generalise to finitely satisfiable types. In general, finitely satisfiable extensions are not unique. In order to have a unique extension we need the additional requirement that in the set of parameters every type is realised. This is statement (a) of the following lemma in the special case that $B = \varnothing$. Statement (b) contains the dual transitivity property which, the notion of a finitely satisfiable type being non-symmetric, also only holds under additional assumptions.

Lemma 2.5.5. *Suppose that every type $q \in S^{<\omega}_{\Delta}(U)$ that is realised in $V \cup A$ is also realised in $V \cup B$.*

(a) *If the types $p_i := \mathrm{tp}_{\Delta}(B \cup \bar{c}_i / V \cup A)$, for $i < 2$, are finitely satisfiable in U and $\mathrm{tp}(\bar{c}_0/V \cup B) = \mathrm{tp}(\bar{c}_1/V \cup B)$, then $p_0 = p_1$.*

(b) *If $\mathrm{tp}_{\Delta}(C \cup B/V \cup A)$ and $\mathrm{tp}_{\Delta}(C/V \cup B)$ are finitely satisfiable in U then so is $\mathrm{tp}_{\Delta}(C/V \cup A \cup B)$.*

Proof. (a) Suppose $p_0 \neq p_1$. Then there exists a formula $\varphi \in \Delta$ and tuples $\bar{b} \subseteq B$, $\bar{a} \subseteq V \cup A$ such that $\mathfrak{M} \vDash \varphi(\bar{b}, \bar{c}_0; \bar{a})$ and $\mathfrak{M} \vDash \neg\varphi(\bar{b}, \bar{c}_1; \bar{a})$. By assumption we can choose tuples $\bar{a}' \subseteq V \cup B$ such that $\mathrm{tp}_{\Delta}(\bar{a}'/U) = \mathrm{tp}_{\Delta}(\bar{a}/U)$. We have $\mathfrak{M} \vDash \neg\varphi(\bar{b}, \bar{c}_1; \bar{a}')$ as, otherwise, p_1 would split over U. Since $p_0|_{V \cup B} = p_1|_{V \cup B}$ it follows that $\mathfrak{M} \vDash \neg\varphi(\bar{b}, \bar{c}_0; \bar{a}')$. Thus, p_0 Δ-splits over U in contradiction to Lemma 2.5.4 (c).

(b) It is sufficient to prove the claim for all finite subsets $\bar{c} \subseteq C$. As the type $\mathrm{tp}_{\Delta}(\bar{c}/V \cup B)$ is finitely satisfiable in U we can use Lemma 2.5.4 (a) to find some tuple $\bar{c}'$ realising $\mathrm{tp}_{\Delta}(\bar{c}/V \cup B)$ such that $\mathrm{tp}_{\Delta}(\bar{c}'/V \cup A \cup B)$ is finitely satisfiable in U. Since $\mathrm{tp}_{\Delta}(B/V \cup A)$ is finitely satisfiable in U Lemma 2.5.4 (b) implies that so is $\mathrm{tp}_{\Delta}(B \cup \bar{c}'/V \cup A)$.

Hence, both $\mathrm{tp}_{\Delta}(B \cup \bar{c}'/V \cup A)$ and $\mathrm{tp}_{\Delta}(B \cup \bar{c}/V \cup A)$ are finitely satisfiable in U and we have $\mathrm{tp}(\bar{c}'/V \cup B) = \mathrm{tp}(\bar{c}/V \cup B)$. By (a) it follows that these types are equal. Thus, $\mathrm{tp}_{\Delta}(\bar{c}'/V \cup A \cup B) = \mathrm{tp}_{\Delta}(\bar{c}/V \cup A \cup B)$. Since the former is finitely satisfiable in U, so is the latter. ☐

The following theorem is one of the main tools to construct finitely satisfiable types.

Theorem 2.5.6 (Shelah). *Let $U \subseteq V$ be sets such that every type over U is realised in V. If $\bar{a} \in \mathbb{M}^{\alpha}$ and $\bar{b} \in \mathbb{M}^{\beta}$ are tuples such that $\mathrm{tp}(\bar{a}/U)$ is finitely satisfiable in U and $\mathrm{tp}(\bar{b}/V)$ is finitely satisfiable in V then there are $\bar{a}', \bar{b}' \subseteq \mathbb{M}$ such that*

- $\mathrm{tp}_{\Delta}(\bar{a}'/U) = \mathrm{tp}_{\Delta}(\bar{a}/U)$,
- $\mathrm{tp}_{\Delta}(\bar{b}'/V) = \mathrm{tp}_{\Delta}(\bar{b}/V)$,
- $\mathrm{tp}_{\Delta}(\bar{a}'/V \cup \bar{b}')$ *is finitely satisfiable in U, and*

- $\mathrm{tp}_\Delta(\bar b'/V \cup \bar a')$ *is finitely satisfiable in* V.

Proof. By Lemma 2.5.4 (a), we can extend the type $\mathrm{tp}(\bar a/U)$ to a complete type $p(\bar x) \in S^\alpha(V)$ that is finitely satisfiable in U. Let $\bar a''$ realise p. Define

$$\begin{aligned}
\Gamma &:= \mathrm{tp}(\bar b/V) \cup \Phi \cup \Psi\,,\\
\Phi &:= \{\, \neg\varphi(\bar x; \bar a'', \bar c) \mid \bar c \subseteq V, \text{ and there is no } \bar d \subseteq V \text{ such that}\\
&\qquad\qquad \mathbb{M} \vDash \varphi(\bar d; \bar a'', \bar c)\,\}\,,\\
\Psi &:= \{\, \neg\psi(\bar x; \bar a'', \bar c) \mid \bar c \subseteq V, \text{ and there is no } \bar d \subseteq U \text{ such that}\\
&\qquad\qquad \mathbb{M} \vDash \psi(\bar b; \bar d, \bar c)\,\}\,.
\end{aligned}$$

If Γ is consistent, then there exists a type $q(\bar y; \bar a'') \in S^\beta(V \cup \bar a'')$ with $q \supseteq \Gamma$. Let $r \in S^{\alpha+\beta}(V)$ be some type with $r \supseteq p(\bar x) \cup q(\bar y; \bar x)$. Any realisation $\bar a'\bar b'$ of r satisfies the above conditions since $\mathrm{tp}(\bar b'/V \cup \bar a') \supseteq \Gamma$.

In order to prove the consistency of Γ let $\Gamma_0 \subseteq \Gamma$ be finite and suppose that

$$\begin{aligned}
\Gamma_0 \cap \mathrm{tp}(\bar b/V) &= \{\vartheta_0(\bar x, \bar c_0), \dots, \vartheta_r(\bar x, \bar c_r)\}\,,\\
\Gamma_0 \cap \Psi &= \{\neg\psi_0(\bar x; \bar a'', \bar c'_0), \dots, \neg\psi_s(\bar x; \bar a'', \bar c'_s)\}\,,\\
\text{and}\qquad \Gamma_0 \cap \Phi &= \{\neg\varphi_0(\bar x; \bar a'', \bar c''_0), \dots, \neg\varphi_t(\bar x; \bar a'', \bar c''_t)\}\,.
\end{aligned}$$

Since every type over U is realised in V we can find a tuple $\bar b^* \bar c^*_0 \dots \bar c^*_s \subseteq V$ realising $\mathrm{tp}(\bar b \bar c'_0 \dots \bar c'_s/U)$. Note that $\mathbb{M} \vDash \neg\psi_k(\bar b^*; \bar d, \bar c^*_k)$, for all $\bar d \subseteq U$, since, otherwise, we would have $\mathbb{M} \vDash \psi_k(\bar b; \bar d, \bar c'_k)$, which implies that $\neg\psi_k \notin \Psi$. Since $\mathrm{tp}(\bar a''/V)$ is finitely satisfiable in U we, therefore, have $\mathbb{M} \vDash \neg\psi_k(\bar b^*; \bar a'', \bar c^*_k)$. Finally, since $\neg\varphi_k(\bar x; \bar a'', \bar c''_k) \in \Phi$ and $\bar b^* \subseteq V$, it follows that $\mathbb{M} \vDash \neg\varphi_k(\bar b^*; \bar a'', \bar c''_k)$. Hence, the tuples $\bar a''$, $\bar b^*$, $\bar c^*_k$, and $\bar c''_k$ realise Γ_0. ☐

The main focus of this section is on indiscernible sequences $(\bar a^\nu)_\nu$ such that, for every index ν, the type $\mathrm{tp}(\bar a^\nu/U \cup \bar a[<\nu])$ is finitely satisfiable in U. Such sequences can be thought of as an analogue of Morley sequences in the unstable context.

Definition 2.5.7. Let $U \subseteq V$ be sets. A *fan* over U/V is an indiscernible sequence $(\bar a^\nu)_{\nu \in I}$ over V such that, for all $\nu \in I$, the type

$$\mathrm{tp}(\bar a^\nu/V \cup \bar a[<\nu])$$

is finitely satisfiable in U.

Example. Consider the set $\mathbb{Z} \times \mathbb{R}$ with two binary relations

$$\begin{aligned}
E &:= \big\{\, (\langle i, x\rangle, \langle i, y\rangle) \bigm| i \in \mathbb{Z},\ x, y \in \mathbb{R} \,\big\}\,,\\
< &:= \big\{\, (\langle i, x\rangle, \langle k, y\rangle) \bigm| x < y,\ i, k \in \mathbb{Z},\ x, y \in \mathbb{R} \,\big\}\,.
\end{aligned}$$

Set $U := \mathbb{Z} \times (0, 1)$ and $V := \mathbb{Z} \times (-\infty, 1)$. For $\nu \in I := (1, \infty) \subseteq \mathbb{R}$, let $\bar a^\nu$ be an enumeration of $\mathbb{Z} \times \{\nu\}$. The sequence $(\bar a^\nu)_{\nu \in I}$ is a fan over U/V.

Lemma 2.5.8 (Shelah [46]). *Let $(\bar{a}^v)_{v \in I}$ be a sequence of α-tuples and V a set. If there exists an ultrafilter* $\mathfrak{u}$ *over U^α such that*

$$\operatorname{tp}(\bar{a}^v / V \cup \bar{a}[<v]) = \operatorname{Av}(\mathfrak{u} / V \cup \bar{a}[<v]), \quad \textit{for all } v \in I,$$

then $(\bar{a}^v)_v$ is indiscernible over V.

Proof. We prove by induction on n that

$$\operatorname{tp}(\bar{a}[\bar{s}]/V) = \operatorname{tp}(\bar{a}[\bar{t}]/V),$$

for all strictly increasing sequences $\bar{s}, \bar{t} \in I^n$. Let $\bar{s} = \bar{s}' s_{n-1}$, $\bar{t} = \bar{t}' t_{n-1}$, and $\bar{c} \subseteq V$. By induction hypotheses it follows that

$$\begin{aligned} & \varphi(\bar{x}_0, \dots, \bar{x}_{n-1}; \bar{c}) \in \operatorname{tp}(\bar{a}[\bar{s}]/V) \\ \text{iff} \quad & \{ \bar{b} \in U^\alpha \mid \mathfrak{M} \models \varphi(\bar{a}[\bar{s}'], \bar{b}; \bar{c}) \} \in \mathfrak{u} \\ \text{iff} \quad & \{ \bar{b} \in U^\alpha \mid \mathfrak{M} \models \varphi(\bar{a}[\bar{t}'], \bar{b}; \bar{c}) \} \in \mathfrak{u} \\ \text{iff} \quad & \varphi(\bar{x}_0, \dots, \bar{x}_{n-1}; \bar{c}) \in \operatorname{tp}(\bar{a}[\bar{t}]/V). \end{aligned}$$

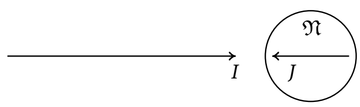
□

A kind of converse to this lemma is given by the next result.

Lemma 2.5.9 (Shelah [46]). *Let $(\bar{a}^v)_{v \in I}$ be an infinite proper indiscernible sequence of α-tuples. We can find a model $\mathfrak{N} \subseteq \mathbb{M}$ of size $|N| = |\Sigma| + |\alpha| + \aleph_0$, where Σ is the signature in question, such that N is disjoint from $\bar{a}[I]$ and, for every $v \in I$, the type* $\operatorname{tp}(\bar{a}^v / N \cup \bar{a}[<v])$ *is finitely satisfiable in N.*

Proof. Let $J := I \cup \{ u_n \mid n < \omega \}$ be a linear order extending I such that

$$v < \dots < u_n < \dots < u_2 < u_1 < u_0, \quad \text{for all } v \in I.$$



Extend $(\bar{a}^v)_{v \in I}$ to an indiscernible sequence $(\bar{a}^v)_{v \in J}$. Let $\mathfrak{M}$ be a model containing $(\bar{a}^v)_{v \in J}$ and let $\mathfrak{M}^+$ be an expansion of $\mathfrak{M}$ by Skolem functions. Since $(\bar{a}^v)_{v \in I}$ is an infinite indiscernible sequence over $N_0 := \bigcup_{n<\omega} \bar{a}^{u_n}$ we can choose the Skolem functions such that the Skolem hull of N_0 is disjoint from $\bar{a}[I]$. We claim that this Skolem hull induces the desired model $\mathfrak{N}$.

To show that $\operatorname{tp}(\bar{a}^s / N \cup \bar{a}[<s])$ is finitely satisfiable in N, let us suppose that

$$\mathfrak{M}^+ \models \varphi(\bar{a}^s, \bar{a}[\bar{v}], \bar{c})$$

where $v_0 < \cdots < v_{n-1} < s$ are indices in I and $\bar{c} \subseteq N$. Fix Skolem terms $\bar{t}$ such that $\bar{c} = \bar{t}(\bar{a}^{u_0}, \dots, \bar{a}^{u_k})$, for some k. Since $(\bar{a}^v)_{v \in J}$ is indiscernible it follows that

$$\mathfrak{M}^+ \vDash \varphi\big(\bar{a}^s, \bar{a}[\bar{v}], \bar{t}(\bar{a}^{u_0}, \dots, \bar{a}^{u_k})\big)$$

implies

$$\mathfrak{M}^+ \vDash \varphi\big(\bar{a}^{u_{k+1}}, \bar{a}[\bar{v}], \bar{t}(\bar{a}^{u_0}, \dots, \bar{a}^{u_k})\big)\,.$$

Since $\bar{a}^{u_{k+1}} \in N$ we are done. □

For every tuple $\bar{a}$ we can create a fan $(\bar{a}^v)_v$ containing $\bar{a}$.

Lemma 2.5.10 (Shelah [48]). *Let $U \subseteq V$ be sets and suppose that* $\operatorname{tp}(\bar{a}/U)$ *is finitely satisfiable in U. For every linear order I, there exists a fan $(\bar{a}^v)_{v \in I}$ over U/V such that* $\operatorname{tp}(\bar{a}^v/U) = \operatorname{tp}(\bar{a}/U)$, *for all v.*

Proof. By compactness, it is sufficient to consider the case that $I = \omega$. Let $\mathfrak{u}$ be the ultrafilter such that $\operatorname{tp}(\bar{a}/U) = \operatorname{Av}(\mathfrak{u}/U)$. By induction on n, we choose tuples $\bar{a}^n$ such that

$$\operatorname{tp}(\bar{a}^n/V \cup \bar{a}^0 \dots \bar{a}^{n-1}) = \operatorname{Av}(\mathfrak{u}/V \cup \bar{a}^0 \dots \bar{a}^{n-1})\,.$$

By Lemma 2.5.8 it follows that $(\bar{a}^n)_{n<\omega}$ is a fan over U/V. □

The following two observations seem to be new.

Lemma 2.5.11. *For all disjoint sets $A, U \subseteq \mathbb{M}$ of size $|U| = \kappa$ and $|A| > 2^{2^\kappa}$, there exists a set U_+ of size $|U_+| = \kappa$ and elements $a, b \in A \setminus U_+$ such that* $\operatorname{tp}(a/U_+ \cup \{b\})$ *is finitely satisfiable in U_+.*

Proof. Fix an enumeration $(a^i)_{i<\lambda}$ of A. By the Theorem of Erdős and Rado we have $(2^{2^\kappa}) \to ((2^\kappa)^+)^2_{2^\kappa}$. Since $\lambda \geq (2^{2^\kappa})^+$ and there are at most 2^κ 2-types over U, we can therefore find a subset $I \subseteq \lambda$ of size $|I| = (2^\kappa)^+$ such that,

$$\operatorname{tp}(a^i a^k/U) = \operatorname{tp}(a^j a^l/U)\,, \quad \text{for all } i < k \text{ and } j < l \text{ in } I\,.$$

Fix indices $s < t$ in I. By compactness there exists an indiscernible sequence $(b^i)_{i<\omega}$ over U such that

$$\operatorname{tp}(b^i b^k/U) = \operatorname{tp}(a^s a^t/U)\,, \quad \text{for all } i < k < \omega\,.$$

Using a suitable U-automorphism we may assume that $b^0 = a^s$ and $b^1 = a^t$. By Lemma 2.5.9 there exists a set $U_+ \subseteq U$ of size $|U_+| = |U|$ that is disjoint from $b[\omega]$ and such that $\operatorname{tp}(b^1/U_+ \cup \{b^0\})$ is finitely satisfiable in U_+. □

Lemma 2.5.12. *Let $(\bar{a}^v)_{v \in I}$ be a sequence of α-tuples and $U \subseteq V$ sets such that, for every $v \in I$,*

$$\mathrm{tp}(\bar{a}^v / V \cup \bar{a}[<v])$$

is finitely satisfiable in U. If $|I| > 2^{2^{|U^\alpha|}}$ then there exists a subset $J \subseteq I$ of size $|J| = |I|$ such that the subsequence $(\bar{a}^v)_{v \in J}$ is indiscernible over V.

Proof. By Lemma 2.5.2 (c), there exist ultrafilters $\mathfrak{u}_v$, for $v \in I$, such that

$$\mathrm{tp}(\bar{a}^v / V \cup \bar{a}[<v]) = \mathrm{Av}(\mathfrak{u}_v / V \cup \bar{a}[<v]) .$$

Since there are only $2^{2^{|U^\alpha|}}$ ultrafilters on U^α it follows that there is a subset $J \subseteq I$ of size $|J| = |I|$ such that $\mathfrak{u}_u = \mathfrak{u}_v$, for all $u, v \in J$. By Lemma 2.5.8 it follows that $(\bar{a}^v)_{v \in J}$ is indiscernible over V. ☐

An important property of fans $(\bar{a}^v)_{v \in I}$ over U/V is the fact that, for every tuple $\bar{b} \subseteq \bar{a}[I]$, the type $\mathrm{tp}(\bar{b}/V)$ is determined by the types $\mathrm{tp}(\bar{b} \cap \bar{a}^v / V)$, for $v \in I$.

Lemma 2.5.13 (Shelah [46]). *Let $(\bar{a}^v)_{v \in I}$ be a fan over U/V. Suppose that every type over U is realised in V. Let $\bar{u}, \bar{v} \in I^n$ be finite strictly increasing tuples and $s, t \in I$ indices with $s \leq \bar{u}\bar{v} \leq t$.*

If $\bar{b}^i \subseteq \bar{a}^{u_i}$ and $\bar{c}^i \subseteq \bar{a}^{v_i}$, for $i < n$, are tuples with

$$\mathrm{tp}_\Delta(\bar{b}^i / V) = \mathrm{tp}_\Delta(\bar{c}^i / V) \qquad \textit{for all } i ,$$

then

$$\begin{aligned} &\mathrm{tp}_\Delta(\bar{b}^0 \dots \bar{b}^{n-1} / V \cup \bar{a}[<s] \cup \bar{a}[>t]) \\ &= \mathrm{tp}_\Delta(\bar{c}^0 \dots \bar{c}^{n-1} / V \cup \bar{a}[<s] \cup \bar{a}[>t]) . \end{aligned}$$

Proof. First, we prove by induction on k that

$$\mathrm{tp}_\Delta(\bar{b}^0 \dots \bar{b}^{k-1} / V) = \mathrm{tp}_\Delta(\bar{c}^0 \dots \bar{c}^{k-1} / V) .$$

By assumption, we have $\mathrm{tp}_\Delta(\bar{b}^0/V) = \mathrm{tp}_\Delta(\bar{c}^0/V)$. Suppose that we have already shown that $\mathrm{tp}_\Delta(\bar{b}^0 \dots \bar{b}^{k-1}/V) = \mathrm{tp}_\Delta(\bar{c}^0 \dots \bar{c}^{k-1}/V)$. By symmetry, we may assume that $v_{k-1} \leq u_{k-1}$. Hence, $u_i, v_i < u_k$, for all $i < k$. Since $\bar{b}^i \subseteq \bar{a}^{u_i}$ and $\bar{c}^i \subseteq \bar{a}^{v_i}$ it follows by indiscernibility that

$$\mathrm{tp}_\Delta(\bar{b}^k \bar{b}^0 \dots \bar{b}^{k-1} / V) = \mathrm{tp}_\Delta(\bar{b}^k \bar{c}^0 \dots \bar{c}^{k-1} / V) .$$

Furthermore, by Lemma 2.5.5 (a), the assumption $\mathrm{tp}_\Delta(\bar{b}^k/V) = \mathrm{tp}_\Delta(\bar{c}^k/V)$ implies that

$$\mathrm{tp}_\Delta(\bar{b}^k / V \cup \bar{c}^0 \dots \bar{c}^{k-1}) = \mathrm{tp}_\Delta(\bar{c}^k / V \cup \bar{c}^0 \dots \bar{c}^{k-1}) .$$

Combining these two equations we have

$$\mathrm{tp}_\Delta(\bar{b}^0 \dots \bar{b}^{k-1}/V) = \mathrm{tp}_\Delta(\bar{c}^0 \dots \bar{c}^{k-1}/V)\,.$$

Having shown that $\mathrm{tp}_\Delta(\bar{b}^0 \dots \bar{b}^{n-1}/V) = \mathrm{tp}_\Delta(\bar{c}^0 \dots \bar{c}^{n-1}/V)$ we can apply Lemma 2.5.5 (a) one more time to conclude that

$$\begin{aligned}&\mathrm{tp}_\Delta(\bar{b}^0 \dots \bar{b}^{n-1}/V \cup \bar{a}[<s] \cup \bar{a}[>t])\\ &= \mathrm{tp}_\Delta(\bar{c}^0 \dots \bar{c}^{n-1}/V \cup \bar{a}[<s] \cup \bar{a}[>t])\,.\end{aligned}$$ □

Corollary 2.5.14. *Let $(\bar{a}^v)_{v\in I}$ be a fan over U/V. Suppose that every type over U is realised in V. For every partition $I = I_0 + I_1 + I_2$ of I into three segments, we have*

$$\mathrm{ti}^n_{\mathrm{FO}}(\bar{a}[I_1]/V \cup \bar{a}[I_0 \cup I_2]) \leq 2^{|V|+|\Sigma|}.$$

Proof. If $\bar{a}, \bar{b} \subseteq \bar{a}[I_1]$ then $\mathrm{tp}(\bar{a}/V) = \mathrm{tp}(\bar{b}/V)$ implies

$$\mathrm{tp}(\bar{a}/V \cup \bar{a}[I_0 \cup I_2]) = \mathrm{tp}(\bar{b}/V \cup \bar{a}[I_0 \cup I_2])\,.$$

Since there are at most $2^{|V|+|\Sigma|}$ n-types over V the claim follows. □

The next lemma provides the connection between finite satisfiability and the relation $\trianglelefteq_U$ introduced in the previous section.

Lemma 2.5.15 (Shelah [46]). *Let $(\bar{a}^v)_{v\in I}$ be fan over U/V with $\alpha := |\bar{a}^v|$. Suppose that every type over U is realised in V and let $p, q \subseteq \alpha$ be sets of indices.*

Then $\mathrm{tp}(\bar{a}^v|_p/V \cup \bar{a}^v|_q)$ is finitely satisfiable in U if and only if $p \ntrianglelefteq_V q$.

Proof. ($\Leftarrow$) Suppose that $s < t$ are indices with

$$\mathrm{tp}(\bar{a}^s|_p\bar{a}^s|_q/V) = \mathrm{tp}(\bar{a}^t|_p\bar{a}^s|_q/V)$$

and let $\varphi(\bar{x}, \bar{a}^v|_q) \in \mathrm{tp}(\bar{a}^v|_p/V \cup \bar{a}^v|_q)$. Then $\varphi(\bar{x}, \bar{a}^s|_q) \in \mathrm{tp}(\bar{a}^t|_p/V \cup \bar{a}^s|_q)$. Since this type is finitely satisfiable in U we can find some tuple $\bar{b} \subseteq U$ such that $\mathfrak{M} \models \varphi(\bar{b}, \bar{a}^s|_q)$. Hence, $\mathrm{tp}(\bar{a}^s|_q/U) = \mathrm{tp}(\bar{a}^v|_q/U)$ implies that $\mathfrak{M} \models \varphi(\bar{b}, \bar{a}^v|_q)$.

($\Rightarrow$) If $\mathrm{tp}(\bar{a}^v|_p/V\cup\bar{a}^v|_q)$ is finitely satisfiable in U then, by indiscernibility, so is $\mathrm{tp}(\bar{a}^s|_p/V\cup\bar{a}^s|_q)$. By definition of a fan $\mathrm{tp}\big(\bar{a}^s|_p\bar{a}^s|_q/V\cup\bar{a}[<s]\big)$ is finitely satisfiable in U. It follows by Lemma 2.5.5 (b) that so is the type

$$\mathrm{tp}\big(\bar{a}^s|_p/V \cup \bar{a}^s|_q \cup \bar{a}[<s]\big)\,.$$

Since, for $t > s$, $\mathrm{tp}\big(\bar{a}[>t]/V\cup\bar{a}^s|_p\bar{a}^s|_q\cup\bar{a}[<s]\big)$ is also finitely satisfiable in U we can use Lemma 2.5.5 (b) again to show that so is

$$\mathrm{tp}\big(\bar{a}^s|_p \cup \bar{a}[>t]/V \cup \bar{a}^s|_q \cup \bar{a}[<s]\big)\,.$$

On the other hand, we know that the type $\operatorname{tp}\big(\bar a^t|_p \cup \bar a[>t]/V \cup \bar a^s|_q \cup \bar a[<s]\big)$ is finitely satisfiable in U, for all $t > s$. Therefore, Lemma 2.5.13 implies that

$$\operatorname{tp}\big(\bar a^s|_p \cup \bar a[>t]/V \cup \bar a[<s]\big) = \operatorname{tp}\big(\bar a^t|_p \cup \bar a[>t]/V \cup \bar a[<s]\big)\,.$$

Hence, it follows from Lemma 2.5.5 (a) that

$$\begin{aligned} &\operatorname{tp}\big(\bar a^s|_p \cup \bar a[>t]/V \cup \bar a^s|_q \cup \bar a[<s]\big) \\ &= \operatorname{tp}\big(\bar a^t|_p \cup \bar a[>t]/V \cup \bar a^s|_q \cup \bar a[<s]\big)\,. \end{aligned}$$

Consequently, we have

$$\operatorname{tp}\big(\bar a^s|_p \bar a^s|_q/V \cup \bar a[<s] \cup \bar a[>t]\big) = \operatorname{tp}\big(\bar a^t|_p \bar a^s|_q/V \cup \bar a[<s] \cup \bar a[>t]\big)\,.$$

□

We use fans as a technical tool to investigate the properties of finitely satisfiable types. The basic idea is as follows. Given some tuple $\bar a$ we construct a fan $(\bar c^\nu)_{\nu\in I}$ over U/V with $\bar c^\circ = \bar a$. By the preceding lemma, $\operatorname{tp}(\bar a|_p/V \cup \bar a|_q)$ is finitely satisfiable in U if and only if $p \ntrianglelefteq_V q$. In this way we can apply the results of Section 2.4 to study finitely satisfiable types.

In the remainder of this section we show that the following relation is a preorder.

Definition 2.5.16. For sets $A, B, U \subseteq \mathbb{M}$, we write

$$A \sqsubseteq_U B \quad :\text{iff} \quad \operatorname{tp}(A/U \cup B) \text{ is not finitely satisfiable in } U\,.$$

Theorem 2.5.17 (Shelah [46]). *If $\mathfrak{M}$ does not admit coding and $A, B \subseteq \mathbb{M}$, $c \in \mathbb{M}$ then $A \not\sqsubseteq_M B$ implies $A \cup \{c\} \not\sqsubseteq_M B$ or $A \not\sqsubseteq_M B \cup \{c\}$.*

Proof. Fix enumerations $\bar a$ of A and $\bar b$ of B. Let $\mathfrak{M}_+ \succ \mathfrak{M}$ be an elementary extension such that every type over M is realised in M_+. Since $\mathfrak{M}$ is a model the type $\operatorname{tp}(\bar b/M)$ is finitely satisfiable in M. Hence, we can use Lemma 2.5.4 (a) to choose a tuple $\bar b'$ realising $\operatorname{tp}(\bar b/M)$ such that $\operatorname{tp}(\bar b'/M_+)$ is finitely satisfiable in M. Let $\bar a'$ be a tuple such that $\operatorname{tp}(\bar a'\bar b'/M) = \operatorname{tp}(\bar a\bar b/M)$. We apply Lemma 2.5.4 (a) again to choose a tuple $\bar a''$ realising $\operatorname{tp}(\bar a'/M \cup \bar b')$ such that $\operatorname{tp}(\bar a''/M_+ \cup \bar b')$ is finitely satisfiable in M. By Lemma 2.5.4 (b), it follows that $\operatorname{tp}(\bar a''\bar b'/M_+)$ is finitely satisfiable in M. Finally, select an element c' such that $\operatorname{tp}(\bar a''\bar b'c'/M) = \operatorname{tp}(\bar a\bar b c/M)$.

Let $(\bar d^\nu)_{\nu\in I}$ be a fan over M/M_+ with $\bar d^\circ = \bar a''\bar b'c'$. By Lemma 2.5.15, we have $\bar a'' \ntrianglelefteq_M \bar b'$. Hence, it follows by Lemma 2.4.4 that $\bar a''c \ntrianglelefteq_M \bar b'$ or $\bar a'' \ntrianglelefteq_M \bar b'c$. By Lemma 2.5.15, this means that at least one of

$$\operatorname{tp}(\bar a''c'/M_+ \cup \bar b') \quad \text{and} \quad \operatorname{tp}(\bar a''/M_+ \cup \bar b'c')$$

is finitely satisfiable in M. Consequently, so is one of

$$\operatorname{tp}(\bar{a}''c'/M \cup \bar{b}') \quad \text{and} \quad \operatorname{tp}(\bar{a}''/M \cup \bar{b}'c')\,.$$

Since $\operatorname{tp}(\bar{a}\bar{b}c/M) = \operatorname{tp}(\bar{a}''\bar{b}'c'/M)$ it follows that one of $\operatorname{tp}(\bar{a}c/M \cup \bar{b})$ and $\operatorname{tp}(\bar{a}/M \cup \bar{b}c)$ is finitely satisfiable in M. ☐

Lemma 2.5.18. *$\bar{a} \not\sqsubseteq_U \{b\}$ and $\bar{a}b \not\sqsubseteq_U \bar{c}$ implies $\bar{a} \not\sqsubseteq_U b\bar{c}$.*

Proof. Fix a set $V \supseteq U$ in which every type over U is realised. By Lemma 2.5.4 (a), we can find a tuple $\bar{a}'$ realising $\operatorname{tp}(\bar{a}/U \cup \{b\})$ such that the type $\operatorname{tp}(\bar{a}'/V \cup \{b\})$ is finitely satisfiable in U. In the same way we obtain a tuple $\bar{a}''b''$ realising $\operatorname{tp}(\bar{a}'b/V)$ such that $\operatorname{tp}(\bar{a}''b''/V \cup \bar{c})$ is finitely satisfiable in U. By Lemma 2.5.5 (b), it follows that $\operatorname{tp}(\bar{a}''/V \cup b''\bar{c})$ is finitely satisfiable in U. Since $\operatorname{tp}(\bar{a}b\bar{c}/U) \subseteq \operatorname{tp}(\bar{a}''b''\bar{c}/V)$ the result follows. ☐

Corollary 2.5.19. *Suppose that $\mathfrak{M}$ does not admit coding.*

(a) *If $\bar{a} \sqsubseteq_M b \sqsubseteq_M \bar{c}$ then $\bar{a} \sqsubseteq_M \bar{c}$.*

(b) *If $\bar{a} \sqsubseteq_M \bar{b}$ then $a_i \sqsubseteq_M \bar{b}$, for some i.*

(c) *If $\bar{a} \sqsubseteq_M \bar{b}$ then $\bar{a} \sqsubseteq_M b_i$, for some i.*

Proof. (a) Suppose that $\bar{a} \not\sqsubseteq_M \bar{c}$. By Theorem 2.5.17, we have $\bar{a}b \not\sqsubseteq_M \bar{c}$ or $\bar{a} \not\sqsubseteq_M b\bar{c}$. It follows that $b \not\sqsubseteq_M \bar{c}$ or $\bar{a} \not\sqsubseteq_M b$.

(b) W.l.o.g. we may assume that $\bar{a}$ and $\bar{b}$ are finite tuples. We prove the claim by induction on $|\bar{a}|$. Suppose that $\bar{a}c \sqsubseteq_M \bar{b}$. As $\bar{a} \not\sqsubseteq_M \bar{b}c$ and $c \not\sqsubseteq_M \bar{b}$ would imply that $\bar{a}c \not\sqsubseteq_M \bar{b}$ it follows that we have $\bar{a} \sqsubseteq_M \bar{b}c$ or $c \sqsubseteq_M \bar{b}$. In the latter case we are done. Assume that $\bar{a} \sqsubseteq_M \bar{b}c$. Together with $\bar{a}c \sqsubseteq_M \bar{b}$ it follows from Theorem 2.5.17 that $\bar{a} \sqsubseteq_M \bar{b}$. By induction hypothesis, there is some $a_i \sqsubseteq_M \bar{b}$.

(c) W.l.o.g. we may assume that $\bar{a}$ and $\bar{b}$ are finite tuples. We prove the claim by induction on $|\bar{b}|$. Suppose that $\bar{a} \sqsubseteq_M \bar{b}c$. If $\bar{a} \sqsubseteq_M c$ then we are done. If $\bar{a}c \sqsubseteq_M \bar{b}$ then Theorem 2.5.17 implies $\bar{a} \sqsubseteq_M \bar{b}$ and, by induction hypothesis, there is some i with $\bar{a} \sqsubseteq_M b_i$. Hence, we may assume that $\bar{a}c \not\sqsubseteq_M \bar{b}$ and $\bar{a} \not\sqsubseteq_M c$. But, by Lemma 2.5.18, this implies that $\bar{a} \not\sqsubseteq_M \bar{b}c$. Contradiction. ☐

Corollary 2.5.20. *If $\mathfrak{M}$ does not admit coding then $\sqsubseteq_M$ forms a preorder on $\mathbb{M} \smallsetminus M$.*

Proof. The reflexivity of $\sqsubseteq_M$ follows immediately form the definition, and we have seen in Corollary 2.5.19 that it is transitive. ☐

2.6 Linear decompositions

In this section we prove that the partition width of any structure $\mathfrak{M}$ that does not admit coding is bounded by $2^{2^{\aleph_0}}$. This is the main result of the first part of this thesis. If we could improve the bound to a finite partition width then this would solve Seese's conjecture.

We will construct the desired partition refinement of $\mathfrak{M}$ inductively from *partial* partition refinements.

Definition 2.6.1. Let $\mathfrak{M}$ be a structure and $A, C \subseteq M$.

(a) A *partial partition refinement* of A is a system $(U_v)_{v \in T}$ of subsets $U_v \subseteq A$ indexed by a tree $T \subseteq 2^{<\alpha}$ with the following properties:

- $U_{\langle\rangle} = A$,
- $U_v = U_{v0} \cup U_{v1}$, for all $v \in T$ (where we set $U_w := \varnothing$, for $w \notin T$),
- $U_v = \bigcap_{u<v} U_u$ if $|v|$ is a limit ordinal.

(b) Let $(U_v)_{v\in T}$ be a partial partition refinement of A. The *n-width* of $(U_v)_v$ over C is the cardinal

$$w_n((U_v)_v/C) := \sup_{v\in T} \mathrm{eti}^n(U_v/C \cup (A \setminus U_v)) .$$

Lemma 2.6.2. *Suppose that $\mathfrak{M}$ is a structure with a finite signature that does not admit coding. Let κ be an infinite cardinal and $A \subseteq M$ a set of size $|A| > 2^{2^\kappa}$ such that*

$$\mathrm{ti}^n_\Delta(A/M \setminus A) \leq \kappa , \quad \text{for all finite sets } \Delta \text{ and all } n < \omega .$$

There exists a partial partition refinement $(U_v)_{v\in T}$ of A such that

- *$w_n((U_v)_v/M \setminus A) \leq 2^{2^\kappa}$, for all n,*
- *if v is a leaf of T then $U_v \subset A$ and $\mathrm{ti}^n_\Delta(U_v/M \setminus U_v) \leq \aleph_0$, for all finite sets Δ of formulae and every $n < \omega$.*

Proof. Fix an increasing sequence $(\Delta_i)_{i<\omega}$ of finite sets $\Delta_i \subseteq \mathrm{FO}$ with union $\bigcup_{i<\omega} \Delta_i = \mathrm{FO}$. By Lemma 1.2.13, we can fix sets $C_i \subseteq M \setminus A$, for $i < \omega$, of size $|C_i| = \kappa$ such that, for $\bar a, \bar b \subseteq A$,

$$\mathrm{tp}_{\Delta_i}(\bar a/C_i) = \mathrm{tp}_{\Delta_i}(\bar b/C_i) \quad \text{implies} \quad \mathrm{tp}_{\Delta_i}(\bar a/M \setminus A) = \mathrm{tp}_{\Delta_i}(\bar b/M \setminus A) .$$

Let $C_\omega := \bigcup_{i<\omega} C_i$ and choose a model $C_* \supseteq C_\omega$ of size $|C_*| = \kappa$. It follows that

$$\mathrm{tp}(\bar a/C_*) = \mathrm{tp}(\bar b/C_*) \quad \text{implies} \quad \mathrm{tp}(\bar a/M \setminus A) = \mathrm{tp}(\bar b/M \setminus A) .$$

By Lemma 2.5.11 we can find a set $C \supseteq C_*$ of size $|C| = \kappa$ and elements $a, b \in A \setminus C$ such that $\mathrm{tp}(a/C \cup \{b\})$ is finitely satisfiable in C. Let $D_0 \supseteq C$ be a set such that every type over C is realised in D_0. We can choose D_0 of size $|D_0| \leq 2^\kappa$. By Lemma 2.5.4 (a) there is an element a' realising $\operatorname{tp}(a/C \cup \{b\})$ such that $\operatorname{tp}(a'/D_0 \cup \{b\})$ is finitely satisfiable in C. Let π be a $(U \cup \{b\})$-automorphism with $\pi(a') = a$ and set $D := \pi[D_0]$. Then $\operatorname{tp}(a/D \cup \{b\})$ is finitely satisfiable in C and every type over C is realised in D.

Fix an enumeration $\bar{a}$ of A and an $|A|$-dense linear order I, i.e., a linear order I such that, for all subsets $X < Y$ of I of size $|X|, |Y| < |A|$, there is some element $i \in I$ with $X < i < Y$. We can use Lemma 2.5.10 to find a fan $(\bar{a}^v)_{v \in I}$ over C/D with $\operatorname{tp}(\bar{a}^v/C) = \operatorname{tp}(\bar{a}/C)$. By applying suitable automorphisms we may assume that $A \subseteq \bar{a}[I]$ and, for all $v \in I$, the set $A_v := \bar{a}^v \cap (A \setminus C)$ is either empty or it consists of a single strong C-class. By Corollary 2.5.14, we have

$$\operatorname{ti}^n\big(\textstyle\bigcup_{v \in H} A_v / D \cup \bigcup_{v \in I \setminus H} A_v\big) \leq 2^{|D|} \leq 2^{2^\kappa},$$

for every convex subset $H \subseteq I$. Furthermore, the fact that $\operatorname{tp}(a/D \cup \{b\})$ is finitely satisfiable in C implies that $a \in A_u$ and $b \in A_v$, for some $u \neq v$. Hence $A_u \subset A$, for all $v \in I$.

Let $\alpha := |I|^+$ and fix an antichain $J \subseteq 2^{<\alpha}$ such that $\langle I, \leq \rangle \cong \langle J, \leq_{\mathrm{lex}} \rangle$. Let $\eta : I \to J$ be the corresponding bijection and let $T \subseteq 2^{<\alpha}$ be the prefix closure of J. For $v \in T$, we set

$$U_v := \bigcup \{ A_v \mid v \leq \eta(u) \}.$$

Then $(U_v)_{v \in T}$ is a partial partition refinement of A such that

$$\operatorname{ti}^n(U_v/M \setminus U_v) = \operatorname{ti}^n\Big(\bigcup_{u \in H} A_u / C \cup \bigcup_{u \in I \setminus H} A_u\Big) \leq 2^{2^\kappa},$$

where $H := \{ u \in I \mid v \leq \eta(u) \}$. Furthermore, if $v \in T$ is a leaf then $v = \eta(u)$, for some $u \in I$, and Theorem 2.4.10 implies that

$$\operatorname{ti}^n_\Delta(U_v/M \setminus U_v) = \operatorname{ti}^n_\Delta(A_u/M \setminus A_u) \leq \aleph_0,$$

for all finite sets Δ of formulae and every $n < \omega$. $\square$

Theorem 2.6.3. *Let $\mathfrak{M}$ be a structure with a finite signature. If $\mathfrak{M}$ does not admit coding then* $\operatorname{pwd} \mathfrak{M} \leq 2^{2^{\aleph_0}}$.

Proof. We construct a partition refinement $(U_v)_v$ of $\mathfrak{M}$ with $\operatorname{pwd}_n(U_v)_v \leq 2^{2^{\aleph_0}}$, for every n. If $|M| \leq 2^{2^{\aleph_0}}$ the claim is trivial. Therefore, we may assume that $|M| > 2^{2^{\aleph_0}}$. By Lemma 2.6.2, there exists a partial partition refinement $(U_v)_{v \in T_0}$ of M of the desired width. If $v \in T_0$ is a leaf then we have $\operatorname{ti}^n_\Delta(U_v/M \setminus U_v) \leq \aleph_0$, for all finite Δ and n, and we can use the lemma again to find a partial partition refinement of U_v of the desired width. This partial partition refinement can be inserted into the first one. We repeat this procedure until we obtain a partial partition refinement $(U_v)_v$ with $|U_v| \leq 2^{2^{\aleph_0}}$, for all leaves v. Then we can use arbitrary partition refinements of the leaves U_v to complete it to a partition refinement of $\mathfrak{M}$. $\square$

In conjunction with Corollary 2.2.5 it follows that there exists a dichotomy between axiomatisable classes with a bounded partition width and those with an unbounded one.

Corollary 2.6.4. *Let T be a complete first-order theory over a finite signature. If T has a model $\mathfrak{M}$ with $\operatorname{pwd}\mathfrak{M} > 2^{2^{\aleph_0}}$ then $\operatorname{pwd}\mathfrak{N}$ is unbounded when $\mathfrak{N}$ ranges over all models of T.*

We have shown that there exists a dichotomy between structures with a definable pairing function and structures with small partition width. This can be seen as a weak form of Seese's conjecture. Unfortunately, the bound on the partition width we obtained in rather high.

Open Problem. *Try to improve the bound of Theorem 2.6.3 to* $\operatorname{pwd}\mathfrak{M} \leq \aleph_0$.

Note that a lower bound is given by the grid $\mathfrak{G} := \langle \mathbb{Z} \times \mathbb{Z}, E \rangle$ where

$$E = \left\{ \langle\langle i, k\rangle, \langle j, l\rangle\rangle \;\middle|\; |i - j| + |k - l| = 1 \right\}.$$

The graph $\mathfrak{G}$ does not admit coding and its partition width is $\aleph_0$.

This example shows that the methods employed in this thesis are not sufficiently strong to prove the original form of Seese's conjecture. Note that in the above example there are no first-order definable pairing functions, but there is an MSO-definable one. Hence, to resolve the conjecture it seems to be necessary to modify the definition of admitting coding to include MSO-definable functions.

3 The Caucal hierarchy

In the second part of the thesis we turn to an investigation of the *Caucal hierarchy* which is obtained by alternated applications of monadic second-order interpretations and the Muchnik construction (see [42, 49, 2, 10]) starting with the class of all finite structures. (Originally, Caucal [15] defined the hierarchy only for graphs where the above operations can be replaced by, respectively, inverse rational mappings and unravellings.) Since these operations preserve both the decidability of the MSO-theory and the finiteness of partition width it follows that every structure in this hierarchy has finite partition width and a decidable monadic theory.

The lowest level of the Caucal hierarchy consists of the class of *prefix-recognisable* (also called *tree-interpretable*) structures. Restricted to graphs this is the class of all graphs that can be obtained from the configuration graph of some pushdown automaton by contracting each ε-transition. Carayol and Wöhrle [13] have extended this characterisation to the whole hierarchy: a graph belongs to the n-th level of the Caucal hierarchy if and only if it can be obtained by contracting ε-transitions from the configuration graph of some higher-order pushdown automaton of level n.

This is our motivation for studying higher-order pushdown automata. We investigate the structure of their configuration graphs. In particular, we study paths in these graphs and we provide operations to decompose and reassemble them. As a technical tool we derive a pumping lemma for higher-order pushdown automata. The material in this chapter is taken from [4].

3.1 Trees and the Caucal hierarchy

To define the Caucal hierarchy we use MSO-interpretations and an operation introduced by Muchnik.

Definition 3.1.1. The *Muchnik iteration* of a Σ-structure $\mathfrak{A}$ is the structure $\mathfrak{A}^* := \langle A^*, \mathrm{suc}, \mathrm{cl}, (R^*)_{R\in\Sigma}\rangle$ where the universe $A^* := A^{<\omega}$ consists of all finite sequence of elements of A and we have

$$\begin{aligned}
\mathrm{suc} &:= \{\, (w, wa) \mid w \in A^*,\ a \in A \,\}\,,\\
\mathrm{cl} &:= \{\, waa \mid w \in A^*,\ a \in A \,\}\,,\\
R^* &:= \{\, (wa_0, \dots, wa_{n-1}) \mid w \in A^*,\ \bar{a} \in R \,\}\,.
\end{aligned}$$

By $\mathfrak{A}^{*n}$ we denote the n-fold iteration of $\mathfrak{A}$

$$\mathfrak{A}^{*0} := \mathfrak{A} \quad \text{and} \quad \mathfrak{A}^{*(n+1)} := (\mathfrak{A}^{*n})^*.$$

Definition 3.1.2. The *Caucal hierarchy* $\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \dots$ is the hierarchy whose n-th level consists of all structures of the form $\mathcal{I}(\mathfrak{A}^{*n})$ where $\mathfrak{A}$ is a finite structure and $\mathcal{I}$ is an MSO-interpretation.

Note that the Muchnik iteration of a structure is a tree with some additional structure on the immediate successors of vertices. To study the expressive power of MSO on iterations Walukiewicz [49] introduced the following kind of tree automaton (see also [2] for an exposition).

Definition 3.1.3. An MSO-*automaton* is a tuple $\mathcal{A} = (Q, \Sigma, \delta, q_{\text{in}}, \Omega)$ where Q is a finite set of states, Σ is the input alphabet, q_{in} is the initial state, $\Omega : Q \to \omega$ a priority function, and $\delta : Q \times \Sigma \to \text{MSO}$ is the transition function.

Such an automaton takes as input a structure $\mathfrak{A}$ and a labelling $\lambda : A^* \to \Sigma$. A run of $\mathcal{A}$ on $\mathfrak{A}$ and λ is a function $\rho : A^* \to Q$ such that

- $\rho(\langle\rangle) = q_{\text{in}}$ and
- for all $w \in A^*$, we have

  $$(\mathfrak{A}, C, \bar{P}) \vDash \delta(\rho(w), \lambda(w))\,,$$

  where, for each $q \in Q$, we have

  $$P_q := \{\, a \in A \mid \rho(wa) = q \,\} \quad \text{and} \quad C := \begin{cases} \{a\} & \text{if } w = w'a\,, \\ \varnothing & \text{if } w = \langle\rangle\,. \end{cases}$$

A run ρ is accepting if it satisfies the parity condition Ω, i.e., on every infinite path the least priority seen infinitely often is even. We say that $\mathcal{A}$ *accepts* a pair $(\mathfrak{A}^*, \lambda)$ if there exists an accepting run of $\mathcal{A}$ on input $\mathfrak{A}$ and λ.

Theorem 3.1.4 (Walukiewicz [49]). *For every* MSO-*formula* $\varphi(\bar{X})$*, we can construct an* MSO-*automaton* $\mathcal{A}$ *such that*

$$\mathfrak{A}^* \vDash \varphi(\bar{P}) \quad \text{iff} \quad \mathcal{A} \textit{ accepts } (\mathfrak{A}^*, \lambda_{\bar{P}})\,,$$

where $\lambda_{\bar{P}}(w) := \{\, i \mid w \in P_i \,\}$.

3.2 Higher-order pushdown automata

We can also characterise the graphs in the Caucal hierarchy in terms of higher-order pushdown automata. These automata recognise sets of finite words instead of trees. We will mainly be interested in their configuration graphs, not in the languages they recognise. The stack of a higher-order pushdown automaton of level n is a list of stacks of level $n-1$. If the innermost stacks, i.e., those of level 1, are words over an alphabet Σ, then we denote the set of level n stacks by Σ^{+n}.

Definition 3.2.1. Let Σ be an alphabet. We define

$$\Sigma^{+0} := \Sigma\,, \qquad \Sigma^{+(n+1)} := (\Sigma^{+n})^{+}\,,$$
$$\Sigma^{*0} := \Sigma\,, \qquad \Sigma^{*(n+1)} := (\Sigma^{+n})^{*}\,.$$

(Note that we use Σ^{+n} instead of Σ^{*n} in the last definition.)

Each word $\xi \in \Sigma^{+n}$ can recursively be factored as

$$\xi = \xi_n a_n\,, \quad a_n = \xi_{n-1} a_{n-1}\,, \dots, \quad a_2 = \xi_1 \xi_0\,,$$

where $\xi_i \in \Sigma^{*i}$ and $a_i \in \Sigma^{+(i-1)}$. We can write such words as

$$\xi_n : \xi_{n-1} : \dots : \xi_1 : \xi_0\,,$$

where $(:) : \Sigma^{*i} \times \Sigma^{+(i-1)} \to \Sigma^{+i}$ with $\xi : a := \xi a$ is the right associative operation that appends a single level i symbol a (i.e., a word of level $i-1$) to a word ξ of level i.

Given a word ξ, we denote by $(\xi)_i$, for $0 \leq i \leq n$, the unique words such that

$$\xi = (\xi)_n : \dots : (\xi)_0\,.$$

Definition 3.2.2. A *pushdown automaton* of level n is a tuple

$$\mathcal{A} = (Q, \Sigma, \Gamma, \Delta, q_0, z, F)$$

where Q is the set of states, Σ the input alphabet, Γ the stack alphabet, $q_0 \in Q$ the initial state, $z \in \Gamma$ the initial stack element, $F \subseteq Q$ the set of accepting states, and

$$\Delta \subseteq Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \times Q \times \mathrm{Op}$$

the transition relation that consists of tuples $(p, a, c, q, \mathrm{op})$ where op is one of the following operations:

$$\begin{aligned}
\mathrm{pop}_k(\xi_n : \dots : \xi_0) &:= \xi_n : \dots : \xi_k\,,\\
\mathrm{push}_a(\xi_n : \dots : \xi_0) &:= \xi_n : \dots : \xi_2 : \xi_1\xi_0 : a\,,\\
\mathrm{clone}_k(\xi_n : \dots : \xi_0) &:= \xi_n : \dots : \xi_{k+1} : (\xi_k : \xi_{k-1} : \dots : \xi_0) : \xi_{k-1} : \dots : \xi_0\,,
\end{aligned}$$

where $\xi_i \in \Gamma^{*i}$ and $a \in \Gamma$. The operation pop_k removes the top symbol from the top most level k stack, push_a adds the symbol a to the top most level 1 stack, and clone_k duplicates the top symbol of the top most level k stack.

Further, we define the projections $\pi : \Gamma^{+n} \times Q \to \Gamma^{+n}$ and $\rho : \Gamma^{+n} \times Q \to Q$ and a function $\mathrm{top} : \Gamma^{+n} \times Q \to \Gamma \times Q$ by

$$\pi(\xi, q) := \xi\,, \quad \rho(\xi, q) := q\,, \quad \text{and} \quad \mathrm{top}(\xi, q) := ((\xi)_0, q)\,.$$

A configuration (ξ, q) of $\mathcal{A}$ consists of a stack content $\xi \in \Gamma^{+n}$ and a state $q \in Q$. We write $(\xi, q) \vdash^a (\zeta, p)$ if $\mathcal{A}$ enters configuration (ζ, p) when reading the letter $a \in \Sigma \cup \{\varepsilon\}$ in configuration (ξ, q), formally,

$$(\xi, q) \vdash^a (\zeta, p) \quad \text{iff} \quad (q, a, (\xi)_0, p, \mathrm{op}) \in \Delta \text{ and } \zeta = \mathrm{op}(\xi)\,.$$

A $(\Gamma^{+n} \times Q)$-labelled path r, i.e., a tree whose domain is linearly ordered by $\leq$, is a *run* of $\mathcal{A}$ if, for every vertex $u \in \mathrm{dom}(r)$ with immediate $\leq$-successor v, we have $r(u) \vdash^a r(v)$, for some $a \in \Sigma$. We do not require that r starts with the initial configuration $(\varepsilon : \cdots : \varepsilon : z, q_0)$. Instead, we only require that the first configuration of r is reachable, that is, there exists a sequence OP of stack operations such that the stack contents of the first configuration is $\mathrm{OP}(\varepsilon : \cdots : \varepsilon : z)$. We will denote the successor function on $\mathrm{dom}(r)$ by σ.

Example. For every n, there exists an automaton $\mathcal{A}_n$ of level $n + 1$ recognising the language

$$L_n := \left\{ a^{\beth_n(k)} \;\middle|\; k < \omega \right\},$$

where $\beth_n(k)$ is the function defined by

$$\beth_0(k) := k \quad \text{and} \quad \beth_{n+1}(k) = 2^{\beth_n(k)}.$$

Informally, the automaton $\mathcal{A}_n$ starts by guessing the number k and writing an encoding of $\beth_n(k)$ onto its stack. Then it enters a loop where in each iteration it decrements the number stored in the stack and reads one input letter. $\mathcal{A}_n$ stops when the number on the stack becomes 0.

How can we encode such huge numbers into a stack of level $n+1$? For the stack alphabet we choose $\Gamma = \{1, \dots, n, a\}$. The bottom of a stack of level i will be marked by the level $i - 1$ word

$$\bar{\imath} := \varepsilon : \cdots : \varepsilon : 12 \dots i \in \Gamma^{+(i-1)}.$$

By induction on n, we define a coding function $\kappa_n : \omega \to \Gamma^{+n}$ based on the binary encoding of integers.

$$\begin{aligned}
\kappa_1(m) &:= \bar{1} a^m\,, \\
\kappa_{n+1}(m) &:= \overline{n+1}\, \kappa_n(i_0) \cdots \kappa_n(i_l)\,, \\
&\qquad\qquad \text{where } m = 2^{i_0} + \cdots + 2^{i_l} \text{ and } i_0 > \cdots > i_l\,.
\end{aligned}$$

Instead of presenting the actual transition table of the automaton we specify it by pseudo-code. We need a predicate $\mathrm{zero}_i(\xi)$ that is true if the top-most level i stack in ξ is empty, and we need a function $\mathrm{dec}_i(\xi)$ that decrements the top-most level i stack of ξ. zero_i can be defined with the help of the markers $\bar{\imath}$.

$$\mathrm{zero}_i(\xi) \quad :\text{iff} \quad (\xi)_0 = i\,.$$

For level 1 the numbers are stored in unary encoding on the stack. Hence, the decrementation procedure only needs to remove one symbol.

$$\mathrm{dec}_1(\xi) := \mathrm{pop}_1(\xi)\,.$$

For $n > 1$, the numbers are stored in binary encoding and $\mathrm{dec}_n(\xi)$ has to distinguish two cases. If the last digit is 1 then we change it to 0. Otherwise, the number ends with a sequence of digits $10\cdots0$ that we have to replace by $01\cdots1$.

$\mathrm{dec}_{n+1}(\xi) :=$ **if** $\mathrm{zero}_n(\xi)$ **then** (∗ last digit is 1 ∗)
  return $\mathrm{pop}_{n+1}(\xi)$
else (∗ last digit is 0 ∗)
  $\xi := \mathrm{dec}_n(\xi)$ (∗ change 10 to 01 ∗)
  while not $\mathrm{zero}_n(\xi)$ **do**
    $\xi := (\mathrm{dec}_n \circ \mathrm{clone}_{n+1})(\xi)$ (∗ change 10 to 11 ∗)
  end
  return ξ
end

The automaton $\mathcal{A}_n$ works as follows. First, it creates the stack content

$$\overline{n+1} : \cdots : \bar{1}\,.$$

Then nondeterministically it performs k push_a-operations. The stack contents now is

$$\overline{n+1} : \cdots : \bar{1}a^k = \kappa_{n+1}(\beth_n(k))\,.$$

Finally, it enters a loop where in each iteration it calls dec_n and it reads one input letter.

Our interest in higher-order pushdown automata stems from the following result.

Theorem 3.2.3 (Carayol, Wöhrle [13]). *A graph $\mathfrak{G}$ belongs to the n-th level $\mathcal{C}_n$ of the Caucal hierarchy if and only if it can be obtained from the configuration graph of a pushdown automaton of level n by contracting all ε-transitions.*

Note that every configuration graph has finite outdegree. Hence, we need the contraction of ε-transitions to obtain graphs of infinite outdegree.

The easy direction of preceding theorem is based on the following lemma which we will need in Section 3.8. Note that we encode a configuration (ξ, q) of a pushdown automaton as a word over the alphabet $\Gamma \cup Q$ by appending q to ξ. The result is the word $\mathrm{push}_q(\xi)$.

Lemma 3.2.4. *Let $\mathcal{A} = (Q, \Sigma, \Gamma, \Delta, q_0, z, F)$ be a pushdown automaton of level n with configuration graph $(C, \vdash)$. Let $\mathfrak{A} := (A, (P_a)_{a \in A})$ be the structure with universe $A := Q \cup \Gamma$ and unary predicates $P_a := \{a\}$, for $a \in A$.*

There exist monadic second-order formulae $\varphi_c(x, y)$, for $c \in \Sigma$, such that

$$\mathfrak{A}^{*n} \vDash \varphi_c(\mathrm{push}_p(\xi), \mathrm{push}_q(\eta)) \quad \text{iff} \quad (\xi, p) \vdash^c (\eta, q)\,,$$

for all $\xi, \eta \in \Gamma^{*n}$ and $p, q \in Q$.

3.3 Graphs of finite outdegree

We start our investigation of the structure of graphs in the Caucal hierarchy by computing a bound on their outdegree. The results in this section will be based on the characterisation of the Caucal hierarchy via MSO-automata. In the following sections we will turn to pushdown automata.

Note that the universe of a structure $\mathfrak{A} \in \mathcal{C}_n$ in the n-th level of the hierarchy has the form $A \subseteq \Gamma^{*n}$, for some finite set Γ. We define a norm $|\xi|_k$ on such sets by taking the maximal length of a level k word contained in ξ.

Definition 3.3.1. Let Γ be a finite set. For $k \leq n$ and $\xi = x_0 \cdots x_{r-1} \in \Gamma^{*n}$ with $x_i \in \Gamma^{+(n-1)}$, we define, by induction on k,

$$|\xi|_k := \begin{cases} 0 & \text{if } r = 0\,, \\ |\xi| & \text{if } k = n\,, \\ \max\{\, |x_i|_k \mid i < r \,\} & \text{if } k < n \text{ and } r > 0\,. \end{cases}$$

Lemma 3.3.2. *Let Γ be a finite set with at least two elements and let $k_1, \dots, k_n$ be numbers. There are less than $|\Gamma|^{k_1 \cdots k_n}$ words $\xi \in \Gamma^{*n}$ such that $|\xi|_i < k_i$, for all $i \leq n$.*

Proof. The claim follows easily by induction on n. For $n = 1$, we have

$$\sum_{i < k_1} |\Gamma|^i = \frac{|\Gamma|^{k_1} - 1}{|\Gamma| - 1} < |\Gamma|^{k_1}$$

words $\xi \in \Gamma^*$ with $|\xi| < k_1$. For $n > 1$, we can employ the induction hypothesis to obtain the bound

$$\sum_{i < k_n} (|\Gamma|^{k_1 \cdots k_{n-1}})^i < |\Gamma|^{k_1 \cdots k_n}. \qquad \square$$

If $\mathfrak{G} = (V, E)$ is a graph in the n-th level of the Caucal hierarchy then, by definition, there exists a finite structure $\mathfrak{A}$ and two MSO-formulae δ and φ such that

$$\begin{aligned} V &= \{\, \xi \in A^{*n} \mid \mathfrak{A}^{*n} \models \delta(\xi) \,\}, \\ E &= \{\, (\xi, \eta) \in A^{*n} \times A^{*n} \mid \mathfrak{A}^{*n} \models \varphi(\xi, \eta) \,\}. \end{aligned}$$

Therefore, we will consider a structure of the form $\mathfrak{A}^{*n}$ and an MSO-formula $\varphi(x, y)$ with two free first-order variables.

Definition 3.3.3. Let $\mathfrak{A}$ be a structure and $\varphi(x, y) \in \mathrm{MSO}$ a formula. The *φ-outdegree* of $a \in A$ in $\mathfrak{A}$ is the number of elements $b \in A$ such that $\mathfrak{A} \models \varphi(a, b)$.

We obtain the following bound on the φ-outdegree.

Theorem 3.3.4. *For every formula $\varphi(x, y) \in \mathrm{MSO}$ and each $n < \omega$, there are constants $c_1, \dots, c_n$ such that, whenever $\mathfrak{A}$ is a finite structure with at least two elements and $a \in A^{*n}$ an element of finite φ-outdegree in $\mathfrak{A}^{*n}$ then*

$$\mathfrak{A}^{*n} \models \varphi(a, b) \quad \textit{implies} \quad |b|_i \leq L_i(a) \quad \textit{for all } i \leq n\,,$$

where

$$L_i(a) := |a|_i + c_i |A|^{L_1(a) \cdots L_{i-1}(a)}.$$

Proof. We prove the claim by induction on n. Let $\mathcal{A} = (Q, \wp[2], \delta, q_{\mathrm{in}}, \Omega)$ be the nondeterministic MSO-automaton corresponding to φ. Since $\mathfrak{A}$ is fixed we will simplify notation by saying that $\mathcal{A}$ accepts a tree $\lambda : A^{*n} \to \wp[2]$ if it accepts the pair $(\mathfrak{A}^{*n}, \lambda)$.

W.l.o.g. we may assume that the set of states $Q = Q_\varnothing \uplus Q_0 \uplus Q_1 \uplus Q_{01}$ is partitioned such that starting in a state $q \in Q_C$ the automaton $\mathcal{A}$ accepts only trees λ where the set of occurring labels is exactly C. (If $\mathcal{A}$ is not of this form then we can construct a new automaton with states $Q \times \wp[2]$.) Furthermore, we assume that there exists a unique state $q_1 \in Q_1$ from which $\mathcal{A}$ accepts the tree λ with

$$\lambda(x) := \begin{cases} \{1\} & \text{if } x = \varepsilon\,, \\ \varnothing & \text{otherwise}\,. \end{cases}$$

Let ρ be an accepting run of $\mathcal{A}$ on the tree $\lambda : A^{*n} \to \wp[2]$. If $\rho(w) \in Q_1$ then we either have $\lambda(w) = \{1\}$ and $\rho(wa) \in Q_\varnothing$, for all $a \in A^{*(n-1)}$, or we have $\lambda(w) = \varnothing$ and there is some $a \in A^{*(n-1)}$ with $\rho(wa) \in Q_1$ and $\rho(wb) \in Q_\varnothing$, for all $b \neq a$. For $p, q \in Q_1$, we define

$$\psi_{pq}(x, y) := \exists C \exists \bar{P}\big(\delta(p, \varnothing)(C, \bar{P}) \wedge C = \{x\} \wedge P_q = \{y\} \wedge \bigwedge_{s \in Q \setminus (Q_\varnothing \cup \{q\})} P_s = \varnothing\big)\,.$$

It follows that, whenever the automaton is in state p at some vertex $wa \in A^{*n}$ with $\lambda(wa) = \varnothing$ then it can go to state q at the vertex wab if and only if $\mathfrak{A} \vDash \psi_{pq}(a, b)$.

Similarly, there exists a formula $\vartheta_{qp_0p_1}(x, y_0, y_1)$, for $q \in Q_{01}$, $p_0 \in Q_0$, and $p_1 \in Q_1$, such that

$$\mathfrak{A} \vDash \vartheta_{qp_0p_1}(a, b_0, b_1)$$

if and only if, whenever the automaton $\mathcal{A}$ is in the state q at some vertex $wa \in A^{*n}$ then it can go into the state p_0 at wab_0 and into the state p_1 at wab_1.

Let $\pi : A^{+n} \to A^{+(n-1)}$ be the projection to the last symbol $\pi(wa) := a$. Fix an element $u \in A^{*n}$ such that the set $V := \{ v \in A^{*n} \mid \mathfrak{A}^* \vDash \varphi(u, v), v \not\leq u \}$ is finite. To each $v \in V$ we associate the maximal sequence $v_0, \dots, v_{m(v)}$ such that $u \sqcap v = v_0 \prec v_1 \prec \cdots \prec v_{m(v)} = v$. By assumption, the set

$$P := \bigcup \{ \pi(v_i) \mid v \in V, \ i \leq m(v) \}$$

is finite. For $v \in V$, we denote the accepting run of $\mathcal{A}$ on the tree $\lambda_{\{u\}\{v\}}$ by ρ_v and we set $p_v := \rho_v(v_0)$ and

$$P_q := \bigcup \{ \pi(v_i) \mid v \in V, \ p_v = q, \ i \leq m(v) \}.$$

Then $P = \bigcup_q P_q$.

We define a formula $\chi_p(x, y)$ such that $\chi_p(\pi(u), y)$ defines in $\mathfrak{A}^{*(n-1)}$ the set P_p. The formula $\chi_p(a, b)$ states that there exist a sequence of elements $d_0, \dots, d_m \in A^{*(n-1)}$, $m < \omega$, and a corresponding sequence of states $p_0, \dots, p_m \in Q_1$ such that

- $d_0 = a$ and $p_0 = p$,
- $d_k = b$, for some $k \leq m$,
- $p_m = q_1$ (q_1 is the state signalling the label 1),
- $\mathfrak{A} \vDash \exists z \bigvee_{q \in Q_0} \vartheta_{p_0 q p_1}(d_0, z, d_1)$, and
- $\mathfrak{A} \vDash \psi_{p_i p_{i+1}}(d_i, d_{i+1})$, for all $0 < i < m$.

Since

$$\left\{ a \in A^{*(n-1)} \,\middle|\, \mathfrak{A} \vDash \chi_{p_v}(\pi(v_0), a) \right\} = P_{p_v} \subseteq P$$

is finite we can apply the induction hypothesis. Hence, there are numbers $c_1, \dots, c_{n-1}$ such that

$$\mathfrak{A}^{*(n-1)} \vDash \chi_{p_v}(\pi(v_0), a) \quad \text{implies} \quad |a|_i \leq L_i(\pi(v_0)),$$

where

$$L_i(a) := |a|_i + c_i |A|^{L_1(a) \cdots L_{i-1}(a)}.$$

It follows that, for $v \in V$ and $i \leq m(v)$, we have

$$|\pi(v_i)|_l \leq \max \{ L_l(\pi(x)) \mid x \leq u \} \leq L_l(u) .$$

Finally, note that, for $v \in V$,

$$\pi(v_i) = \pi(v_l), \quad \text{for } i < l, \qquad \text{implies} \qquad \rho(v_i) \neq \rho(v_l),$$

since, otherwise, the path $\pi(v_i), \dots, \pi(v_{l-1})$ can be repeated an arbitrary number of times and χ_{p_v} defines an infinite set. It follows that

$$m(v) \leq |Q_1| \cdot |A|^{L_1(u) \cdots L_{n-1}(u)}.$$

Consequently, setting $c_n := |Q_1|$ we have

$$|v|_n \leq |u|_n + c_n \cdot |A|^{L_1(u) \cdots L_{n-1}(u)} = L_n(u),$$

and $\quad |v|_i \leq L_i(u), \quad$ for $i < n$. □

Corollary 3.3.5. *Let $\mathfrak{A}$ be a finite structure and $\varphi(x, y) \in \mathrm{MSO}$ some formula that defines a relation $R := \varphi^{\mathfrak{A}^{*n}}$ of finite outdegree on $\mathfrak{A}^{*n}$. If $u_0, u_1, \dots \in A^{*n}$ is an R-path then we have*

$$|u_k|_i \leq |u_0|_i + \beth_{i-1}(\mathcal{O}(k + |u_0|_1 + \dots + |u_0|_{i-1})), \qquad \textit{for all } i \leq n .$$

Proof. By the preceding theorem, we have

$$|u_k|_1 \leq |u_{k-1}|_1 + c_1 \leq |u_0|_1 + c_1 k \leq |u_0|_1 + \beth_0(\mathcal{O}(k)),$$

and, for $i > 1$, it follows by induction that

$$\begin{aligned} |u_k|_i &\leq |u_{k-1}|_i + c_i |A|^{L_1(u_{k-1}) \cdots L_{i-1}(u_{k-1})} \\ &\leq |u_0|_i + \sum_{l<k} c_i |A|^{L_1(u_l) \cdots L_{i-1}(u_l)}. \end{aligned}$$

Since

$$\begin{aligned} &L_1(u_l) \cdots L_{i-1}(u_l) \\ &\leq \big(|u_0|_1 + \beth_0(\mathcal{O}(l))\big) \cdots \big(|u_0|_{i-1} + \beth_{i-2}(\mathcal{O}(l + |u_0|_1 + \dots + |u_0|_{i-2}))\big) \\ &\leq (|u_0|_1 + \dots + |u_0|_{i-1} + \beth_{i-2}(\mathcal{O}(l + |u_0|_1 + \dots + |u_0|_{i-2})))^{i-1} \\ &\leq (|u_0|_1 + \dots + |u_0|_{i-1} + \beth_{i-2}(\mathcal{O}(k + |u_0|_1 + \dots + |u_0|_{i-2})))^{i-1} \\ &\leq (k + |u_0|_1 + \dots + |u_0|_{i-1} + \beth_{i-2}(\mathcal{O}(k + |u_0|_1 + \dots + |u_0|_{i-2} + |u_0|_{i-1})))^{i-1} \\ &\leq \beth_{i-2}(\mathcal{O}(k + |u_0|_1 + \dots + |u_0|_{i-2} + |u_0|_{i-1}))^{i-1} \\ &\leq \beth_{i-2}(\mathcal{O}(k + |u_0|_1 + \dots + |u_0|_{i-2} + |u_0|_{i-1})) \end{aligned}$$

it follows that

$$\begin{aligned}|u_k|_i &\le |u_0|_i + \sum_{l<k} c_i 2^{\beth_{i-2}(\mathcal{O}(k+|u_0|_1+\dots+|u_0|_{i-2}+|u_0|_{i-1}))} \\ &\le |u_0|_i + c_i k \beth_{i-1}(\mathcal{O}(k+|u_0|_1+\dots+|u_0|_{i-2}+|u_0|_{i-1})) \\ &\le |u_0|_i + \beth_{i-1}(\mathcal{O}(k+|u_0|_1+\dots+|u_0|_{i-2}+|u_0|_{i-1}))\,. \end{aligned}$$ □

Corollary 3.3.6. *Let $\mathfrak{A}$ be a finite structure and $\varphi(x,y) \in$ MSO some formula that defines a relation of finite outdegree on $\mathfrak{A}^{*n}$. The k-neighbourhood*

$$N_k(u) := \{\, v \in A^{*n} \mid d(u,v) \le k \,\}$$

of an element $u \in A^{*n}$ is bounded by

$$|N_k(u)| \le \beth_n(\mathcal{O}(k+|u|_1+\dots+|u|_n))\,.$$

Proof. If $d(u,v) \le k$ then we know by the preceding corollary that

$$|v|_i \le |u|_i + \beth_{i-1}(\mathcal{O}(k+|u|_1+\dots+|u|_{i-1}))\,.$$

It therefore follows from Lemma 3.3.2 that there are less than

$$\begin{aligned} & |A|^{(|u|_1+\mathcal{O}(k))\cdots(|u|_n+\beth_{n-1}(\mathcal{O}(k+|u|_1+\dots+|u|_{n-1})))} \\ \le\ & |A|^{(|u|_1+\dots+|u|_n+\beth_{n-1}(\mathcal{O}(k+|u|_1+\dots+|u|_{n-1})))^n} \\ \le\ & |A|^{\beth_{n-1}(\mathcal{O}(k+|u|_1+\dots+|u|_{n-1}+|u|_n))^n} \\ \le\ & |A|^{\beth_{n-1}(\mathcal{O}(k+|u|_1+\dots+|u|_{n-1}+|u|_n))} \\ =\ & \beth_n(\mathcal{O}(k+|u|_1+\dots+|u|_n)) \end{aligned}$$

such words v. □

Corollary 3.3.7. *Let $\mathfrak{A}$ be a finite structure $\varphi(x,y) \in$ MSO, and $u \in A^{*n}$. If the φ-outdegree of u in $\mathfrak{A}^{*n}$ is finite then it is bounded by*

$$\beth_n(\mathcal{O}(|u|_1+\dots+|u|_n))\,.$$

Example. Let

$$T_m := \{\, 0^n i \in \omega^* \mid n < \omega,\ i < \beth_m(n) \,\}$$

and let $E \subseteq T_m \times T_m$ be the immediate successor relation. It follows by results of Caucal [15] that $(T_m, E) \in \mathcal{C}_{m+2}$. On the other hand, the results above imply that the tree (T_{2k}, E) is not contained in the k-th level of the Caucal hierarchy. Otherwise, let $w_n \in A^{*k}$ be the word encoding the element $0^n \in T_{2k}$. By Corollary 3.3.5, we have

$$|w_n|_i \le \beth_{i-1}(\mathcal{O}(n))\,.$$

According to Corollary 3.3.7 the outdegree of w_n is therefore bounded by $\beth_{2k-1}(\mathcal{O}(n))$. Contradiction.

Similarly, if we define $T_\omega := \{\, 0^n i \in \omega^* \mid i < \beth_{2n}(n) \,\}$ then (T_ω, E) is not contained in any level of the hierarchy.

3.4 Substitution of stacks

After having studied the degree of vertices in a graph of the Caucal hierarchy we now turn to the investigation of paths in such graphs, or rather of runs of higher-order pushdown automata. For the remainder of the article we fix a pushdown automaton $\mathcal{A}$ of level n. Let us introduce some additional notation. If r is a run and $x \in \mathrm{dom}(r)$ then the *operation at* x is the operation op such that $\pi r(\sigma x) = \mathrm{op}(\pi r(x))$. We call pop_1 and push_a a *level* 1 *operation* and, for $k > 1$, pop_k and clone_k a *level* k *operation*. A push(1)-*operation* is an operation of the form push_a and, for $k > 1$, we call clone_k a push(k)-*operation*.

We start by showing how to replace in a given run the bottom part of all stacks by some other stack content such that the resulting sequence of configurations still forms a run. To do so we define a variant of the prefix relation $\xi \triangleleft_k \zeta$ saying that some stack content ξ is contained in a larger stack ζ. In the constructions of the following sections we will need to also consider operations and relations on just the bottom levels of a stack. Therefore, we have to define all notions dependent on a parameter k.

Definition 3.4.1. For words $\xi, \eta \in \Gamma^{+n}$ and $k \geq 1$, we define the prefix relation $\xi \triangleleft_k \eta$ by induction on n.

If $n < k$, in particular if $n = 0$, then $\xi \triangleleft_k \eta$ always holds. For $n \geq k$, suppose that $\xi = \xi' : x$. We define $\xi \triangleleft_k \eta$ iff there are symbols $y_0, \dots, y_r \in \Gamma^{+(n-1)}$, $0 \leq r < \omega$, such that

$$\eta = \xi' y_0 \dots y_r \quad \text{and} \quad x \triangleleft_k y_i \,, \quad \text{for all } i \leq r\,.$$

For notational convenience, if r is a run and $x, y \in \mathrm{dom}(r)$, we define

$$x \triangleleft_k y \quad :\text{iff} \quad \pi r(x) \triangleleft_k \pi r(y)\,.$$

Example. We have

$$(ab : a) : a : a \triangleleft_1 (ab : a)(aa : b) : ab : a\,.$$

The following easy observations will frequently be used in the proofs below.

Lemma 3.4.2. *If we have* $\xi_n : \dots : \xi_0 \triangleleft_k \xi_n \eta : \zeta_{n-1} : \dots : \zeta_0$ *and* $\eta \neq \varepsilon$ *then* $\xi_n : \dots : \xi_0 \triangleleft_k \xi_n \eta$.

Proof. Suppose that $\eta = y_0 \dots y_m$. Then

$$\xi_n : \dots : \xi_0 \triangleleft_k \xi_n \eta : \zeta_{n-1} : \dots : \zeta_0$$

implies $\xi_{n-1} : \dots : \xi_0 \triangleleft_k y_i$, for all $i \leq m$. Hence,

$$\xi_n : \dots : \xi_0 \triangleleft_k \xi_n y_0 \dots y_m = \xi_n \eta\,. \qquad \square$$

The following technical lemma can be used to infer $\eta \lhd_k \zeta$ from $\eta \lhd_{k+1} \zeta$.

Lemma 3.4.3. *Suppose that $\eta, \zeta \in \Gamma^{+n}$ are words with $\eta \lhd_{k+1} \zeta$. If there exists some word ξ with*

$$\xi \lhd_k \eta\,, \quad \xi \lhd_k \zeta\,, \quad \textit{and} \quad (\xi)_k = (\eta)_k\,,$$

then we have $\eta \lhd_k \zeta$.

Proof. We prove the claim by induction on $n - k$. If $k = n$ then we have $(\eta)_n = (\xi)_n \leq (\zeta)_n$ which implies $\eta \lhd_n \zeta$. Suppose that $k < n$ and let

$$\xi = x_0 \cdots x_r\,, \quad \eta = y_0 \cdots y_s\,, \quad \zeta = z_0 \cdots z_t\,, \qquad \text{for } x_i, y_i, z_i \in \Gamma^{+(n-1)}.$$

Let $s \leq i \leq t$. $\eta \lhd_{k+1} \zeta$ implies that

$$y_0 \cdots y_{s-1} = z_0 \cdots z_{s-1} \quad \text{and} \quad y_s \lhd_{k+1} z_i\,.$$

Since $x_r \lhd_k y_s$, $x_r \lhd_k z_i$, and $(x_r)_k = (\xi)_k = (\eta)_k = (y_s)_k$ we can apply the induction hypothesis and it follows that $y_s \lhd_k z_i$, for all $s \leq i \leq t$. Hence, we have $\eta \lhd_k z$. ☐

If $\xi \lhd_k \eta$ then we can replace ξ by some other value ζ.

Definition 3.4.4. Let $\xi, \eta, \zeta \in \Gamma^{+n}$ where $\xi = \xi' : x$ and $\zeta = \zeta' : z$. If $\xi \lhd_k \eta$, say, $\eta = \xi' y_0 \dots y_m$, for $y_i \in \Gamma^{+(n-1)}$, we define, by induction on n, the *substitution*

$$\eta[\xi/\zeta]_k := \begin{cases} \eta & \text{if } k > n\,, \\ \zeta' y_0[x/z]_k \cdots y_m[x/z]_k & \text{if } k \leq n\,. \end{cases}$$

We extend this operation to configurations $(\eta, q) \in \Gamma^{+n} \times Q$ by setting

$$(\eta, q)[\xi/\zeta]_k := (\eta[\xi/\zeta]_k, q)\,.$$

Note that this definition ensures that $\xi \lhd_k \eta$ implies $\zeta \lhd_k \eta[\xi/\zeta]$.

Example. Let

$$\begin{aligned} \xi &:= (ab : a) : a : a\,, \\ \eta &:= (ab : a)(aa : b) : ab : a\,, \\ \zeta &:= (ba : b) : bb : c\,. \end{aligned}$$

We have $\xi \lhd_1 \eta$ and

$$\eta[\xi/\zeta]_1 = (ba : b)(bba : b) : bbb : a\,.$$

The above recursive definitions of $\lhd_k$ and $\eta[\xi/\zeta]_k$ were chosen to be compatible with the pushdown operations as stated in the following important lemma. It states that, if ξ is a prefix of η and the operation op does not delete too much of η then ξ is also a prefix of $\mathrm{op}(\eta)$ and op commutes with the substitution $[\xi/\zeta]_k$.

Lemma 3.4.5. *Let* $\mathrm{op} \in \{\mathrm{push}_b, \mathrm{clone}_j, \mathrm{pop}_j\}$ *be a pushdown operation,* $1 \leq k \leq n$*, and let* $\xi, \eta, \zeta \in \Gamma^{+n}$ *be words. If*

$$\xi \lhd_k \eta \quad \textit{and} \quad |(\mathrm{op}(\eta))_i| \geq |(\xi)_i|\,, \quad \textit{for all } i \geq k\,,$$

then we have

$$\xi \lhd_k \mathrm{op}(\eta) \quad \textit{and} \quad \mathrm{op}(\eta[\xi/\zeta]_k) = (\mathrm{op}(\eta))[\xi/\zeta]_k\,.$$

Proof. We prove the claims by induction on n. Clearly, we only need to consider the case that $k \leq n$. Let

$$\xi = x_0 \cdots x_r\,, \quad \eta = y_0 \cdots y_s\,, \quad \zeta = z_0 \cdots z_t\,, \qquad \text{for } x_i, y_i, z_i \in \Gamma^{+(n-1)}.$$

(A) First we consider the case that $\mathrm{op} = \mathrm{push}_b$. For $n = k = 1$, we have

$$\begin{aligned}
\mathrm{push}_b(\eta[\xi/\zeta]_1) &= \mathrm{push}_b(z_0 \cdots z_{t-1} y_r \cdots y_s) \\
&= z_0 \cdots z_{t-1} y_r \cdots y_s b \\
&= (y_0 \cdots y_s b)[\xi/\zeta]_1 \\
&= (\mathrm{push}_b(\eta))[\xi/\zeta]_1\,,
\end{aligned}$$

and, for $n > 1$,

$$\begin{aligned}
&\mathrm{push}_b(\eta[\xi/\zeta]_k) \\
&\quad = \mathrm{push}_b(z_0 \cdots z_{t-1} y_r[x_r/z_t]_k \cdots y_s[x_r/z_t]_k) \\
&\quad = z_0 \cdots z_{t-1} y_r[x_r/z_t]_k \cdots y_{s-1}[x_r/z_t]_k (\mathrm{push}_b(y_s[x_r/z_t]_k)) \\
&\quad = z_0 \cdots z_{t-1} y_r[x_r/z_t]_k \cdots y_{s-1}[x_r/z_t]_k (\mathrm{push}_b(y_s))[x_r/z_t]_k \\
&\quad = (y_0 \cdots y_{s-1} \mathrm{push}_b(y_s))[\xi/\zeta]_k \\
&\quad = (\mathrm{push}_b(\eta))[\xi/\zeta]_k\,.
\end{aligned}$$

(B) Suppose that $\mathrm{op} = \mathrm{clone}_j$. For $n = j$, we have

$$\begin{aligned}
\mathrm{clone}_j(\eta[\xi/\zeta]_k) &= \mathrm{clone}_j(z_0 \cdots z_{t-1} y_r[x_r/z_t]_k \cdots y_s[x_r/z_t]_k) \\
&= z_0 \cdots z_{t-1} y_r[x_r/z_t]_k \cdots y_s[x_r/z_t]_k y_s[x_r/z_t]_k \\
&= (y_0 \cdots y_s y_s)[\xi/\zeta]_k \\
&= (\mathrm{clone}_j(\eta))[\xi/\zeta]_k\,,
\end{aligned}$$

and, for $n > j$,

$$\begin{aligned}
&\operatorname{clone}_j(\eta[\xi/\zeta]_k) \\
&\quad = \operatorname{clone}_j(z_0 \cdots z_{t-1} y_r[x_r/z_t]_k \cdots y_s[x_r/z_t]_k) \\
&\quad = z_0 \cdots z_{t-1} y_r[x_r/z_t]_k \cdots y_{s-1}[x_r/z_t]_k(\operatorname{clone}_j(y_s[x_r/z_t]_k)) \\
&\quad = z_0 \cdots z_{t-1} y_r[x_r/z_t]_k \cdots y_{s-1}[x_r/z_t]_k(\operatorname{clone}_j(y_s))[x_r/z_t]_k \\
&\quad = (y_0 \cdots y_{s-1}\operatorname{clone}_j(y_s))[\xi/\zeta]_k \\
&\quad = (\operatorname{clone}_j(\eta))[\xi/\zeta]_k\,.
\end{aligned}$$

(c) Finally, consider the case that op $= \operatorname{pop}_j$. Since

$$r = |(\xi)_n| \leq |(\operatorname{pop}_n(\eta))_n| = s - 1\,,$$

we have, for $n = j$,

$$\begin{aligned}
\operatorname{pop}_n(\eta[\xi/\zeta]_k) &= \operatorname{pop}_n(z_0 \cdots z_{t-1} y_r[x_r/z_t]_k \cdots y_{s-1}[x_r/z_t]_k y_s[x_r/z_t]_k) \\
&= z_0 \cdots z_{t-1} y_r[x_r/z_t]_k \cdots y_{s-1}[x_r/z_t]_k \\
&= (y_0 \cdots y_{s-1})[\xi/\zeta]_k \\
&= (\operatorname{pop}_n(\eta))[\xi/\zeta]_k\,,
\end{aligned}$$

and, for $n > j$,

$$\begin{aligned}
&\operatorname{pop}_j(\eta[\xi/\zeta]_k) \\
&= \operatorname{pop}_j(z_0 \cdots z_{t-1} y_r[x_r/z_t]_k \cdots y_s[x_r/z_t]_k) \\
&= z_0 \cdots z_{t-1} y_r[x_r/z_t]_k \cdots y_{s-1}[x_r/z_t]_k(\operatorname{pop}_j(y_s[x_r/z_t]_k)) \\
&= z_0 \cdots z_{t-1} y_r[x_r/z_t]_k \cdots y_{s-1}[x_r/z_t]_k(\operatorname{pop}_j(y_s))[x_r/z_t]_k \\
&= (y_0 \cdots y_{s-1}\operatorname{pop}_j(y_s))[\xi/\zeta]_k \\
&= (\operatorname{pop}_j(\eta))[\xi/\zeta]_k\,.
\end{aligned}$$

(d) In all cases we have $\xi \triangleleft_k \operatorname{op}(\eta)$ since $(\operatorname{op}(\eta))[\xi/\zeta]_k$ is defined. □

By induction, it follows that each transition of a run can be lifted from η to $\eta[\xi/\zeta]_k$ as long as the word ξ is still contained in η.

Corollary 3.4.6. *Let $\xi, \zeta, \eta, \eta' \in \Gamma^{+n}$ be words such that $|(\eta')_i| \geq |(\xi)_i|$, for all $i \geq k$. Then*

$$\xi \triangleleft_k \eta \quad \textit{and} \quad (\eta, q) \vdash^a (\eta', q')$$

implies

$$\xi \triangleleft_k \eta' \quad \textit{and} \quad (\eta[\xi/\zeta]_k, q) \vdash^a (\eta'[\xi/\zeta]_k, q')\,.$$

Proof. Let $\delta = (q, c, a, q', \mathrm{op}) \in \Delta$ be the transition witnessing $(\eta, q) \vdash^a (\eta', q')$. By definition, we have $(\eta[\xi/\zeta]_k)_0 = (\eta)_0$. Hence

$$\mathrm{top}(\eta[\xi/\zeta]_k, q) = \mathrm{top}(\eta, q) = (a, q)$$

and we can apply δ to $(\eta[\xi/\zeta]_k, q)$. The resulting configuration (μ, q') has the stack contents

$$\mu = \mathrm{op}(\eta[\xi/\zeta]_k) = (\mathrm{op}(\eta))[\xi/\zeta]_k = \eta'[\xi/\zeta]_k .$$

The relation $\xi \lhd_k \eta' = \mathrm{op}(\eta)$ follows immediately from the preceding lemma. □

In particular, if we have a run such that the stack content ξ of the first configuration is never touched then we can replace ξ by an arbitrary other word ζ and we obtain again a valid run.

Lemma 3.4.7. *Let r be a run and $x \in \mathrm{dom}(r)$ its first vertex. Suppose that*

$$\xi := \pi r(x) \lhd_k \pi r(y) , \qquad \text{for all } y \in \mathrm{dom}(r) .$$

If $\zeta \in \Gamma^{+n}$ is an arbitrary word then the function r' defined by

$$r'(y) := r(y)[\xi/\zeta]_k , \qquad \text{for } y \in \mathrm{dom}(r) ,$$

forms a valid run.

Proof. We can use Corollary 3.4.6 to prove, by induction on $\leq$, that

$$\xi \lhd_k \pi r(y) \quad \text{and} \quad r(y)[\xi/\zeta]_k \vdash r(\sigma y)[\xi/\zeta]_k . \qquad □$$

3.5 Weak domination

In this section we introduce the weak domination order $\sqsubseteq_k^*$ which will be our main tool for decomposing runs.

Definition 3.5.1. (a) For $\xi, \zeta \in \Gamma^{+n}$ and $0 \leq k \leq n$, we say that ξ *weakly k-dominates* ζ, written $\xi \sqsubseteq_k \zeta$, if there exists a sequence POP of pop-operations such that

$$\mathrm{pop}_k(\xi) = \mathrm{pop}_k(\mathrm{POP}(\zeta)) .$$

(b) If r is a run and $x, y \in \mathrm{dom}(r)$ then we define

$$x \sqsubseteq_k y \quad :\text{iff} \quad \pi r(x) \sqsubseteq_k \pi r(y) ,$$
$$\text{and} \quad x \sqsubseteq_k^* y \quad :\text{iff} \quad x \leq y \quad \text{and} \quad x \sqsubseteq_k z \quad \text{for all } x \leq z \leq y .$$

The greatest lower $\sqsubseteq_k^*$-bound of x and y will be denoted by $x \sqcap_k y$.

(c) Let r be a run and $x \in \mathrm{dom}(r)$. By $\omega_k(x)$ we denote the $\leq$-minimal element $y \in \mathrm{dom}(r)$ such that $x \leq y$ and $x \not\sqsubseteq_k^* y$. Note that $\omega_k(x)$ might be undefined.

Lemma 3.5.2. $(\mathrm{dom}(r), \sqsubseteq_k^*)$ *is a forest.*

Remark. Note that the original ordering $\leq$ of a run r coincides with the ordering we obtain when traversing the forest $(\mathrm{dom}(r), \sqsubseteq_k^*)$ in "prefix ordering" (which is not related to the prefix order $\preceq$). This is the same as the lexicographic ordering $\leq_{\mathrm{lex}}$ of $(\mathrm{dom}(r), \sqsubseteq_k^*)$ which in this case is defined by

$$x \leq_{\mathrm{lex}} y \quad \text{iff} \quad x \sqsubseteq_k^* y \text{ or } u < v \text{ where } u \text{ and } v \text{ are the immediate } \sqsubseteq_k^*\text{-successors of } x \sqcap_k y \text{ with } u \sqsubseteq_k^* x \text{ and } v \sqsubseteq_k^* y\,.$$

In particular, if $x \sqsubseteq_k^* y$ and $x \not\sqsubseteq_k^* z$ then $z < x \leq y$ or $x \leq y < z$.

Example. Consider the run

$$\begin{aligned}\varepsilon : \varepsilon : a &\vdash \varepsilon : \varepsilon : ab \vdash \varepsilon : ab : ab \vdash \varepsilon : ab : a \vdash (ab : a) : ab : a \\ &\vdash (ab : a) : \varepsilon : ab \vdash (ab : a) : \varepsilon : a \vdash \varepsilon : ab : a \vdash \varepsilon : \varepsilon : ab \\ &\vdash \varepsilon : \varepsilon : a\end{aligned}$$

where we have left out the states for simplicity. The weak domination orderings $\sqsubseteq_3^*$, $\sqsubseteq_2^*$ and $\sqsubseteq_1^*$ are shown in Figure 3.1.

Lemma 3.5.3. *Let $\xi, \eta \in \Gamma^{+n}$. If $\xi \lhd_k \eta$ then $\xi \sqsubseteq_k \eta$.*

Proof. Let $\xi = x_0 \cdots x_r$ and $\eta = y_0 \cdots y_s$, for $x_i, y_i \in \Gamma^{+(n-1)}$. We prove the claim by induction on n. If $n = k$ then

$$\mathrm{pop}_k(\xi) = x_0 \cdots x_{r-1} = y_0 \cdots y_{r-1} = (\mathrm{pop}_k)^{s-r+1}(\eta)\,.$$

For $n > k$, we have, by definition of $\lhd_k$,

$$\xi = x_0 \cdots x_r \lhd_k y_0 \cdots y_r = (\mathrm{pop}_n)^{s-r}(\eta)\,.$$

By induction hypothesis, there exists a sequence POP of pop-operations such that

$$\mathrm{pop}_k(x_r) = \mathrm{pop}_k(\mathrm{POP}(y_r))\,.$$

It follows that

$$\mathrm{pop}_k(\xi) = \big(\mathrm{pop}_k \circ \mathrm{POP} \circ \mathrm{pop}_n^{s-r}\big)(\eta)\,. \qquad \square$$

In the following sequence of lemmas we relate the structure of the weak dominance order to the stack contents of the underlying run. First, we consider $\leq$-successors that are not $\sqsubseteq_k^*$-successors.

Lemma 3.5.4. *Let r be a run and x, $y \in \mathrm{dom}(r)$ vertices such that $x \sqsubseteq_k y$ and $x \not\sqsubseteq_k \sigma y$. Then $\pi r(\sigma y) = \mathrm{pop}_l \pi r(x)$, for some $l \geq k$.*

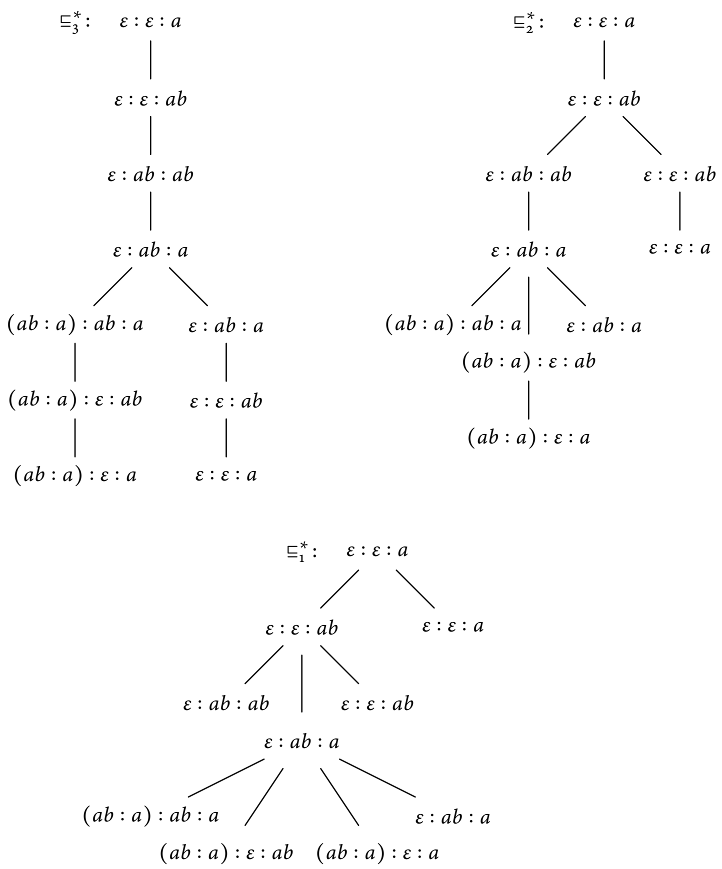


Figure 3.1: The weak domination orders $\sqsubseteq_3^*$, $\sqsubseteq_2^*$ and $\sqsubseteq_1^*$.

Proof. Let $\pi r(x) = \xi_n : \cdots : \xi_0$. Since $x \sqsubseteq_k y$ we have, for some $i \geq k$,

$$\pi r(y) = \xi_n : \cdots : \xi_{i+1} : \xi_i \eta_i : \eta_{i-1} : \cdots : \eta_0 \,,$$

where either $\xi_i : \cdots : \xi_0 \sqsubseteq_k \xi_i \eta_i$, or $i = k$ and $\eta_k = \varepsilon$. Since $x \not\sqsubseteq_k \sigma y$ there exist some index $l \geq i \geq k$ such that

$$\pi r(\sigma y) = \mathrm{pop}_l \pi r(y) = \xi_n : \cdots : \xi_l = \mathrm{pop}_l \pi r(x) \,. \qquad \square$$

A configuration with several immediate $\sqsubseteq_k^*$-successors must perform a clone_i-operation and the stack contents of the successors have a certain format.

Lemma 3.5.5. *Let r be a run, $k > 1$, and $x \in \mathrm{dom}(r)$ a vertex with several immediate $\sqsubseteq_k^*$-successors $y_0, \dots, y_m$, $m \geq 1$. Set $\xi_n : \cdots : \xi_1 = \pi r(x)$.*
There exists an index $i \geq k$ satisfying the following conditions.

(a) *There is a* $\mathrm{push}(i)$*-operation at x.*

(b) *There are indices*

$$1 = l(0) \leq k \leq l(1) \leq \cdots \leq l(m) \leq i$$

and words $\zeta_0 \sqsubseteq_0 \xi_{l(0)}, \dots, \zeta_m \sqsubseteq_0 \xi_{l(m)}$ such that, for all $s < m$, we have

$$\pi r(y_s) = \xi_n : \cdots : \xi_{i+1} : (\xi_i : \cdots : \xi_1) : \xi_{i-1} : \cdots : \xi_{l(s)+1} : \zeta_s$$

and $\pi r(y_{s+1}) = \mathrm{pop}_l \pi r(y_s)$, for some $k \leq l \leq i$.

(c) *$y_s \sqsubseteq_i^* y_t$, for all $s \leq t < m$, and $y_s \sqsubseteq_i^* y_m$ iff $\pi r(y_m) \neq \pi r(x)$.*

(d) *$x \sqsubseteq_l^* y_s$, for all $s \leq m$ and every $l \leq n$. Furthermore, $y_0, \dots, y_m$ are immediate $\sqsubseteq_l^*$-successors of x, for all $l \leq k$.*

Proof. (a) If $\pi r(\sigma x) = \mathrm{pop}_i \pi r(x)$, for some i, then $x \sqsubseteq_k^* z$ implies $\sigma x \sqsubseteq_k^* z$. Hence, x has at most one immediate $\sqsubseteq_k^*$-successor. The same is the case for a $\mathrm{push}(i)$-operation with with $i < k$.

(b) We proceed by induction on s. For $s = 0$, the claim follows from (a) since $y_0 = \sigma x$. Suppose that $s > 0$ and

$$\pi r(y_{s-1}) = \xi_n : \cdots : \xi_{i+1} : (\xi_i : \cdots : \xi_1) : \xi_{i-1} : \cdots : \xi_{l(s-1)+1} : \zeta_{s-1} \,,$$

where $l(s-1) \leq i$.

(A) If $\pi r(\sigma^{-1} y_s) = \pi r(y_{s-1})$ then $x \sqsubseteq_k^* y_s$ and $\sigma^{-1} y_s \not\sqsubseteq_k^* y_s$ imply that,

$$\pi r(y_s) = \mathrm{pop}_l \pi r(\sigma^{-1} y_s) = \mathrm{pop}_l \pi r(y_{s-1}) \,,$$

for some $k \leq l \leq i$. Hence, if $l > l(s-1)$ then

$$\pi r(y_s) = \xi_n : \cdots : \xi_{i+1} : (\xi_i : \cdots : \xi_1) : \xi_{i-1} : \cdots : \xi_l\,,$$

and, for $l \leq l(s-1)$, we have

$$\pi r(y_s) = \xi_n : \cdots : \xi_{i+1} : (\xi_i : \cdots : \xi_1) : \xi_{i-1} : \cdots : \xi_{l(s-1)+1} : \zeta_s\,,$$

where $\zeta_s := \mathrm{pop}_l(\zeta_{s-1})$.

(B) If $\pi r(\sigma^{-1} y_s) \neq \pi r(y_{s-1})$ we fix the maximal index h such that

$$(\pi r(\sigma^{-1} y_s))_h \neq (\pi r(y_{s-1}))_h\,.$$

We claim that $h < i$. Suppose otherwise. Since $y_{s-1} \sqsubseteq^*_k \sigma^{-1} y_s$ we have

$$\pi r(\sigma^{-1} y_s) = \xi_n : \cdots : \xi_{h+1} : \xi_h \eta_h : \eta_{h-1} : \cdots : \eta_1$$

for some words $\eta_h, \dots, \eta_1$ such that

$$\begin{aligned} &\xi_h : \cdots : \xi_{i+1} : (\xi_i : \cdots : \xi_1) : \xi_{i-1} : \cdots : \xi_{l(s-1)+1} : \zeta_{s-1} \\ \sqsubseteq^*_k\ &\xi_h \eta_h : \eta_{h-1} : \cdots : \eta_1\,. \end{aligned}$$

Furthermore, by choice of h we have $\eta_h \neq \varepsilon$, and if $h = i$ then

$$\eta_h = (\xi_{i-1} : \cdots : \xi_1)\eta'_h\,,$$

for some $\eta'_h \neq \varepsilon$. Hence,

$$(*) \quad \xi_h : \cdots : \xi_{i+1} : (\xi_i : \cdots : \xi_1) : \xi_{i-1} : \cdots : \xi_{l(s-1)+1} : \zeta_{s-1} \sqsubseteq^*_k \xi_h \eta_h\,.$$

Since $x \sqsubseteq^*_k y_s$ and $\sigma^{-1} y_s \not\sqsubseteq^*_k y_s$ it follows that

$$\pi r(y_s) = \mathrm{pop}_j \pi r(\sigma^{-1} y_s) = \xi_n : \cdots : \xi_{h+1} : \xi_h \eta_h : \eta_{h-1} : \cdots : \eta_j\,,$$

for some $k \leq j \leq h$. But $(*)$ implies

$$\begin{aligned} \pi r(y_{s-1}) &= \xi_n : \cdots : \xi_{i+1} : (\xi_i : \cdots : \xi_1) : \xi_{i-1} : \cdots : \xi_{l(s-1)+1} : \zeta_{s-1} \\ &\sqsubseteq^*_k \xi_n : \cdots : \xi_{h+1} : \xi_h \eta_h : \eta_{h-1} : \cdots : \eta_j = \pi r(y_s)\,, \end{aligned}$$

that is, $y_{s-1} \sqsubseteq^*_k y_s$. Contradiction.

(C) Consequently, we have $h < i$. If $h > l(s-1)$ then $y_{s-1} \sqsubseteq^*_k \sigma^{-1} y_s$ implies

$$\begin{aligned} \pi r(\sigma^{-1} y_s) = \xi_n : \cdots : \xi_{i+1} : (\xi_i : \cdots : \xi_1) : \xi_{i-1} : \cdots : \xi_{h+1} : \\ \xi_h \eta_h : \eta_{h-1} : \cdots : \eta_1\,. \end{aligned}$$

Again, by $x \sqsubseteq^*_k y_s$ and $\sigma^{-1} y_s \not\sqsubseteq^*_k y_s$ it follows that

$$\pi r(y_s) = \mathrm{pop}_l \pi r(\sigma^{-1} y_s)$$

for some $k \leq l \leq i$. If $l \leq h$ then

$$\begin{aligned}\pi r(y_s) = \xi_n : \dots : \xi_{i+1} : (\xi_i : \dots : \xi_1) : \xi_{i-1} : \dots : \xi_{h+1} :\\ \xi_h \eta_h : \eta_{h-1} : \dots : \eta_l\end{aligned}$$

and as above it follows that $y_{s-1} \sqsubseteq_k^* y_s$. Contradiction. Therefore, $l > h$ and

$$\pi r(y_s) = \xi_n : \dots : \xi_{i+1} : (\xi_i : \dots : \xi_1) : \xi_{i-1} : \dots : \xi_l = \mathrm{pop}_l r(y_{s-1})$$

as desired.

It remains to consider the case that $h \leq l(s-1)$. Let $\zeta_s = \mu_{l(s-1)} : \dots : \mu_1$. Since $y_{s-1} \sqsubseteq_k^* \sigma^{-1} y_s$ we have

$$\begin{aligned}\pi r(\sigma^{-1} y_s) = \xi_n : \dots : \xi_{i+1} : (\xi_i : \dots : \xi_1) : \xi_{i-1} : \dots : \xi_{l(s-1)+1} :\\ \mu_{l(s-1)} : \dots : \mu_{h+1} : \mu_h \eta_h : \eta_{h-1} : \dots : \eta_1 ,\end{aligned}$$

As above, there is some $h < l \leq i$ such that $\pi r(y_s) = \mathrm{pop}_l \pi r(\sigma^{-1} y_s)$ which implies

$$\begin{aligned}\pi r(y_s) &= \xi_n : \dots : \xi_{i+1} : (\xi_i : \dots : \xi_1) : \xi_{i-1} : \dots : \xi_{l(s-1)+1} : \mu_{l(s-1)} : \dots : \mu_l\\ &= \mathrm{pop}_l \pi r(y_{s-1}) .\end{aligned}$$

(D) Finally, if $s < t$ then $y_s \not\sqsubseteq_k^* y_t$ implies that $l(s) \leq l(t)$ and $l(t) \geq k$.

(c) By induction on t, we have $y_s \sqsubseteq_i^* y_{t-1} \sqsubseteq_k^* \sigma^{-1} y_t$ which implies $y_s \sqsubseteq_i^* \sigma^{-1} y_t$. By (b), we also have $y_s \sqsubseteq_i y_t$. Together it follows that $y_s \sqsubseteq_i^* y_t$. If $\pi r(y_m) \neq \pi r(x)$ then $x \sqsubseteq_k^* y_m$ implies

$$\pi r(y_m) = \xi_n : \dots : \xi_{i+1} : (\xi_i : \dots : \xi_1) : \eta_{i-1} : \dots : \eta_1 ,$$

and the claim follows as above.

(d) By (a), we have $x \sqsubseteq_l^* y_s$, for all $s \leq m$ and every $l \leq n$. Furthermore, if there were some element $x \sqsubset_l^* z \sqsubset_l^* y_s$, for $l \leq k$, then this would imply $x \sqsubset_k^* z \sqsubset_k^* y_s$ which is impossible. □

Finally, we collect some basic facts about the function ω_k.

Lemma 3.5.6. *Let r be a run, $x \in \mathrm{dom}(r)$, and $y := \omega_k(x)$. The element $x \sqcap_k y$ is the immediate $\sqsubseteq_k^*$-predecessor of y and*

$$\pi r(y) = \mathrm{pop}_l \pi r(x) \qquad \textit{for some } l \geq k .$$

Proof. Suppose that there is some element z such that $x \sqcap_k y \sqsubset_k^* z \sqsubset_k^* y$. Then $x < z < y$ and, by choice of y, we have $x \sqsubseteq_k^* z$. Hence, $x \sqsubseteq_k^* z \sqsubseteq_k^* y$. A contradiction. The second claim is a special case of Lemma 3.5.4. □

Lemma 3.5.7. *Let r be a run and $x \in \mathrm{dom}(r)$. If $i < k$ then*

$$\omega_k(x) = \omega_i(x) \quad \textit{or} \quad \omega_k(x) = \omega_k(\omega_i(x))\,.$$

Proof. Let $y := \omega_i(x)$ and $z := \omega_k(x)$. If $z < y$ then $x \sqsubseteq_i^* z$ which implies $x \sqsubseteq_k^* z$. A contradiction.

Suppose that $y < z$. By Lemma 3.5.6, there exist indices $l \geq i$ and $m \geq k$ such that

$$\pi r(y) = \mathrm{pop}_l \pi r(x) \qquad \text{and} \qquad \pi r(z) = \mathrm{pop}_m \pi r(x)\,.$$

If $y < z$ then we have $l < k$. Consequently,

$$\pi r(z) = \mathrm{pop}_m \pi r(x) = \mathrm{pop}_m \pi r(y)\,,$$

and it follows that $y \not\sqsubseteq_k^* z$. Hence, $\omega_k(y) \leq z$. On the other hand, we have

$$\pi r(\omega_k(y)) = \mathrm{pop}_h \pi r(y) = \mathrm{pop}_h \pi r(x)\,, \qquad \text{for some } h \geq k\,.$$

Therefore, we have $x \not\sqsubseteq_k^* \omega_k(y)$ which implies $z \leq \omega_k(y)$. Together, it follows that $z = \omega_k(y)$. □

3.6 Strong domination and holes

Remember that we want to decompose a given run r into parts such that in each subrun s we can apply a substitution, that is, if x is the first element of $\mathrm{dom}(s)$ we would like to have $x \lhd_k y$, for all $y \in \mathrm{dom}(s)$. Therefore, we define a second domination order by combining the relations $\lhd_k$ and $\sqsubseteq_k^*$.

Definition 3.6.1. For a run r, elements $x, y \in \mathrm{dom}(r)$, and a number $1 \leq k \leq n$, we define the *strong domination order* $\leqq_k$ by

$$\begin{aligned} x \leqq_k y \quad &:\text{iff} \quad x \sqsubseteq_k^* y \quad \text{and} \\ &\phantom{:\text{iff}} \quad x \lhd_i z \quad \text{for all } i \geq k \text{ and } x \sqsubseteq_i^* z \sqsubseteq_i^* y\,. \end{aligned}$$

The greatest lower $\leqq_k$-bound of x and y will be denoted by $x \sqcap_k y$.

Example. Figure 3.2 shows the strong domination orderings $\leqq_2$ and $\leqq_1$ corresponding to the run whose weak domination order is depicted in Figure 3.1.

Let us collect some basic properties of the strong domination order.

Lemma 3.6.2. *Let $x \leqq_k y$. We have $x \leqq_k \sigma y$ iff $x \sqsubseteq_k \sigma y$ and $x \lhd_k \sigma y$.*

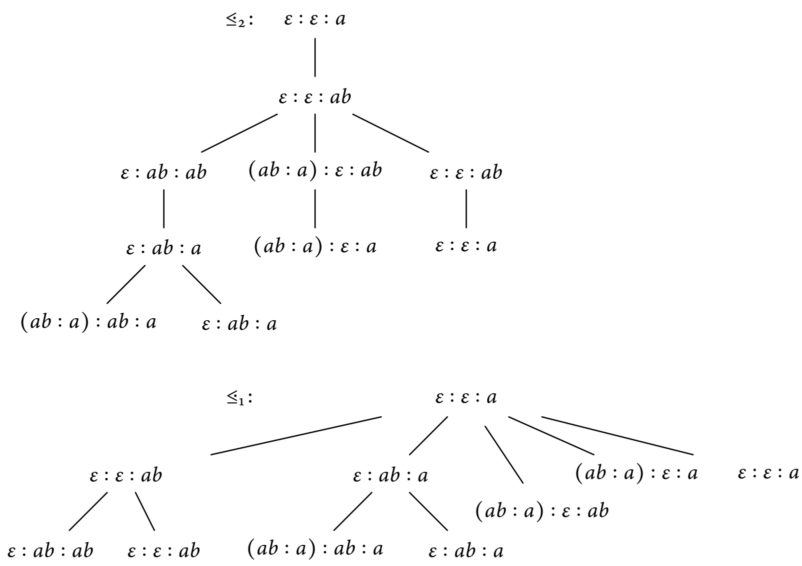


Figure 3.2: The strong domination orders $\lesssim_2$ and $\lesssim_1$.

Proof. ($\Rightarrow$) follows immediately from the definition.

($\Leftarrow$) Suppose $x \not\lesssim_k \sigma y$. By definition, we either have $x \not\sqsubseteq_k^* \sigma y$ or there is some $x \sqsubseteq_i^* z \sqsubseteq_i^* \sigma y$, for $i \geq k$, with $x \ntriangleleft_i z$. In the first case, $x \sqsubseteq_k^* y$ implies $x \not\sqsubseteq_k y$. For the second case, note that, if $z \sqsubset_i^* \sigma y$ then $z \sqsubseteq_i^* y$, and $x \lesssim_i y$ implies $x \triangleleft_i z$. Consequently, $z = \sigma y$ and $x \ntriangleleft_k \sigma y$. ☐

Lemma 3.6.3. *Suppose that* $x \lesssim_{k+1} \sigma x$ *and* $x \not\lesssim_k \sigma x$.

(a) *There is a* pop_k*-operation at* x.

(b) *There is a* $\mathrm{push}(i)$*-operation at* $w := x \sqcap_k \sigma x$, *for some* $i \geq k$.

(c) *If* $u \in \mathrm{dom}(r)$ *is some element with* $u \lesssim_k x$ *and* $u \not\lesssim_k \sigma x$ *then there are words* $\xi_n, \dots, \xi_0$ *and* $\mu_n, \dots, \mu_{k+1}$ *such that*

$$\pi r(u) = \xi_n : \cdots : \xi_0 \quad \textit{and} \quad \pi r(\sigma x) = \xi_n \mu_n : \cdots : \xi_{k+1} \mu_{k+1} : \xi_k \,.$$

Proof. (a) Since $x \lesssim_{k+1} \sigma x$ and $x \not\lesssim_k \sigma x$ we have

$$\pi r(\sigma x) = \mathrm{pop}_k(\pi r(x)) \,.$$

(b) If the operation at w were a $\mathrm{push}(i)$ or a pop_i with $i < k$ then $w \lesssim_k x, \sigma x$ would imply $\sigma w \lesssim_k x, \sigma x$ and we would have $w \neq x \sqcap_k \sigma x$. If there were a pop_i-operation at w with $i \geq k$ then w would have no $\lesssim_k$-successor. Consequently, the operation at w is a $\mathrm{push}(i)$ with $i \geq k$.

(c) Let $\pi r(u) = \xi_n : \cdots : \xi_0$. $u \lesssim_k x$ implies $u \triangleleft_k x$. Hence, there are words $\mu_n : \cdots : \mu_0$ such that

$$\pi r(x) = \xi_n \mu_n : \cdots : \xi_k \mu_k : \mu_{k-1} : \cdots : \mu_0 \,,$$

and $\quad \pi r(\sigma x) = \mathrm{pop}_k \pi r(x) = \xi_n \mu_n : \cdots : \xi_k \mu_k \,.$

We claim that $\mu_k = \varepsilon$. Suppose otherwise. Then $u \lhd_k \sigma x$ and it follows that $u \not\sqsubseteq_k^* \sigma x$. Since $u \sqsubseteq_k^* x$ this implies $u \not\trianglelefteq_k \sigma x$. Consequently, $\mu_i = \varepsilon$, for all $k \leq i \leq n$. Contradiction. ☐

We will study decompositions of a run into parts of the following form.

Definition 3.6.4. For a run r and a vertex $x \in \mathrm{dom}(r)$ we define

$$D_k(x) := \{\, y \in \mathrm{dom}(r) \mid x \trianglelefteq_k y \,\}\,,$$
$$E_k(x) := \{\, y \in \mathrm{dom}(r) \mid x \sqsubseteq_k^* y \,\}\,.$$

Remark. Note that $D_k(x)$ is an initial segment of $E_k(x)$.

Lemma 3.6.5. *$x \trianglelefteq_k y$ iff $D_k(y) \subseteq D_k(x)$.*

Proof. ($\Leftarrow$) By definition, $y \in D_k(y) \subseteq D_k(x)$ implies $x \trianglelefteq_k y$.
($\Rightarrow$) If $z \in D_k(y)$ then $y \trianglelefteq_k z$. Hence, $x \trianglelefteq_k y \trianglelefteq_k z$ and $z \in D_k(x)$. ☐

It will turn out that a good way to construct such a decomposition is by considering subruns whose domain is of the form $D_k(v)$. But in doing so we face the problem that such subruns might contain *holes*, that is, there might be vertices $x, y \in D_k(v)$, $x \prec y$, such that all vertices $x \prec z \prec y$ are not contained in $D_k(v)$. In the remainder of this section we study the structure of such a hole.

Definition 3.6.6. Let r be a run, $v \in \mathrm{dom}(r)$, and $1 \leq k \leq n$.
(a) If z is the $\preceq$-maximal element of $E_k(v)$ we define

$$\Omega_k(v) := \big\{(*, \rho r(z))\big\} \cup \big\{\, (h, q) \;\big|\; r(z) \vdash (\mathrm{pop}_h \pi r(v), q),\ h \geq k \,\big\}\,.$$

(b) $D_k(v)$ has a *hole* at x if $x \in D_k(v)$ and $\sigma x \in E_k(v) \smallsetminus D_k(v)$. In this case we define

$$H(x) := \{\, y \in \mathrm{dom}(r) \mid z \in E_k(v) \smallsetminus D_k(v) \text{ for all } x \prec z \preceq y \,\}\,.$$

We say that the hole is *between* x and y if

$$H(x) = \{\, z \mid x \prec z \prec y \,\}\,.$$

If such an element y exists then we call the hole *properly terminated*. The maximal element y such that

$$\{\, z \mid x \prec z \prec y \,\} \subseteq H(x)$$

is the *end point* of the hole. Note that the end point is contained in $H(x)$ if and only if the hole is not properly terminated.

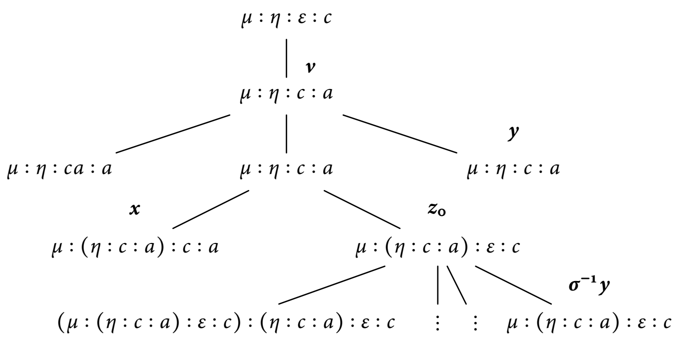


Figure 3.3: A hole in $D_1(v)$ between x and y.

(c) An *exit point* of $D_k(v)$ is a $\leq_k$-minimal element of $E_k(v) \setminus D_k(v)$. The set of all exit points of $D_k(v)$ is denoted by $X_k(v)$. The *order* of an exit point x is the number k such that

$$\pi r(x) = \mathrm{pop}_k(\pi r(\sigma^{-1}(x)))\,,$$

and its *type* is the triple

$$(k, \rho r(x), \Omega_{k+1}(x))$$

where k is the order of x.

(d) Suppose that there is a hole in $D_k(v)$ at x with end point y. We define the *principal sequence* $z_0, \dots, z_m$ of this hole and the *associated sequence* $l(0), \dots, l(m)$ of indices inductively as follows. $z_0 := \sigma x$ and $l(0)$ is the index such that $\pi r(z_0) = \mathrm{pop}_{l(0)} \pi r(x)$. Suppose that z_j and $l(j)$ are already defined. If $z_j \not\sqsubseteq^*_{l(j)+1} y$ then we define $z_{j+1} := \omega_{l(j)+1}(z_j)$, and $l(j+1)$ is the index such that $\pi r(z_{j+1}) = \mathrm{pop}_{l(j+1)} \pi r(z_j)$. We continue this construction until we reach a vertex with $z_j \sqsubseteq^*_{l(j)+1} y$.

If $z_j \neq y$ then we call the element z_j a *principal exit point* of $D_k(v)$. Its *order* is the number $l(j)$. By $P_{kl}(v)$ we denote the set of all principal exit points of $D_k(v)$ of order l.

(e) Suppose there is a hole at x with principal sequence $z_0, \dots, z_m$ and associated sequence of indices $l(0), \dots, l(m)$. Set $h := m - 1$ if the hole is properly terminated and $h := m$, otherwise. The *type* of the hole is the sequence

$$\big(l(0), \rho r(z_0), \Omega_{l(0)+1}(z_0)\big), \dots, \big(l(h), \rho r(z_h), \Omega_{l(h)+1}(z_h)\big)\,,$$

of the types of $z_0, \dots, z_h$

Lemma 3.6.7. *Let r be a run, $v \in \mathrm{dom}(r)$ and suppose that there is a hole in $D_k(v)$ at x.*

$$H(x) = \dot{\bigcup} \{\, D_k(z) \mid z \in H(x) \cap X_k(v) \,\}$$

and $E_k(v) = D_k(v) \cup \dot{\bigcup} \{ D_k(z) \mid z \in X_k(v) \}$.

Proof. Since the second equation follows from the first one we only need to prove the first equation.

($\subseteq$) If $y \in H(x)$ then $z \leq_k y$, for some exit point z. If $z \notin H(x)$ then we have $z \prec x \prec y$ and $z \sqsubseteq_k^* y$, and it follows that $z \sqsubseteq_k^* x$. Hence, $x \in D_k(v)$ implies $z \in D_k(v)$. A contradiction.

($\supseteq$) Let $y \in D_k(z)$ for some exit point $z \in H(x)$. Then $z \sqsubseteq_k^* y$ and $z \in E_k(v) \setminus D_k(v)$ implies $y \in E_k(v) \setminus D_k(v)$. It remains to show that there is no element $w \in D_k(v)$ with $x \prec w \preceq y$. Suppose otherwise. Since $z \in H(x)$ we have $z \prec w \preceq y$. Hence, $z \sqsubseteq_k^* y$ implies $z \sqsubseteq_k^* w$. But $v \sqsubseteq_k^* z \sqsubseteq_k^* w$ and $v \leq_k w$ implies $v \leq_k z$. A contradiction. ☐

The following lemma investigates the structure of a hole and it clarifies the role of the principal sequence.

Lemma 3.6.8. *Let r be a run, $v \in \mathrm{dom}(r)$, $1 \leq k \leq n$. Suppose that there is a hole in $D_k(v)$ at x with end point y, let $z_0, \dots, z_m$ be its principal sequence, and $l(0), \dots, l(m)$ the sequence of indices such that*

$$\pi r(z_j) = \mathrm{pop}_{l(j)} \pi r(z_{j-1}) .$$

Suppose that $\pi r(v) = \xi_n : \dots : \xi_0$ and $\pi r(x) = \xi_n \eta_n : \dots : \xi_k \eta_k : \eta_{k-1} : \dots : \eta_0$.

(a) *If $z_j \neq y$ then $z_j \in E_k(v) \setminus D_k(v)$.*

(b) *$k \leq l(0) < \dots < l(m)$, in particular $m < n$.*

(c) *If u_j is the immediate $\sqsubseteq_k^*$-predecessor of z_j then $u_j \in D_k(v)$.*

(d) *We have*

$$\pi r(z_j) = \mathrm{pop}_{l(j)} \pi r(x) = \xi_n \eta_n : \dots : \xi_{l(j)+1} \eta_{l(j)+1} : \xi_{l(j)} \eta_{l(j)} .$$

Furthermore, if $z_j \neq y$ then $\eta_{l(j)} = \varepsilon$.

(e) *If the hole is properly terminated then $z_m = y$.*

Proof. (a) $x \prec z_j \prec y$ implies, by definition of y, that

$$z_j \in H(x) \subseteq E_k(v) \setminus D_k(v) .$$

(b) Since $v \lhd_k x$ and $v \ntriangleleft_k \sigma x$ we have, by Lemma 3.6.3 (a),

$$\pi r(z_0) = \pi r(\sigma x) = \mathrm{pop}_{l(0)} \pi r(x) \qquad \text{with } l(0) \geq k .$$

Furthermore, Lemma 3.5.6 implies that $l(j+1) \geq l(j) + 1$, for $j < m$.

(c) We claim that $v \sqsubseteq_k^* u_{j+1} \sqsubseteq_k^* u_j$, for all $j < m$. Then the result follows by induction on j since $v \sqsubseteq_k^* u_0 \sqsubseteq_k^* x \in D_k(v)$ implies $u_0 \in D_k(v)$ and $v \sqsubseteq_k^* u_{j+1} \sqsubseteq_k^* u_j \in D_k(v)$ implies $u_{j+1} \in D_k(v)$.

Note that $v \leq x < z_{j+1}$ and $v \sqsubseteq_k^* z_{j+1}$ implies $v \sqsubset_k^* z_{j+1}$ and, hence, $v \sqsubseteq_k^* u_{j+1}$. Therefore, we only need to prove that $u_{j+1} \sqsubseteq_k^* u_j$.

By Lemma 3.5.6, the immediate $\sqsubseteq_{l(j)+1}^*$-predecessor of z_{j+1} is

$$w_{j+1} := z_j \sqcap_{l(j)+1} z_{j+1} \,.$$

As w_{j+1} has at least two immediate $\sqsubseteq_{l(j)+1}^*$-successors it follows by Lemma 3.5.5 (d) that z_{j+1} is an immediate $\sqsubseteq_l^*$-successor of w_{j+1}, for all $l \leq l(j) + 1$. Because $k \leq l(j) + 1$ we therefore have $u_{j+1} = w_{j+1} = z_j \sqcap_{l(j)+1} z_{j+1}$. Consequently, we have $u_{j+1} < z_j < z_{j+1}$ and, together with $u_{j+1} \sqsubseteq_k^* z_{j+1}$, it follows that $u_{j+1} \sqsubset_k^* z_j$. Hence, by definition of u_j, we have $u_{j+1} \sqsubseteq_k^* u_j$.

(d) First, consider the case that $l(j) = 1$. By (b), this implies $k = 1$ and $j = 0$. Since $z_0 = \sigma x$ we have $\pi r(z_0) = \mathrm{pop}_{l(0)} \pi r(x)$, by definition of $l(0)$. Finally, we have $\eta_1 = \varepsilon$, by Lemma 3.6.3 (c).

For $l(j) > 1$, we prove the claim by induction on j. For $j = 0$, we have, by definition,

$$\pi r(z_0) = \pi r(\sigma x) = \mathrm{pop}_{l(0)} \pi r(x) = \xi_n \eta_n : \dots : \xi_{l(0)} \eta_{l(0)} \,,$$

and, for $j > 0$, the induction hypothesis implies that

$$\begin{aligned} \pi r(z_j) &= \mathrm{pop}_{l(j)} \pi r(z_{j-1}) \\ &= \mathrm{pop}_{l(j)} (\xi_n \eta_n : \dots : \xi_{l(j-1)} \eta_{l(j-1)}) \\ &= \xi_n \eta_n : \dots : \xi_{l(j)} \eta_{l(j)} \,. \end{aligned}$$

Suppose that $\eta_{l(j)} \neq \varepsilon$. We claim that $z_j = y$.

$$\xi_n : \dots : \xi_0 \lhd_k \pi r(x) = \xi_n \eta_n : \dots : \xi_k \eta_k : \eta_{k-1} : \dots : \eta_0$$

implies, by Lemma 3.4.2, that

$$\xi_n : \dots : \xi_0 \lhd_k \xi_n \eta_n : \dots : \xi_{l(j)} \eta_{l(j)} = \pi r(z_j) \,.$$

Furthermore, by (c), we have $u_j \in D_k(v)$ for the immediate $\sqsubseteq_k^*$-predecessor u_j of z_j. Together with $z_j \in E_k(v)$ it therefore follows that $z_j \in D_k(v)$. This implies $z_j = y$.

(e) Suppose that $z_m \neq y$. We define a sequence $w_0, \dots, w_s$ of vertices as follows. Set $w_0 := z_m$. For $j > 0$, fix the maximal index h such that $w_{j-1} \not\sqsubseteq_h^* y$ and let $w_j := \omega_h(w_{j-1})$. The construction stops when we reach a vertex $w_s \sqsubseteq_k^* y$. Since the hole is properly terminated we have $y \in D_k(v)$. Hence, $v \sqsubseteq_k^* w_s \sqsubseteq_k^* y$ implies $w_s \in D_k(v)$ and it follows that $w_s = y$.

Let $l := l(m)$. We prove by induction on j that

$$\pi r(w_j) = \xi_n\eta_n : \cdots : \xi_{l+1}\eta_{l+1} : \mu_j\,, \quad \text{for some } \mu_j \sqsubseteq_0 \xi_l \text{ with } \mu_j \neq \varepsilon\,.$$

For $j = 0$, we have $\pi r(w_0) = \pi r(z_m)$ and $\mu_0 = \xi_l$ as desired. By Lemma 3.5.6, for every $j > 0$, there is some index h such that

$$\pi r(w_j) = \mathrm{pop}_h \pi r(w_{j-1})\,.$$

If $h > l$ then

$$\pi r(w_j) = \mathrm{pop}_h(\xi_n\eta_n : \cdots : \xi_{l+1}\eta_{l+1} : \mu_{j-1}) = \xi_n\eta_n : \cdots : \xi_h\eta_h\,,$$

which implies $z_m \not\sqsubseteq^*_h w_j$. Hence, $z_m \not\sqsubseteq^*_{l+1} w_j$ and, therefore, $z_m \not\sqsubseteq^*_{l(m)+1} y$. Contradiction. Thus, we have $h \leq l$ and

$$\begin{aligned}\pi r(w_j) &= \mathrm{pop}_h(\xi_n\eta_n : \cdots : \xi_{l+1}\eta_{l+1} : \mu_{j-1}) \\ &= \xi_n\eta_n : \cdots : \xi_{l+1}\eta_{l+1} : \mu_j\end{aligned}$$

with $\mu_j = \mathrm{pop}_h(\mu_{j-1})$.

Since $\mu_s \sqsubseteq_0 \xi_l$ implies $\xi_l : \cdots : \xi_0 \ntriangleleft_k \mu_s$, it follows that

$$\xi_n : \cdots : \xi_0 \ntriangleleft_k \xi_n\eta_n : \cdots : \xi_{l+1}\eta_{l+1} : \mu_s = \pi r(y)$$

in contradiction to $y \in D_k(v)$. □

Lemma 3.6.9. *Every principal exit point is an exit point.*

Proof. Let $z \in P_{kl}(v)$. Clearly, $z \in E_k(v) \setminus D_k(v)$. Suppose there is some $y \in E_k(v) \setminus D_k(v)$ with $y \lessdot_k z$. By Lemma 3.6.8 (c), $y \sqsubset^*_k z$ implies $y \in D_k(v)$. Contradiction. □

3.7 Expansion sequences

In order to perform the pumping construction in the next section we need to find a pair of vertices $u \lessdot_1 v$ with certain properties. As an intermediate step to prove the existence of such pairs we show in the current section that, if the run is long enough then we can find arbitrary long chains $u_0 \lessdot_1 \cdots \lessdot_1 u_m$.

In order to prove the existence of long chains $u_0 \lessdot_1 \cdots \lessdot_1 u_m$ it is sufficient to bound the branching factor of the forest $(\mathrm{dom}(r), \leq_1)$. To do so we employ the following device.

Definition 3.7.1. Let r be a run. An *expansion sequence* of r is a sequence of injections $t_k \to \cdots \to t_n$ between forests where $t_n := r$ and, for $i < n$, we have $t_i := (C, \leq_{i+1})$ where $C \subseteq \mathrm{dom}(t_{i+1})$ is a maximal chain in t_{i+1}.

We want to prove that each forest in an expansion sequence is binary. The following lemmas collect basic properties about the vertices in such a forest.

Lemma 3.7.2. *Let $t_k \to \cdots \to t_n$ be an expansion sequence of r and let $x \in \mathrm{dom}(t_k)$. If y is an immediate successor of x with $(\pi t_k(y))_{k+1} = (\pi t_k(x))_{k+1}$ then there exist no immediate successors z of x with $y < z$.*

Proof. Denote the first embedding by $\iota : t_k \to t_{k+1}$. We show that, for all $z \in \mathrm{dom}(t_k)$ with $x \leq_{k+1} z$, we have $y \leq_{k+1} z$. The proof proceeds by induction on the number of elements w such that $\iota y \leq w \leq \iota z$.

Since

$$x \lhd_{k+1} y \lhd_{k+2} z\,, \quad x \lhd_{k+1} z\,, \quad \text{and} \quad (\pi t_k(x))_{k+1} = (\pi t_k(y))_{k+1}\,,$$

it follows by Lemma 3.4.3 that $y \lhd_{k+1} z$. Consequently, $y \sqsubseteq_{k+1} z$ and, by induction hypothesis, we have $y \sqsubseteq^*_{k+1} z$.

Let w be some element such that $y \sqsubseteq^*_{k+1} w \sqsubseteq^*_{k+1} z$. We have to show that $y \lhd_{k+1} w$. Since $x \sqsubseteq^*_{k+1} w \sqsubseteq^*_{k+1} z$ and $x \leq_{k+1} z$ we have $x \lhd_{k+1} w$. Similarly, $y \sqsubseteq^*_{k+2} w \sqsubseteq^*_{k+2} z$ implies $y \lhd_{k+2} w$. Since $(\pi t_k(x))_{k+1} = (\pi t_k(y))_{k+1}$ we can again apply Lemma 3.4.3 to infer that $y \lhd_{k+1} w$. Together with $y \leq_{k+2} z$ it therefore follows that $y \leq_{k+1} z$. □

Lemma 3.7.3. *Let $t_k \to \cdots \to t_n$ be an expansion sequence of r. Denote the embedding $t_k \to t_n$ by ι and let $x \in \mathrm{dom}(t_k)$.*

(a) *If the operation at x is a level i operation with $i \leq k$ and x has an immediate successor y then $\iota y = \sigma\iota x$. In particular, y is the only immediate successor of x.*

(b) *If there is a pop_i-operation at x with $i > k$ then x is a leaf.*

Proof. (a) follows from Lemma 3.7.2 by induction on k, and (b) follows immediately from the definition. □

Lemma 3.7.4. *Let $t_k \to \cdots \to t_n$ be an expansion sequence of r and $x \in \mathrm{dom}(t_k)$ a vertex with several immediate successors $y_0, \dots, y_{m-1}$, $m \geq 2$.*

(a) *The operation at x is a $\mathrm{push}(k+1)$-operation.*

(b) *There are words $\xi_n, \dots, \xi_0$ and $\mu_n, \dots, \mu_{k+2}$ such that*

$$\begin{aligned}
\pi t_k(x) &= \xi_n : \cdots : \xi_0\,,\\
\pi t_k(y_0) &= \begin{cases} \mathrm{clone}_{k+1}(\pi t_k(x)) & \text{if } k > 0\,,\\ \mathrm{push}_a(\pi t_k(x)) & \text{if } k = 0\,,\end{cases}\\
\pi t_k(y_1) &= \xi_n\mu_n : \cdots : \xi_{k+2}\mu_{k+2} : \xi_{k+1} : \cdots : \xi_0\,.
\end{aligned}$$

(c) *x has exactly two immediate successors.*

Proof. We prove the claims by induction on k. Denote the embedding $t_k \to t_i$ by ι_i and set $C := \mathrm{rng}(\iota_{k+1})$.

(a) Lemma 3.7.3 (a) and (b) imply that there is a push(i)-operation at x with $i > k$. Suppose that $i > k+1$. Let z be the element such that $\iota_{i-1}z$ is the immediate successor of $\iota_{i-1}x$. By construction of t_k, z is the first immediate successor of x. By induction hypothesis we have $(\pi t_k(z))_{k+1} = (\pi t_k(x))_{k+1}$. Therefore, it follows from Lemma 3.7.2 that z is also the last immediate successor of x. Hence, x has only one immediate successor. Contradiction.

(b) By Lemma 3.7.3 (a), we know that $\iota_n y_0 = \sigma\iota_n x$. Hence, (a) implies that

$$\pi t_k(y_0) = \begin{cases} \mathrm{clone}_{k+1}(\pi t_k(x)) & \text{if } k > 0\,, \\ \mathrm{push}_a(\pi t_k(x)) & \text{if } k = 0\,. \end{cases}$$

By construction of t_k, $\iota_{k+1}y_1$ is the minimal element of $C \smallsetminus \{\iota_{k+1}x\}$ such that

$$y_0 \not\trianglelefteq_{k+1} y_1\,.$$

Let z be the element such that $\iota_{k+1}z$ is the immediate predecessor of $\iota_{k+1}y_1$ in C. Since $\iota_{k+1}z$ is not a leaf of t_{k+1}, Lemma 3.7.3 (b) implies that the operation at z is not a pop_i with $i > k+1$. Since

$$y_0 \trianglelefteq_{k+1} z\,,$$

the operation at z must therefore be a pop_{k+1} and, by Lemma 3.7.3 (a), we have $\iota_n y_1 = \sigma\iota_n z$. Furthermore, it follows that there are words $\mu_n, \dots, \mu_0$ such that

$$\pi t_k(z) = \xi_n\mu_n : \dots : \xi_{k+2}\mu_{k+2} : \xi_{k+1}(\xi_k : \dots : \xi_0)\mu_{k+1} : \mu_k : \dots : \mu_0\,.$$

Consequently, $y_0 \not\trianglelefteq_{k+1} y_1$ implies that $\mu_{k+1} = \varepsilon$ and

$$\pi t_k(y_1) = \mathrm{pop}_{k+1}(\pi t_k(z)) = \xi_n\mu_n : \dots : \xi_{k+2}\mu_{k+2} : \xi_{k+1} : \dots : \xi_0\,.$$

(c) By (b) and Lemma 3.7.2 it follows that y_1 is the last immediate successor of x. □

Corollary 3.7.5. *Every forest in an expansion sequence is binary.*

Using this corollary we can prove that every sufficiently long run contains a sequence $u_0 \trianglelefteq_1 \dots \trianglelefteq_1 u_m$.

Lemma 3.7.6. *Let t be a binary tree with $|\mathrm{dom}(t)| \geq 2^m$ vertices. Then there exists a chain $C \subseteq \mathrm{dom}(t)$ of size $|C| > m$.*

Proof. If every chain is of size at most m then $\mathrm{dom}(t) \subseteq \{0,1\}^{<m}$ which implies

$$|\mathrm{dom}(t)| \leq \sum_{i<m} 2^i = 2^m - 1\,.$$

Contradiction. ☐

We only consider the case of runs starting at the initial configuration. This ensures that the expansion sequence constructed below consists of trees instead of forests. The restriction will be lifted below.

Lemma 3.7.7. *Let r be a run that starts at the initial configuration. For every set $M \subseteq \mathrm{dom}(r)$ of size $|M| \geq \beth_n(m)$ there exists a sequence $u_0 \leq_1 \dots \leq_1 u_m$ of vertices of length strictly greater than m such that,*

$$M \cap (D_1(u_i) \setminus D_1(u_{i+1})) \neq \varnothing\,, \quad \textit{for all } i < m\,.$$

Proof. We construct an expansion sequence $t_0 \to \dots \to t_n$ and two sequences $C_0, \dots, C_n$ and $M_0, \dots, M_n$ of sets as follows. We start with $t_n := r$ and $M_n := M$. To construct t_k suppose that we have already defined $t_{k+1} = (\mathrm{dom}(t_{k+1}), \leq)$ and a subset $M_{k+1} \subseteq \mathrm{dom}(t_{k+1})$. Choose a chain $C'_{k+1} \subseteq M_{k+1}$ of maximal length in the tree $(M_{k+1}, \leq)$, and let $C_{k+1} \subseteq \mathrm{dom}(t_{k+1})$ be a maximal chain in t_{k+1} with $C'_{k+1} \subseteq C_{k+1}$. We set

$$t_k := (C_{k+1}, \leq_{k+1}) \quad \text{and} \quad M_k := C_{k+1} \cap \{\, u \wedge v \mid u, v \in M_{k+1} \,\}\,,$$

where $\wedge$ denotes the greatest lower bound in t_k. Finally, we also choose some chain $C'_0 \subseteq M_0$ of maximal length and a corresponding maximal chain $C_0 \subseteq \mathrm{dom}(t_0)$ with $C'_0 \subseteq C_0$.

Let x be the first element of $\mathrm{dom}(r)$. Since x is initial we have $\pi r(x) = \varepsilon : \dots : \varepsilon : a$, for some letter a, which implies, by Corollary 3.4.6, that $x \lhd_1 y$, for all $y \in \mathrm{dom}(r)$. Therefore, x is the unique minimal element of each t_k and all t_k are binary trees. Since the sets M_k are closed under greatest lower bounds it follows that the subforests induced by them also form binary trees. Consequently, we can apply the preceding lemma. By induction on k, it follows that $|C'_k| > \beth_k(m)$, for $k < n$. Let $u_0 < \dots < u_m$ be an enumeration of (a subset of) C'_0. The sequence $\iota_n u_0, \dots, \iota_n u_m$ has the desired property. ☐

By an automaton construction we can generalise this result to arbitrary runs. Unfortunately, this introduces a dependence on the size of the stack contents of the first configuration.

Definition 3.7.8. For $\xi = x_0 \dots x_m \in \Gamma^{+n}$ we define, by induction on n,

$$\|\xi\| := \begin{cases} |\xi| & \text{if } n = 1\,, \\ \sum_{i \leq m} \|x_i\| & \text{if } n > 1\,. \end{cases}$$

Corollary 3.7.9. *Let r be a run with first element w and set $k := 2\|\pi r(w)\|$. For every set $M \subseteq \operatorname{dom}(r)$ of size $|M| \geq \beth_n(m+k)$ there exists a sequence $u_0 \trianglelefteq_1 \cdots \trianglelefteq_1 u_m$ of vertices of length strictly greater than m such that,*

$$M \cap (D_1(u_i) \smallsetminus D_1(u_{i+1})) \neq \varnothing\,, \quad \text{for all } i < m\,.$$

Proof. Let $\xi := \pi r(w)$. There exists a sequence op of at most $k := 2\|\xi\|$ stack operations such that $\xi := \operatorname{op}(\varepsilon : \cdots : \varepsilon : a)$. We construct an automaton $\mathcal{B}$ by modifying the given automaton $\mathcal{A}$ such that, starting at the initial configuration $\mathcal{B}$ executes the operations op until it reaches the configuration $r(w)$. Then it continues in exactly the same way as $\mathcal{A}$ would. Let $r' = sr$ be the run of $\mathcal{B}$ starting at the initial configuration. The preceding lemma implies that there exists a sequence $u_0 \trianglelefteq_1 \cdots \trianglelefteq_1 u_{m+k}$ with the desired properties in $\operatorname{dom}(r')$. Since $|\operatorname{dom}(s)| = k$ it follows that $u_i \in \operatorname{dom}(r)$, for $i \geq k$. Hence, $u_k \trianglelefteq_1 \cdots \trianglelefteq_1 u_{m+k}$ is the desired sequence. □

3.8 A pumping lemma

Using the structure theory developed in Sections 3.4 to 3.7 we prove a pumping lemma for higher-order pushdown automata. For the construction below we need to find two vertices $u \vartriangleleft_1 v$ such that the same types of holes appear in $D_1(u)$ and in $D_1(v)$. Such vertices u, v will be called a *pumping pair*. The formal definition is based on the equivalence relation $\sim_{km}$.

Definition 3.8.1. (a) Let $\xi = \xi_n : \cdots : \xi_k$. We define the set

$$\tilde{\chi}_k(\xi) \subseteq \Gamma^{*n} \times \cdots \times \Gamma^{*(k+1)} \times Q \times \{*, k+1, \dots, n\} \times Q$$

by the following conditions. For $l \in \{k+1, \dots, n\}$, we have

$$(\mu_n, \dots, \mu_{k+1}, p, l, q) \in \tilde{\chi}_k(\xi)\,,$$

iff there is a run r and an element $x \in \operatorname{dom}(r)$ such that

$$r(x) = (\xi_n \mu_n : \cdots : \xi_{k+1}\mu_{k+1} : \xi_k,\ p)\,,$$
$$\text{and } r(\omega_{k+1}(x)) = (\xi_n \mu_n : \cdots : \xi_l \mu_l,\ q)\,,$$

and we have

$$(\mu_n, \dots, \mu_{k+1}, p, *, q) \in \tilde{\chi}_k(\xi)$$

iff there is a run r and elements $x, y \in \operatorname{dom}(r)$ such that $y \in E_{k+1}(x)$,

$$r(x) = (\xi_n \mu_n : \cdots : \xi_{k+1}\mu_{k+1} : \xi_k,\ p)\,, \quad \text{and} \quad \rho r(y) = q\,.$$

(b) For $\xi, \zeta \in \Gamma^{+n}$ and $k, m \le n$, we define an equivalence relation $\sim_{km}$ by

$$\begin{aligned} \xi \sim_{km} \zeta \quad &:\text{iff} \quad \text{for all } \mu_i \in \Gamma^{*i},\ p, q \in Q, \text{ and } l \in \{*, k+1, \dots, n\}, \\ &\qquad (\mu_n^\xi, \dots, \mu_{k+1}^\xi, p, l, q) \in \tilde{\chi}_k(\mathrm{pop}_k(\xi)) \\ &\qquad \Leftrightarrow (\mu_n^\zeta, \dots, \mu_{k+1}^\zeta, p, l, q) \in \tilde{\chi}_k(\mathrm{pop}_k(\zeta)), \end{aligned}$$

where, for $\lambda = \lambda_n : \dots : \lambda_0$, we set

$$\mu_i^\lambda := \begin{cases} \varepsilon & \text{if } \mu_i = \varepsilon, \\ \mu_i[\varepsilon : \dots : \varepsilon : (\mu_i)_0 / \varepsilon : \lambda_{i-1} : \dots : \lambda_0]_m & \text{otherwise}. \end{cases}$$

(c) Let r be a run. Two vertices $u, v \in \mathrm{dom}(r)$ form a *pumping pair* if

$$u \lessdot_1 v, \quad \rho r(u) = \rho r(v), \quad \text{and} \quad \pi r(u) \sim_{k1} \pi r(v), \quad \text{for all } k \le n.$$

Given a pumping pair $u \lessdot_1 v$ we can perform the following pumping construction.

Lemma 3.8.2. *Let r be a run with a pumping pair $u \lessdot_1 v$ and suppose*

$$\pi r(u) = \xi = \xi_n : \dots : \xi_0 \quad \text{and} \quad \pi r(v) = \zeta = \zeta_n : \dots : \zeta_0.$$

There exists a run s whose first configuration is the same as that of r and there are vertices $u', v', w' \in \mathrm{dom}(s)$ such that

$$\pi s(u') = \xi, \quad \pi s(v') = \zeta, \quad \pi s(w') = \zeta[\xi/\zeta]_1,$$

$u' \lessdot_1 v'$ form a pumping pair, and $|D_1(v')| = |D_1(u)|$.

Proof. Define

$$s_0 := r|_{\mathrm{dom}(r) \smallsetminus E_1(v)}, \quad \text{and} \quad s_1 := (r|_{D_1(u)})[\xi/\zeta]_1.$$

Let u' be the copy of u in $\mathrm{dom}(s_0)$ and let v' and w' be the copies of, respectively, u and v in $\mathrm{dom}(s_1)$. For each principal exit x of some hole in $\mathrm{dom}(s_1) = D_1(u)$ we construct a run s_x of the same type as x. We obtain the desired run s by inserting s_1 into s_0 and each s_x into the corresponding hole of s_1.

It remains to find s_x. If x is of order k then, by Lemma 3.6.8 (d), there are words $\mu_n, \dots, \mu_{k+1}$ such that

$$\pi r(x) = \xi_n \mu_n : \dots : \xi_{k+1}\mu_{k+1} : \xi_k.$$

Since $\xi \sim_{k1} \zeta$ we can find a run s_x of the same type as x such that

$$\pi s_x(y) = \zeta_n \tilde{\mu}_n : \dots : \zeta_{k+1}\tilde{\mu}_{k+1} : \zeta_k,$$

where y is the first element of $\mathrm{dom}(s_x)$ and

$$\tilde{\mu}_i := \begin{cases} \varepsilon & \text{if } \mu_i = \varepsilon, \\ \mu_i[\varepsilon : \xi_{i-1} : \dots : \xi_0 / \varepsilon : \zeta_{i-1} : \dots : \zeta_0]_1 & \text{otherwise}. \end{cases} \qquad \square$$

It remains to prove the existence of a pumping pair. We start by showing that $\tilde{\chi}_k(\xi)$ is closed under $\sim_{i,k+1}$.

Lemma 3.8.3. *Let $\xi = \xi_n : \dots : \xi_k \in \Gamma^{+n}$ and $\mu_i, \eta_i \in \Gamma^{+i}$, for $k < i \leq n$. If*

$$\xi_n\mu_n : \dots : \xi_{k+1}\mu_{k+1} : \xi_k \sim_{i,k+1} \xi_n\eta_n : \dots : \xi_{k+1}\eta_{k+1} : \xi_k\,,$$

for all $k < i \leq n$, then we have

$$(\mu_n, \dots, \mu_{k+1}, p, l, q) \in \tilde{\chi}_k(\xi) \quad \text{iff} \quad (\eta_n, \dots, \eta_{k+1}, p, l, q) \in \tilde{\chi}_k(\xi)\,.$$

Proof. Let r be a run of minimal length witnessing the fact that

$$(\mu_n, \dots, \mu_{k+1}, p, l, q) \in \tilde{\chi}_k(\xi)\,.$$

Denote the first and last elements of $\operatorname{dom}(r)$ by x and y, respectively. By minimality of r, we have

$$r(x) = (\xi_n\mu_n : \dots : \xi_{k+1}\mu_{k+1} : \xi_k,\ p)$$

and either

$$\begin{aligned} & l \neq *\,, \quad y = \omega_{k+1}(x)\,, \quad \text{and} \quad r(y) = (\xi_n\mu_n : \dots : \xi_l\mu_l,\ q)\,, \\ \text{or} \quad & l = *\,, \quad y \in E_{k+1}(x)\,, \quad \text{and} \quad \rho r(y) = q\,. \end{aligned}$$

We construct a witness s for

$$(\eta_n, \dots, \eta_{k+1}, p, l, q) \in \tilde{\chi}_k(\xi)$$

as follows. Let

$$t := (r|_{D_{k+1}(x)})[\xi_n\mu_n : \dots : \xi_{k+1}\mu_{k+1} : \xi_k / \xi_n\eta_n : \dots : \xi_{k+1}\eta_{k+1} : \xi_k]_{k+1}\,.$$

If $l \neq *$ then we add the element y as last element to t by setting

$$t(y) := (\xi_n\eta_n : \dots : \xi_l\eta_l, q)\,.$$

Clearly, t is a partial run of the right type with

$$t(x) = (\xi_n\eta_n : \dots : \xi_{k+1}\eta_{k+1} : \xi_k, p)\,.$$

If t does not contain holes then we have already found the desired witness.

Suppose that there is a hole in $\operatorname{dom}(t) = D_{k+1}(x)$ and let w be one of its principal exits. If w is of order i then

$$\pi r(w) = \xi_n\mu_n\beta_n : \dots : \xi_{i+1}\mu_{i+1}\beta_{i+1} : \xi_i\mu_i\,,$$

for some words $\beta_n, \ldots, \beta_{i+1}$. We construct a run t_w of the same type as w that be inserted into t to fill the hole. Since

$$\xi_n\mu_n : \cdots : \xi_{k+1}\mu_{k+1} : \xi_k \sim_{i,k+1} \xi_n\eta_n : \cdots : \xi_{k+1}\eta_{k+1} : \xi_k$$

there exists a run t_w with first and last element u and v, respectively, such that

$$\pi t_w(u) = \xi_n\eta_n\tilde{\beta}_n : \cdots : \xi_{i+1}\eta_{i+1}\tilde{\beta}_{i+1} : \xi_i\eta_i\,,$$

where

$$\tilde{\beta}_j := \begin{cases} \varepsilon & \text{if } \beta_j = \varepsilon\,, \\ \beta_j[\varepsilon : \xi_{j-1}\mu_{j-1} : \cdots : \xi_{k+1}\mu_{k+1} : \xi_k/\varepsilon : \xi_{j-1}\eta_{j-1} : \cdots : \xi_{k+1}\eta_{k+1} : \xi_k]_{k+1} \\ & \text{otherwise}\,. \end{cases}$$

Furthermore, if $l \neq *$ then

$$\pi t_w(v) = \xi_n\eta_n\tilde{\beta}_n : \cdots : \xi_h\eta_h\tilde{\beta}_h\,,$$

and, otherwise, we have $\rho t_w(v) = \rho r(v)$. □

We can use the preceding result to compute a bound on the index of $\sim_{km}$.

Lemma 3.8.4. *The index of* $\sim_{km}$ *is bounded by*

$$|\Gamma^{+n}/\sim_{km}| \leq \beth_{n-k+1}\big(3^{n-k}|Q|^2(n-k+1)!\big)\,.$$

Proof. Let $s := |Q|$. We prove the claim by induction on k. For $k = n$, we have

$$\tilde{\chi}_n(\xi) \subseteq Q \times \{*\} \times Q$$

which implies $\xi \sim_{nm} \zeta$ iff $\tilde{\chi}_n(\xi) = \tilde{\chi}_n(\zeta)$. Hence, there are at most 2^{s^2} $\sim_{nm}$-classes.

Suppose that $k < n$. For $\lambda = \lambda_n : \cdots : \lambda_k \in \Gamma^{+n}$ and $\mu_i, \eta_i \in \Gamma^{*i}$, we define

$$\begin{aligned} & (\mu_n, \ldots, \mu_{k+1}) \equiv_\lambda (\eta_n, \ldots, \eta_{k+1}) \\ \text{iff} \quad & \lambda_n\mu_n : \cdots : \lambda_{k+1}\mu_{k+1} : \lambda_k \sim_{i,k+1} \lambda_n\eta_n : \cdots : \lambda_{k+1}\eta_{k+1} : \lambda_k\,, \text{ for all } i > k\,. \end{aligned}$$

By Lemma 3.8.3, $(\mu_n, \ldots, \mu_{k+1}) \equiv_\lambda (\eta_n, \ldots, \eta_{k+1})$ implies

$$(\mu_n, \ldots, \mu_{k+1}, p, l, q) \in \tilde{\chi}_k(\lambda) \quad \text{iff} \quad (\eta_n, \ldots, \eta_{k+1}, p, l, q) \in \tilde{\chi}_k(\lambda)\,.$$

By induction hypothesis, there are at most

$$\begin{aligned} \prod_{i=1}^{n-k} \beth_i(3^{i-1}s^2 i!) &\leq \beth_{n-k}\big((n-k)3^{n-k-1}s^2(n-k)!\big) \\ &\leq \beth_{n-k}\big(3^{n-k-1}s^2(n-k+1)!\big) \end{aligned}$$

$\equiv_\lambda$-classes. Set

$$\begin{aligned}
&(\mu_n, \dots, \mu_{k+1}) \equiv (\eta_n, \dots, \eta_{k+1}) \\
\text{iff}\quad &(\mu_n^\xi, \dots, \mu_{k+1}^\xi) \equiv_{\mathrm{pop}_k(\xi)} (\eta_n^\xi, \dots, \eta_{k+1}^\xi) \\
\text{and}\ &(\mu_n^\zeta, \dots, \mu_{k+1}^\zeta) \equiv_{\mathrm{pop}_k(\zeta)} (\eta_n^\zeta, \dots, \eta_{k+1}^\zeta)\,,
\end{aligned}$$

where, as above,

$$\mu_i^\xi := \begin{cases} \varepsilon & \text{if } \mu_i = \varepsilon\,, \\ \mu_i[\varepsilon : \dots : \varepsilon : (\mu_i)_0/\varepsilon : \xi_{i-1} : \dots : \xi_0]_m & \text{otherwise}\,. \end{cases}$$

By Lemma 3.8.3, we have $\xi \sim_{km} \zeta$ iff, for every $\equiv$-class $[\mu_n, \dots, \mu_{k+1}]$ we have

$$\begin{aligned}
&(\mu_n^\xi, \dots, \mu_{k+1}^\xi, p, l, q) \in \tilde{\chi}_k(\mathrm{pop}_k(\xi)) \\
\text{iff}\quad &(\mu_n^\zeta, \dots, \mu_{k+1}^\zeta, p, l, q) \in \tilde{\chi}_k(\mathrm{pop}_k(\zeta))\,.
\end{aligned}$$

Hence, there are at most

$$\begin{aligned}
2^{\beth_{n-k}(3^{n-k-1}s^2(n-k+1)!)^2 \cdot s^2 \cdot (n-k+1)} &\leq \beth_{n-k+1}\big(3 \cdot 3^{n-k-1}s^2(n-k+1)!\big) \\
&= \beth_{n-k+1}\big(3^{n-k}s^2(n-k+1)!\big)
\end{aligned}$$

$\sim_{km}$-classes. □

The existence of a pumping pair immediately follows from the previous lemma and Corollary 3.7.9.

Lemma 3.8.5. *Let r be a run with first element w and set $k := 2\|\pi r(w)\|$. For every set $M \subseteq \mathrm{dom}(r)$ of size*

$$|M| \geq \beth_{2n}\big(n3^{n-1}|Q|^3 n! + k\big)$$

there exists a pumping pair $u \lessdot_1 v$ such that

$$M \cap (D_1(u) \setminus D_1(v)) \neq \varnothing\,.$$

Proof. By Corollary 3.7.9, there exists a sequence $u_0 \lessdot_1 \dots \lessdot_1 u_m$ of length strictly greater than

$$m := \beth_n\big(n3^{n-1}|Q|^3 n!\big) \geq |Q| \cdot \prod_{1 \leq i \leq n} \beth_{n-i+1}\big(3^{n-i}|Q|^2(n-i+1)!\big)$$

such that

$$M \cap (D_1(u_i) \setminus D_1(u_{i+1})) \neq \varnothing\,, \quad \text{for all } i < m\,.$$

By Lemma 3.8.4, it therefore follows that there are two indices $i < j$ such that u_i and u_j form a pumping pair. □

We apply the technical Lemma 3.8.2 to show that, if there exists a run of a certain length then there are infinitely many different runs.

Theorem 3.8.6 (Pumping Lemma). *Suppose that $\mathcal{A}$ is a pushdown automaton of level n and let r be a run of $\mathcal{A}$ with first element w.*

(a) *If*

$$|\mathrm{dom}(r)| \geq \beth_{2n}\big(n3^{n-1}|Q|^3 n! + 2\|\pi r(w)\|\big)$$

then there exists a sequence $r_0, r_1, \dots$ of runs, each starting with w, where $r_0 = r$ and

$$|\mathrm{dom}(r_i)| < |\mathrm{dom}(r_{i+1})|\,, \qquad \textit{for all } i < \omega\,.$$

(b) *Similarly, if r contains at least*

$$\beth_{2n}\big(n3^{n-1}|Q|^3 n! + 2\|\pi r(w)\|\big)$$

non-ε-transitions then there exists a sequence $r_0, r_1, \dots$ of runs, each starting at w, where $r_0 = r$ and r_{i+1} contains more non-ε-transitions than r_i.

Proof. (a) Let $M := \mathrm{dom}(r)$. By Lemma 3.8.5, there exists a pumping pair $u \lessdot_1 v$ in r. We define a sequence of runs $r'_0, r'_1, \dots$ inductively. For each run r_i, we will also choose a pumping pair $u_i \lessdot_1 v_i$. We start with $r'_0 := r$, $u_0 := u$, and $v_0 := v$. Suppose that r'_i is already defined. By Lemma 3.8.2, we can construct a new run r'_{i+1} that contains elements u_{i+1} and v_{i+1} such that $u_{i+1} \lessdot_1 v_{i+1}$ forms a pumping pair and $|D_1(v_{i+1})| = |D_1(u_i)| > |D_1(v_i)|$. To obtain the desired sequence $r_0, r_1, \dots$ we delete from $r'_0, r'_1, \dots$ all runs r'_i such that $|\mathrm{dom}(r'_l)| \geq |\mathrm{dom}(r'_i)|$, for some $l < i$. The condition $|D_1(v_i)| < |D_1(v_{i+1})|$ ensures that the resulting sequence is still infinite.

(b) Let $M \subseteq \mathrm{dom}(r)$ be the set of all configurations with an outgoing non-ε-transition. If we perform the same construction as in the proof of (a) we obtain a sequence of runs r_i, $i < \omega$, such that the number of non-ε-transitions in each run is strictly increasing. ☐

Corollary 3.8.7. *Let $\mathcal{A}$ be a pushdown automaton of level n. If $\mathcal{A}$ accepts a word of length at least*

$$\beth_{2n}\big(n3^{n-1}|Q|^3 n!\big)$$

then the language recognised by $\mathcal{A}$ is infinite.

One immediate consequence of this theorem is the fact that finiteness is decidable for languages recognised by a higher-order pushdown automaton.

Corollary 3.8.8. *The problem whether the language recognised by a given higher-order pushdown automaton is finite is decidable.*

We apply the theorem to prove that a given graph does not belong to a certain level of the Caucal hierarchy.

Example. Let $\mathfrak{T}_k := (T_k, \preceq)$ where $T_k := \{\, 0^i 1^l \mid i < \omega,\ l < \beth_k(i)\,\}$. We claim that $\mathfrak{T}_{3n} \notin \mathcal{C}_n$. For a contradiction, suppose otherwise. By Theorem 3.2.3, there exists a pushdown automaton $\mathcal{A}$ of level n whose configuration graph becomes isomorphic to $\mathfrak{T}_{3n}$ when we contract all ε-transitions. Furthermore, we can use Lemma 3.2.4 to find a finite structure $\mathfrak{A}$ with universe $Q \cup \Gamma$ such that the configuration graph of $\mathcal{A}$ is definable in $\mathfrak{A}^{*n}$.

Let $w_k \in A^{*n}$ be the word encoding the element $0^k 1 \in T_{3n}$. In the same way as in the example on page 78 it follows that

$$|w_k|_i \leq \beth_{i-1}(\mathcal{O}(k)) .$$

Hence, $\|w_k\| \leq \beth_{n-1}(\mathcal{O}(k))$. The unique path starting at w_k has length $\beth_{3n}(k) - 1$. Thus, the run of $\mathcal{A}$ corresponding to this path has at least that much non-ε-transitions. Since

$$\begin{aligned}\beth_{2n}\big(n3^{n-1}|Q|^3 n! + 2\|w_k\|\big) &\leq \beth_{2n}\big(n3^{n-1}|Q|^3 n! + 2\beth_{n-1}(\mathcal{O}(k))\big)\\ &\leq \beth_{3n-1}(\mathcal{O}(k))\\ &\leq \beth_{3n}(k) - 1\end{aligned}$$

it follows from part (b) of the theorem that, for large enough k, there are runs starting at w_k with arbitrarily many non-ε-transitions. But this implies that $\mathfrak{T}_{3n}$ contains arbitrarily long paths starting at w_k. Contradiction.

Conclusion

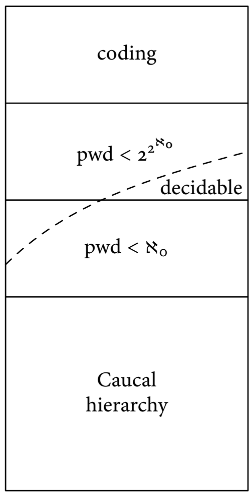
In this thesis we have studied and classified monadic second-order theories. Our main interest was the dividing line between theories simple enough to allow for a structure theory and those whose complexity inhibits analysis. In the first part we have shown that the lack of definable pairing functions is one indication of simplicity. In fact, we have proved a dichotomy: either a given first-order theory is complex, meaning that

- every model of a given first-order theory admits a definable pairing function and
- the partition width of these models is unbounded,

or the theory is simple, that is,

- no model has a definable pairing function with infinite domain and
- the partition width is bounded by $2^{2^{\aleph_0}}$.

Based on this result we obtain the following landscape of monadic second-order theories. (The diagram is imprecise since there are structures admitting coding with partition width less than $2^{2^{\aleph_0}}$.)



In the second part of the thesis we collected technical results that can be used to determine the level of a given structure in the Caucal hierarchy. On the one hand, we employed MSO-automata to compute bounds on the outdegree of vertices. On the other hand, we performed a detailed investigation of configuration graphs of higher-order pushdown automata in order to obtain bounds on the length of paths in such a structure.

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