Simple Monadic Theories and Partition Width

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1 Introduction

Over the last decades the beginnings of a model theory for monadic second-order logic have emerged. After seminal papers by Büchi [12], Läuchli [33], Rabin [36], and Shelah [41] a thorough investigation of the monadic theory of linear orders was performed by Gurevich and Shelah [30, 31]. General monadic theories and their model theory were studied by Baldwin and Shelah in [1, 42, 43].

A second development advancing the model theory for monadic second-order logic consists in the work on graph grammars initiated by Courcelle. The main subject of this line of work is the study of graph operations that are compatible with monadic second-order theories [20, 22, 23, 26, 34] (see [9] for an overview). Noteworthy recent developments include the *Muchnik iteration* [40, 46, 2, 13, 11] and *set interpretations* [17]. Such operations give rise to graph algebras and the corresponding notions of recognisable sets and equational sets [29, 10]. Furthermore, one can use these operations to define hierarchical decompositions of graphs and the corresponding complexity measures, like tree width, clique width, and partition width [25, 21, 27, 4, 24, 6]. Finally, operations can also be used to construct finite presentations of infinite graphs via regular terms [18, 19, 15, 16, 14, 5, 7]. As monadic second-order logic is more expressive than first-order logic, it is unsurprising that most structures possess an extremely complicated monadic second-order theory. Fortunately, there remain structures where the theory is simple enough for the existence of a structure theory.

The prime example of such a structure is the infinite binary tree which, according to Rabin's theorem, has a decidable monadic theory. Starting from this result we can obtain further structures with a manageable theory by applying operations that preserve decidability of the MSO-theory, like monadic second-order interpretations or disjoint unions. We can also consider other trees than the complete binary one. Although their monadic theories can become highly undecidable there still exists a structure theory for structures interpretable in them (see [6, 4]).

On the other extreme there are structures in which one can define arbitrarily large grids or pairing functions. Their monadic theories are very complex since they can encode arithmetic or even full second-order logic. In particular, there is no hope for a structure theory for such structures.

According to a conjecture of Seese [39] these cases form a dichotomy: either a structure is interpretable in some tree or we can define arbitrarily large grids. For graphs (or structures with relations of arity at most 2) a variant of this conjecture has been solved by Courcelle and Oum [28]. But the general case of arbitrary structures is still open.

Building on techniques developed in [8, 3], we approached Seese's conjecture by considering a weaker statement about *first-order* theories and applying standard tools from first-order model theory. Instead of grids we consider first-order definable pairing functions and we will prove that every structure where there is no pairing function is tree-like (as defined below).

The article is organised as follows. We start in Section 2 by introducing the notion of partition width which is used to define what we mean by 'tree-like'. Section 3 summarises the results of [3] about indiscernible sequences.

In Section 4 we continue the investigation of indiscernible sequences in structures without definable pairing functions. Section 5 contains an overview over the notion of finite satisfiability (without stability assumption). In Section 6 we finally show that every structure without definable pairing functions has bounded partition width and, hence, is tree-like.

2 Partition width

Let us recall some basic definitions and fix our notation. We write $[n]$ for the set $\{0, \dots, n-1\}$. We tacitly identify tuples $\bar{a} = a_0 \dots a_{n-1} \in A^n$ with functions $[n] \to A$ and frequently we do not distinguish between a tuple $\bar{a}$ and the set $\{a_0, \dots, a_{n-1}\}$ of its components. This allows us to write $\bar{a} \subseteq A$ or $\bar{a}|_I$ for $I \subseteq [n]$.

We use the words 'tuple' and 'sequence' synonymously. In particular, tuples may be infinite.

$2^{<\alpha}$ denotes the set of all binary sequences of length less than α and $\preceq$ is the prefix ordering on such sequences

$$x \preceq y \quad :\text{iff} \quad y = xz \text{ for some } z\,.$$

The empty sequence is denoted by $\langle\rangle$.

We start by defining what we consider as 'tree-like'. In the literature several notions have been proposed that measure how much a structure resembles a tree. The most prominent one is *tree width,* which was first introduced by Halin [32] and which plays an important role in the proof of the Graph Minor Theorem by Robertson and Seymour [37]. This measure is closely related to guarded second-order logic. For studying monadic second-order logic more appropriate complexity measures are *clique width,* introduced by Courcelle, Engelfriet, and Rozenberg in [25], and its variant *rank width,* defined by Oum and Seymour [35]. These measures have only been defined for graphs, but there are generalisations of clique width to arbitrary structures. The notion we will use is *partition width* introduced in [6, 4]. Correspondingly we consider a structure to be tree-like if it admits a hierarchical decomposition of the following kind.

Definition 2.1. A *partition refinement* of a structure $\mathfrak{M}$ is a system $(U_v)_{v\in T}$ of subsets $U_v \subseteq M$ indexed by a tree $T \subseteq 2^{<\alpha}$ with the following properties:

- $U_{\langle\rangle} = M$,
- for every element $a \in M$, there exists a vertex $v \in T$ with $U_v = \{a\}$,
- $U_v = U_{v0} \mathbin{\dot\cup} U_{v1}$, for all $v \in T$ (where we set $U_w := \varnothing$, for $w \notin T$),
- $U_v = \bigcap_{u<v} U_u$ if $|v|$ is a limit ordinal.

Example. (a) A natural partition refinement for a linear order $\langle A, <\rangle$ consists of a recursive division into intervals.

(b) For a tree $\langle 2^{<\alpha}, \preceq\rangle$, we can take as components all sets of the form $U_v := \{\, x \in 2^{<\alpha} \mid v \preceq x \,\}$ and all singletons.

Clearly, every structure has partition refinements. In order to define when a structure is tree-like we introduce a complexity measure for partition refinements based on the number of types realised in each component.

Definition 2.2. (a) The *atomic type* of a tuple $\bar a$ over a set U is

$$\operatorname{atp}(\bar a/U) := \big\{\, \varphi(\bar x, \bar c) \;\big|\; \bar c \subseteq U\,,\ \varphi \text{ a literal with } \mathfrak{M} \models \varphi(\bar a, \bar c) \,\big\}\,.$$

For a set Δ of formulae, we denote the *Δ-type* of $\bar a$ over U by $\operatorname{tp}_\Delta(\bar a/U)$. Furthermore, we define its *external type* by

$$\operatorname{etp}(\bar a/U) := \operatorname{atp}(\bar a/U) \smallsetminus \operatorname{atp}(\bar a)\,.$$

(b) For a set Δ of formula we define the n-ary *Δ-type index* of a set A over U by

$$\operatorname{ti}^n_\Delta(A/U) := |A^n/{\approx_U}|\,,$$

where $\approx_U$ is the equivalence relation

$$\bar a \approx_U \bar b \quad :\text{iff} \quad \operatorname{tp}_\Delta(\bar a/U) = \operatorname{tp}_\Delta(\bar b/U)\,.$$

If Δ is the set of all quantifier-free formulae then we write $\operatorname{ati}^n(A/U)$ instead of $\operatorname{ti}^n_\Delta(A/U)$.

Similarly, we define the *external type index* of A over U by

$$\operatorname{eti}^n(A/U) := |A^n/{\simeq_U}|\,,$$

where

$$\bar a \simeq_U \bar b \quad :\text{iff} \quad \operatorname{etp}(\bar a/U) = \operatorname{etp}(\bar b/U)\,.$$

Definition 2.3. (a) Let $(U_v)_{v\in T}$ be a partition refinement of $\mathfrak{M}$. The *n-ary partition width* of $(U_v)_v$ is

$$\operatorname{pwd}_n (U_v)_{v\in T} := \sup_{v\in T} \operatorname{eti}^n(U_v/M \smallsetminus U_v)\,.$$

(b) For an infinite cardinal κ we write $\operatorname{pwd}\mathfrak{M} < \kappa$ if there exists a partition refinement $(U_v)_v$ of $\mathfrak{M}$ with $\operatorname{pwd}_n(U_v)_v < \kappa$, for all $n < \omega$. If $\operatorname{pwd}\mathfrak{M} \nless \kappa$ we write $\operatorname{pwd}\mathfrak{M} \geq \kappa$. We say that $\mathfrak{M}$ has *finite partition width* if $\operatorname{pwd}\mathfrak{M} < \aleph_0$.

We will consider a structure to be tree-like if it has finite partition width.

Example. The partition refinements for linear orders and trees given in the above example have n-ary partition width 1, for every n. Hence, linear orders and trees are tree-like. Grids are a prime example of structures that are not tree-like. We will show in Lemma 3.2 below that every grid has a large partition width.

We can transfer bounds on the partition width from a structure $\mathfrak{M}$ to its substructures since each partition refinement of $\mathfrak{M}$ induces partition refinements of the substructures of $\mathfrak{M}$ whose width does not increase.

Lemma 2.4. *If $\mathfrak{M} \subseteq \mathfrak{N}$ and $\operatorname{pwd} \mathfrak{N} < \kappa$ then $\operatorname{pwd} \mathfrak{M} < \kappa$.*

We will consider a structure to be tree-like if it has finite partition width. The following result shows that, for finite signatures, this notion coincides with the interpretability in some tree.

Theorem 2.5 ([6, 4]). *Let $\mathfrak{M}$ be a structure with finite signature. $\mathfrak{M}$ has finite partition width if and only if there exist an ordinal α, a set $P \subseteq 2^{<\alpha}$, and a monadic second-order interpretation $\mathcal{I}$ with*

$$\mathfrak{M} \cong \mathcal{I}(2^{<\alpha}, \preceq, P) .$$

We conclude this section with two technical results which will be used below.

Lemma 2.6. *Let $\kappa := \operatorname{ti}^n_\Delta(A/U)$. There exists a set $U_0 \subseteq U$ of size $|U_0| \leq \kappa + \aleph_0$ such that, for all $\bar{a}, \bar{b} \in A^n$,*

$$\operatorname{tp}_\Delta(\bar{a}/U_0) = \operatorname{tp}_\Delta(\bar{b}/U_0) \quad \text{implies} \quad \operatorname{tp}_\Delta(\bar{a}/U) = \operatorname{tp}_\Delta(\bar{b}/U) .$$

Proof. Fix a sequence $(\bar{a}^\alpha)_{\alpha<\kappa}$ of tuples $\bar{a}^\alpha \in A^n$ such that, for every $\bar{b} \in A^n$, there exists a unique index α with

$$\operatorname{tp}_\Delta(\bar{a}^\alpha/U) = \operatorname{tp}_\Delta(\bar{b}/U) .$$

By induction on α, we will define finite sets $C_\alpha \subseteq U$ such that, for all $\beta < \alpha$,

$$\operatorname{tp}_\Delta(\bar{a}^\alpha/C_{<\alpha}) \neq \operatorname{tp}_\Delta(\bar{a}^\beta/C_{<\alpha}) ,$$

where $C_{<\alpha} := \bigcup_{i<\alpha} C_i$. Then the set $U_0 := C_{<\kappa}$ has the desired properties.

To define C_α we consider two cases. If there is no index $\beta < \alpha$ with

$$\operatorname{tp}_\Delta(\bar{a}^\alpha/C_{<\alpha}) = \operatorname{tp}_\Delta(\bar{a}^\beta/C_{<\alpha})$$

then we can simply set $C_\alpha := \varnothing$. Otherwise, there is exactly one such index β. Since

$$\operatorname{tp}_\Delta(\bar{a}^\alpha/U) \neq \operatorname{tp}_\Delta(\bar{a}^\beta/U)$$

there are some formula $\varphi(\bar{x}, \bar{y}) \in \Delta$ and parameters $\bar{c} \subseteq U$ with

$$\mathfrak{M} \vDash \varphi(\bar{a}, \bar{c}) \leftrightarrow \neg\varphi(\bar{b}, \bar{c}) .$$

We set $C_\alpha := \bar{c}$. □

Lemma 2.7 ([3]). *Let κ be an infinite cardinal, Δ a set of formulae of size $|\Delta| \leq \kappa$, and $A, B \subseteq M$ sets. If $\operatorname{ti}^n_\Delta(A/B) > 2^\kappa$ then there exist a formula $\varphi(\bar{x}, \bar{y}) \in \Delta$, a number $m < \omega$, and tuples $\bar{a}^v \in A^n$ and $\bar{b}^v \in B^m$, for $v < \kappa^+$, such that*

$$\mathfrak{M} \vDash \varphi(\bar{a}^u, \bar{b}^u) \leftrightarrow \neg\varphi(\bar{a}^v, \bar{b}^u) , \quad \text{for all } u < v .$$

3 Coding and indiscernibles

In the presence of a definable pairing function the monadic second-order theory of a structure is quite complicated. It is highly unlikely that we can develop a structure theory covering such structures. In [1] Baldwin and Shelah started an investigation of the monadic theories of structures without pairing function. Continuing their work we studied indiscernible sequences in [3]. Let us briefly summarise these results.

Definition 3.1. A structure $\mathfrak{M}$ *admits coding* if there exist an elementary extension $\mathfrak{N} \succeq \mathfrak{M}$, unary predicates $\bar{P}$, and infinite sets $A, B, C \subseteq N$ such that in the structure $(\mathfrak{N}, \bar{P})$ there exists a first-order definable bijection $A \times B \to C$.

It is not difficult to show that structures admitting coding have large partition width. A weak version of the converse will be established in Theorem 6.3.

Lemma 3.2 ([6, 4, 8]). *If $\mathfrak{M}$ admits coding then*

(a) $\operatorname{pwd} \mathfrak{M} \geq \aleph_0$,

(b) *for every cardinal κ, there exists an elementary extension $\mathfrak{N} \succeq \mathfrak{M}$ with $\operatorname{pwd} \mathfrak{N} \geq \kappa$.*

A first simple criterion for coding is the independence property.

Lemma 3.3 (Baldwin, Shelah [1]). *If $\mathfrak{M}$ has the independence property then it admits coding.*

It turns out that in structures which do not admit coding indiscernible sequences are well-behaved.

Theorem 3.4 ([3]). *Suppose that $\mathfrak{M}$ does not admit coding. Let $(\bar{a}^v)_{v\in I}$ be an indiscernible sequence over U. For every element c, there exist a linear order $J \supseteq I$ and an indiscernible sequence $(\bar{b}^v c^v)_{v\in I}$ over U such that $\bar{b}^v = \bar{a}^v$, for $v \in I$, and $c = c^v$, for some $v \in J$.*

Corollary 3.5 ([3]). *Suppose that $\mathfrak{M}$ does not admit coding. Let $(\bar{a}^v)_{v\in I}$ be an indiscernible sequence over U. For every element c such that $(\bar{a}^v)_v$ is not indiscernible over $U \cup \{c\}$, there exist a linear order $J \supseteq I$, an indiscernible sequence $(\bar{b}^v)_{v\in J}$ with $\bar{b}^v = \bar{a}^v$, for $v \in I$, and a unique index $s \in J$ such that*

$$\mathfrak{M} \vDash \varphi(c, \bar{b}[\bar{u}]) \leftrightarrow \varphi(c, \bar{b}[\bar{v}]),$$

for all formulae φ over U and all tuples $\bar{u}, \bar{v} \subseteq J$ with $\operatorname{ord}(s\bar{u}) = \operatorname{ord}(s\bar{v})$.

Definition 3.6. Let $\varphi(\bar{x})$ be a formula and $(\bar{a}^v)_{v\in I}$ a sequence. We define

$$[\![\varphi(\bar{a}^v)]\!]_{v\in I} := \{\, v \in I \mid \mathbb{M} \vDash \varphi(\bar{a}^v) \,\}.$$

Corollary 3.7 ([3]). *Suppose that $\mathfrak{M}$ does not admit coding and let $(\bar{a}^v)_{v\in I}$ be an indiscernible sequence over U where the order I has no minimal and no maximal element.*

For each element c and all formulae $\varphi(x, \bar{y})$ over U, one of the following cases holds:

- $\left|[\![\varphi(c, \bar{a}^v)]\!]_v\right| \leq 1$
- $\left|[\![\neg\varphi(c, \bar{a}^v)]\!]_v\right| \leq 1$
- $[\![\varphi(c, \bar{a}^v)]\!]_v$ *is an initial segment of I.*
- $[\![\varphi(c, \bar{a}^v)]\!]_v$ *is a final segment of I.*

4 Indiscernibles and the commutation order

In [3] we introduced an equivalence relation $\asymp_U \subseteq \alpha \times \alpha$ on the components of an indiscernible sequence of α-tuples. In the present section we define a linear preorder $\trianglelefteq_U$ refining this relation. Let us recall the results of [3] (see also [8]).

Definition 4.1. Let $(\bar{a}^v)_{v\in I}$ be a sequence of α-tuples indexed by a linear order I.

(a) We denote the *order type* of $\bar{v} \in I^m$ by $\operatorname{ord}(\bar{v})$ and its *equality type* by $\operatorname{equ}(\bar{v})$. For sets $C, D \subseteq I$, we write $C < D$ if $c < d$, for all $c \in C$ and $d \in D$. Analogously, we define $\bar{u} < \bar{v}$ for tuples $\bar{u}, \bar{v} \subseteq I$.

(b) The sequence $(\bar{a}^v)_v$ is *proper* if $\bar{a}^u \cap \bar{a}^v = \emptyset$, for $u \neq v$.

(c) For $\bar{v} \in I^m$, we set

$$\bar{a}[\bar{v}] := (\bar{a}^{v_0}, \dots, \bar{a}^{v_{m-1}}).$$

For $J \subseteq I$ and $s \in I$ we define

$$\bar{a}[J] := (\bar{a}^v)_{v\in J} \quad \text{and} \quad \bar{a}[<s] := (\bar{a}^v)_{v<s}.$$

The terms $\bar{a}[>s]$, $\bar{a}[\leq s]$, and so on, are defined analogously.

(d) For $\bar{v} \in I^\alpha$, we set

$$\bar{a}\langle\bar{v}\rangle := (a_i^{v_i})_{i<\alpha}.$$

Definition 4.2. (a) Let $\varphi(\bar{x}^0, \dots, \bar{x}^{k-1})$ be a formula where each $\bar{x}^i$ is an α-tuple of variables. A sequence $(\bar{a}^v)_{v\in I}$ of α-tuples is *φ-indiscernible* if, for all indices $\bar{u}^i, \bar{v}^i \in I^\alpha$, $i < k$, with $\operatorname{ord}(\bar{u}^0 \dots \bar{u}^{k-1}) = \operatorname{ord}(\bar{v}^0 \dots \bar{v}^{k-1})$, we have

$$\mathfrak{M} \vDash \varphi\big(\bar{a}\langle\bar{u}^0\rangle, \dots, \bar{a}\langle u^{k-1}\rangle\big) \leftrightarrow \varphi\big(\bar{a}\langle\bar{v}^0\rangle, \dots, \bar{a}\langle v^{k-1}\rangle\big).$$

If Δ is a set of such formulae we call $(\bar{a}^v)_{v\in I}$ *Δ-indiscernible* if it is φ-indiscernible, for every $\varphi \in \Delta$.

We adopt the usual convention of working in a sufficiently saturated monster model $\mathbb{M}$ into which we can embed every model $\mathfrak{M}$ under consideration. All elements and sets are tacitly assumed to be contained in $\mathbb{M}$. By an *U-automorphism*, we mean an automorphism π of $\mathbb{M}$ with $\pi|_U = \mathrm{id}_U$. We will frequently use the following standard facts from model theory.

Lemma 4.3. *Let $(\bar{a}^v)_{v\in I}$ be an infinite indiscernible sequence over U and let P be its minimal U-partition. For every family $(\beta_p)_{p\in P}$ of strictly increasing maps $\beta_p : I \to I$, there exists a U-automorphism π such that*

$$\pi(\bar{a}^v|_p) = \bar{a}^{\beta_p(v)}|_p.$$

Lemma 4.4. *Let $(\bar{a}^v)_{v\in I}$ be an indiscernible sequence over U. For every order embedding $\alpha : I \to J$ there exists an indiscernible sequence $(\bar{b}^v)_{v\in J}$ over U such that $\bar{b}^{\alpha(v)} = \bar{a}^v$, for $v \in I$.*

The relation $\{\, \bar{a}^v \mid v \in I \,\}$ is usually not definable but we can define relations $\{\, \bar{a}^v|_p \mid v \in I \,\}$ for certain subsets $p \subseteq \alpha$.

Definition 4.5. (a) A *partition* of a set X is a set $P \subseteq \wp(X)$ such that $X = \bigcup P$ and $p \cap q = \emptyset$, for distinct $p, q \in P$.

(b) Every partition P on X induces the equivalence relation

$$x \approx_P y \quad :\text{iff} \quad \text{there is some } p \in P \text{ with } x, y \in p.$$

(c) We order partitions P and Q of X by

$$P \sqsubseteq Q \quad :\text{iff} \quad \approx_P \subseteq \approx_Q .$$

Definition 4.6. Let $(\bar a^\nu)_{\nu\in I}$ be a sequence of α-tuples and let $\varphi(\bar x^0,\dots,\bar x^k)$ be a formula where each $\bar x^i$ is an α-tuple of variables. A *φ-partition* of $(\bar a^\nu)_{\nu\in I}$ is a partition P of α such that

$$\mathfrak{M} \vDash \varphi\big(\bar a\langle \bar u^0\rangle,\dots,\bar a\langle \bar u^k\rangle\big) \leftrightarrow \varphi\big(\bar a\langle \bar v^0\rangle,\dots,\bar a\langle \bar v^k\rangle\big),$$

for all indices $\bar u^i, \bar v^i \in I^\alpha$, $i \le k$, such that

$$\operatorname{ord}(\bar u^0|_p \dots \bar u^k|_p) = \operatorname{ord}(\bar v^0|_p \dots \bar v^k|_p), \quad \text{for every } p \in P .$$

Let Δ be a set of formulae. A *Δ-partition* is a partition P that is a φ-partition, for every $\varphi \in \Delta$.

Theorem 4.7 ([3]). *For every infinite Δ-indiscernible sequence $(\bar a^\nu)_{\nu\in I}$, there exists a unique minimal Δ-partition P.*

Definition 4.8. Let $(\bar a^\nu)_{\nu\in I}$ be an infinite Δ-indiscernible sequence of α-tuples and let P be the minimal Δ-partition of α corresponding to $(\bar a^\nu)_\nu$.

(a) The elements of P are called *Δ-classes.*

(b) We set $\asymp_\Delta := \approx_P$. Two indices i and k are *Δ-dependent* if $i \asymp_\Delta k$. Otherwise, they are *Δ-independent.*

(c) If Δ is the set of all first-order formulae over U we also also speak of *U-partitions, U-classes, U-independent* indices, etc. and we write $\asymp_U$ instead of $\asymp_\Delta$.

Remark. Let $(\bar a^\nu)_{\nu\in I}$ be an infinite indiscernible sequence over U. For every U-class p, the sequence $(\bar a^\nu|_p)_{\nu\in I}$ is indiscernible over $U \cup \bar a|_{\alpha\setminus p}[I]$.

Theorem 4.9 ([3]). *Let $(\bar a^\nu)_{\nu\in I}$ be an infinite φ-indiscernible sequence of α-tuples and suppose that φ has r free variables. For each φ-class p and every finite subset $q \subseteq p$, there exists a formula $\chi_q(\bar x; \bar y, \bar z, \bar Z)$ with the following property.*

If $\bar s, \bar t \in I^r$ are strictly increasing r-tuples with $\bar s < \bar t$ and

$$A_i := \{\, a_i^\nu \mid \nu \in I,\ \bar s < \nu < \bar t \,\}, \quad \textit{for } i \in p,$$

then we have

$$\mathfrak{M} \vDash \chi_q(\bar c; \bar a[\bar s], \bar a[\bar t], \bar A) \quad \text{iff} \quad \bar c = \bar a^\nu|_q \textit{ for some } \nu \in I \textit{ with } \bar s < \nu < \bar t .$$

In the absence of coding the relation of Δ-dependence is 'local' in the sense that whether or not $i \asymp_\Delta k$ holds only depends on the sequence $(a_i^\nu a_k^\nu)_{\nu\in I}$, not on all of $(\bar a^\nu)_\nu$.

Proposition 4.10 ([3]). *Suppose that $\mathfrak{M}$ does not admit coding. Let $(\bar a^\nu)_{\nu\in I}$ be an indiscernible sequence over U with $|\bar a^\nu| = \alpha$, and let $N \subseteq \alpha$. If P is the U-partition of $(\bar a^\nu)_\nu$ then the U-partition of $(\bar a^\nu|_N)_\nu$ is $\{\, p \cap N \mid p \in P \,\}$.*

The following criterion for coding will be used in Lemma 4.15 below.

Lemma 4.11 (Shelah [43]). *Let $(\bar a^\nu)_{\nu\in I}$ be an infinite indiscernible sequence over U. Suppose that there exists a U-class p, an element $c \in \mathbb{M}$, a formula ψ over U, and indices $s < t$ such that*

- $\mathfrak{M} \vDash \psi(c, \bar a^s|_p, \bar a^t|_p)$,
- $\mathfrak{M} \vDash \neg\psi(c, \bar a^s|_p, \bar a^\nu|_p)$ *for infinitely many $\nu > t$,*
- $\mathfrak{M} \vDash \neg\psi(c, \bar a^\nu|_p, \bar a^t|_p)$ *for infinitely many $\nu < s$.*

Then $\mathfrak{M}$ admits coding.

The results of [3] indicate that the relation $\asymp_U$ partitioning an indiscernible sequence into its U-classes is well-behaved for structures that do not admit coding. In this section we introduce a refinement $\trianglelefteq_U$ of $\asymp_U$ and we show that it linearly preorders each U-class, provided that the structure in question does not admit coding.

Definition 4.12. Suppose that $(\bar a^\nu)_{\nu\in I}$ is an indiscernible sequence of α-tuples over U. For sets $p, q \subseteq \alpha$ of indices, we define $p \trianglelefteq_U q$ iff, for some/all $s < t$ in I, we have

$$\operatorname{tp}\big(\bar a^s|_p \bar a^s|_q / U \cup \bar a[<s] \cup \bar a[>t]\big) \neq \operatorname{tp}\big(\bar a^t|_p \bar a^s|_q / U \cup \bar a[<s] \cup \bar a[>t]\big).$$

For single indices $i, k < \alpha$, we write $i \trianglelefteq_U k$ instead of $\{i\} \trianglelefteq_U \{k\}$.

We start by showing that the U-classes are exactly the connected components of this relation.

Lemma 4.13. *Let $(\bar a^\nu)_\nu$ be an indiscernible sequence of α-tuples over U. For $i, k < \alpha$, we have*

$$i \asymp_U k \quad \text{iff} \quad i \trianglelefteq_U k \textit{ or } k \trianglelefteq_U i .$$

Proof. ($\Leftarrow$) follows immediately from the definition of $\asymp_U$.

($\Rightarrow$) Suppose that $i \ntrianglelefteq_U k$ and $k \ntrianglelefteq_U i$. We have to show that $i \not\asymp_U k$, i.e.,

$$\operatorname{tp}(a_i[\bar u]a_k[\bar v]/U) = \operatorname{tp}(a_i[\bar s]a_k[\bar t]/U)\,,$$

for all $\bar u, \bar v, \bar s, \bar t \subseteq I$ with $\operatorname{ord}(\bar u) = \operatorname{ord}(\bar s)$ and $\operatorname{ord}(\bar v) = \operatorname{ord}(\bar t)$. As usual we only need to consider the case that $\bar u$ and $\bar v$ differ at only one component. Hence, consider indices

$$u_0 < \dots < u_{m-1} < s < t < v_0 < \dots < v_{n-1}\,.$$

It is sufficient to show that

$$\begin{aligned}\operatorname{tp}(a_i[\bar u s \bar v]a_k[\bar u t \bar v]/U) &= \operatorname{tp}(a_i[\bar u s \bar v]a_k[\bar u s \bar v]/U)\\ &= \operatorname{tp}(a_i[\bar u t \bar v]a_k[\bar u s \bar v]/U)\,.\end{aligned}$$

For the first equation, note that $i \ntrianglelefteq_U k$ implies

$$\operatorname{tp}(a_i^t a_k^s/U \cup a_i[\bar u \bar v] \cup a_k[\bar u \bar v]) = \operatorname{tp}(a_i^s a_k^s/U \cup a_i[\bar u \bar v] \cup a_k[\bar u \bar v])\,.$$

Similarly, $k \ntrianglelefteq_U i$ implies that

$$\operatorname{tp}(a_i^s a_k^t/U \cup a_i[\bar u \bar v] \cup a_k[\bar u \bar v]) = \operatorname{tp}(a_i^s a_k^s/U \cup a_i[\bar u \bar v] \cup a_k[\bar u \bar v])\,,$$

as desired. ☐

Lemma 4.14. *Let $(\bar a^v)_v$ be an indiscernible sequence over U.*

(a) *$p \trianglelefteq_U q$ implies that $p_+ \trianglelefteq_U q_+$, for all $p_+ \supseteq p$ and $q_+ \supseteq q$.*

(b) *If $p \ntrianglelefteq_U q \cup r$ and $q \ntrianglelefteq_U r$ then $p \cup q \ntrianglelefteq_U r$.*

(c) *If $p \cup q \ntrianglelefteq_U r$ and $p \ntrianglelefteq_U q$ then $p \ntrianglelefteq_U q \cup r$.*

Proof. (a) follows immediately from the definition.

(b) For $s < v < t$, we have

$$\begin{aligned}&\operatorname{tp}\big(\bar a^s|_p \bar a^s|_q \bar a^s|_r / U \cup \bar a[<s] \cup \bar a[>t]\big)\\ &= \operatorname{tp}\big(\bar a^t|_p \bar a^s|_q \bar a^s|_r / U \cup \bar a[<s] \cup \bar a[>t]\big) && (p \ntrianglelefteq_U q \cup r)\\ &= \operatorname{tp}\big(\bar a^t|_p \bar a^v|_q \bar a^s|_r / U \cup \bar a[<s] \cup \bar a[>t]\big) && (q \ntrianglelefteq_U r)\\ &= \operatorname{tp}\big(\bar a^v|_p \bar a^v|_q \bar a^s|_r / U \cup \bar a[<s] \cup \bar a[>t]\big) && (p \ntrianglelefteq_U q)\\ &= \operatorname{tp}\big(\bar a^t|_p \bar a^t|_q \bar a^s|_r / U \cup \bar a[<s] \cup \bar a[>t]\big)\,,\end{aligned}$$

as desired.

(c) For $s < v < t$, we have

$$\begin{aligned}&\operatorname{tp}\big(\bar a^s|_p \bar a^s|_q \bar a^s|_r / U \cup \bar a[<s] \cup \bar a[>t]\big)\\ &= \operatorname{tp}\big(\bar a^v|_p \bar a^v|_q \bar a^s|_r / U \cup \bar a[<s] \cup \bar a[>t]\big) && (p \cup q \ntrianglelefteq_U r)\\ &= \operatorname{tp}\big(\bar a^t|_p \bar a^v|_q \bar a^s|_r / U \cup \bar a[<s] \cup \bar a[>t]\big) && (p \ntrianglelefteq_U q)\\ &= \operatorname{tp}\big(\bar a^t|_p \bar a^s|_q \bar a^s|_r / U \cup \bar a[<s] \cup \bar a[>t]\big)\,, && (q \ntrianglelefteq_U r)\end{aligned}$$

as desired. ☐

Lemma 4.15. *Suppose that $\mathfrak M$ does not admit coding and let $(\bar a^v)_{v \in I}$ be an indiscernible sequence of α-tuples over U. Let $p, q \subseteq \alpha$ and $i \in \alpha$. If $p \ntrianglelefteq_U q$ then $p \cup \{i\} \ntrianglelefteq_U q$ or $p \ntrianglelefteq_U q \cup \{i\}$.*

Proof. W.l.o.g. assume that I is dense. Fix $s < t$ in I. Since $p \ntrianglelefteq_U q$ we have

$$\operatorname{tp}(\bar a^t|_p \bar a^s|_q / U \cup \bar a[<s] \cup \bar a[>t]) = \operatorname{tp}(\bar a^s|_p \bar a^s|_q / U \cup \bar a[<s] \cup \bar a[>t])\,.$$

Hence, there exists an element c such that

$$\operatorname{tp}(\bar a^t|_p \bar a^s|_q c / U \cup \bar a[<s] \cup \bar a[>t]) = \operatorname{tp}(\bar a^s|_p \bar a^s|_q a_i^s / U \cup \bar a[<s] \cup \bar a[>t])\,.$$

For a contradiction, suppose that $p \cup \{i\} \trianglelefteq_U q$ and $p \trianglelefteq_U q \cup \{i\}$. Then there are formulae $\varphi(\bar x, \bar y, z)$ and $\psi(\bar x, \bar y, z)$ over $U \cup \bar a[<s] \cup \bar a[>t]$ such that, for $s < v \le t$,

$$\begin{aligned}&\mathfrak M \vDash \varphi(\bar a^s|_p, \bar a^s|_q, a_i^s)\,, && \mathfrak M \vDash \psi(\bar a^s|_p, \bar a^s|_q, a_i^s)\,,\\ &\mathfrak M \nvDash \varphi(\bar a^v|_p, \bar a^s|_q, a_i^v)\,, && \mathfrak M \nvDash \psi(\bar a^v|_p, \bar a^s|_q, a_i^s)\,.\end{aligned}$$

Let u_0 be the maximal index $u < s$ such that an element of $\bar a^u$ appears in φ or ψ, and let u_1 be the minimal index $u_1 > t$ appearing in φ or ψ. Then

$$\mathfrak M \nvDash \varphi(\bar a^t|_p, \bar a^s|_q, a_i^t) \quad \text{implies} \quad \mathfrak M \nvDash \varphi(\bar a^s|_p, \bar a^v|_q, a_i^s) \quad \text{for } u_0 < v < s\,.$$

Setting $\chi := \varphi \wedge \psi$ it follows that

$$\begin{aligned}&\mathfrak M \vDash \chi(\bar a^s|_p, \bar a^s|_q, a_i^s)\,,\\ &\mathfrak M \nvDash \chi(\bar a^v|_p, \bar a^s|_q, a_i^s)\,, && \text{for } s < v < u_1\,,\\ &\mathfrak M \nvDash \chi(\bar a^s|_p, \bar a^v|_q, a_i^s)\,, && \text{for } u_0 < v < s\,.\end{aligned}$$

By choice of c this implies that

$$\begin{aligned}
&\mathfrak{M} \vDash \chi(\bar{a}^t|_p, \bar{a}^s|_q, c)\,, \\
&\mathfrak{M} \nvDash \chi(\bar{a}^v|_p, \bar{a}^s|_q, c)\,, \qquad \text{for } t < v < u_1\,, \\
&\mathfrak{M} \nvDash \chi(\bar{a}^t|_p, \bar{a}^v|_q, c)\,, \qquad \text{for } u_0 < v < s\,.
\end{aligned}$$

Therefore, we can use Lemma 4.11 to conclude that $\mathfrak{M}$ admits coding. Contradiction. □

Corollary 4.16. *Suppose that $\mathfrak{M}$ does not admit coding and let $(\bar{a}^v)_v$ be an indiscernible sequence of α-tuples over U.*

(a) *$p \trianglelefteq_U i \trianglelefteq_U q$ implies $p \trianglelefteq_U q$, for $p, q \subseteq \alpha$ and $i \in \alpha$.*

(b) *$\trianglelefteq_U$ linearly preorders every U-class.*

Proof. (a) Suppose that $p \ntrianglelefteq_U q$. Then we have $p \cup \{i\} \ntrianglelefteq_U q$ or $p \ntrianglelefteq_U q \cup \{i\}$, by Lemma 4.15. In the former case, it follows by monotonicity that $i \ntrianglelefteq_U q$ while in the latter case we have $p \ntrianglelefteq_U i$.

(b) $\trianglelefteq_U$ is clearly reflexive. In (a) we have shown that it is transitive. Hence, $\trianglelefteq_U$ is a preorder. To show that it is linear on each U-class note that $i \asymp_U k$ implies $i \trianglelefteq_U k$ or $k \trianglelefteq_U i$. □

Corollary 4.17. *Suppose that $\mathfrak{M}$ does not admit coding and let $(\bar{a}^v)_v$ be an indiscernible sequence over U.*

(a) *$p \trianglelefteq_U q$ if and only if $i \trianglelefteq_U q$, for some $i \in p$.*

(b) *$i \trianglelefteq_U q$ if and only if $i \trianglelefteq_U k$, for some $k \in q$.*

(c) *$p \trianglelefteq_U q$ if and only if $i \trianglelefteq_U k$, for some $i \in p$ and $k \in q$.*

Proof. (a) By monotonicity it follows that $p \ntrianglelefteq_U q$ implies $i \ntrianglelefteq_U q$ for all $i \in p$. We prove the converse by induction on $|p|$. Suppose that $p \cup \{i\} \trianglelefteq_U q$. If $p \trianglelefteq_U q$ then the claim follows by induction hypothesis. Hence, we may assume that $p \ntrianglelefteq_U q$. Since $p \cup \{i\} \trianglelefteq_U q$ it follows by Lemma 4.15 that $p \ntrianglelefteq_U q \cup \{i\}$. If $i \ntrianglelefteq_U q$ then we would have $p \cup \{i\} \ntrianglelefteq_U q$, by Lemma 4.14 (b). Consequently, we have $i \trianglelefteq_U q$.

(b) The proof is analogous to (a). By monotonicity, $i \ntrianglelefteq_U q$ implies $i \ntrianglelefteq_U k$ for all $k \in q$. We prove the converse by induction on $|q|$. Suppose that $i \trianglelefteq_U q \cup \{k\}$. If $i \trianglelefteq_U q$ then the claim follows by induction hypothesis. Hence, we may assume that $i \ntrianglelefteq_U q$. By Lemma 4.15, it follows that $\{i, k\} \ntrianglelefteq_U q$. If $i \ntrianglelefteq_U k$ then we would have $i \ntrianglelefteq_U q \cup \{k\}$, by Lemma 4.14 (c). Consequently, we have $i \trianglelefteq_U k$.

(c) follows immediately from (a) and (b). □

Since $\trianglelefteq_U$ is a preorder on each $\asymp_U$-class it follows that we can divide each U-class into the classes of this preorder which we call *strong U-classes.*

Definition 4.18. Let $(\bar{a}^v)_{v \in I}$ be an indiscernible sequence of α-tuples over U. A *strong U-class* is an equivalence class for the relation

$$\{\, \langle i, k\rangle \in \alpha \times \alpha \mid i \trianglelefteq_U k \text{ and } k \trianglelefteq_U i \,\}\,.$$

We have shown above that every U-class is partitioned into one or several strong U-classes that are linearly ordered by $\trianglelefteq_U$. Sets of the form $\bar{a}^v|_p$, for a U-class p, will be the building blocks of the partition refinement we will construct in Section 6. To compute the width of the resulting partition refinement we have to bound the type index $\mathrm{ti}^n_\Delta(\bar{a}^v|_p / U \cup \bar{a}[\neq v])$ of such sets. This will be done in the next theorem. Let us start with two technical lemmas that are needed in its proof.

Lemma 4.19. *Suppose that there are formulae φ, ψ_k, and ψ^*_k, monadic parameters $\bar{P}$, and sequences $(a^v)_{v\in I}$, $(\bar{b}^n)_{n\in N}$, $(\bar{c}^{vn})_{v\in I, n\in N}$, $(\bar{c}^{v*})_{v\in I}$, and $\bar{d}$ satisfying the following conditions:*

- *The sequence $(a^v \bar{c}^{v*} (\bar{c}^{vn})_n)_{v\in I}$ is indiscernible over $(\bar{b}^n)_n \cup \bar{d}$.*
- *I and N are infinite.*
- *There is some $\sigma \in \{=, \neq, <, >, \leq, \geq\}$ such that*

$$\mathfrak{M} \vDash \varphi(\bar{b}^i, \bar{c}^{vk}, \bar{c}^{v*}, \bar{d}) \quad \text{iff} \quad i\ \sigma\ k\,.$$

- *There are relations $\rho_k \in \{=, \leq, \geq\}$ such that*

$$\mathfrak{M} \vDash \psi_k(c^{ui}_k, a^v, \bar{d}) \quad \text{iff} \quad u\ \rho_k\ v\,.$$

- $\mathfrak{M} \vDash \psi^*_k(c^{u*}_k, a^v, \bar{d}, \bar{P}) \quad \text{iff} \quad u = v\,.$

Then $\mathfrak{M}$ admits coding.

Proof. Set

$$\begin{aligned}
A &:= \{\, a^v \mid v \in I \,\}\,, &\quad C^*_k &:= \{\, c^{v*}_k \mid v \in I \,\}\,, \\
B_k &:= \{\, b^n_k \mid n \in N \,\}\,, &\quad C^v_k &:= \{\, c^{vn}_k \mid n \in N \,\}\,,
\end{aligned}$$

and $C_k := \bigcup_v C^v_k$. The formula

$$\vartheta^*(x, \bar{z}) := Ax \wedge \bigwedge_k [C^*_k z_k \wedge \psi^*_k(z_k, x, \bar{d}, \bar{P})]$$

satisfies

$$\mathfrak{M} \models \vartheta^*(a, \bar{c}) \quad \text{iff} \quad a = a^\nu \text{ and } \bar{c} = \bar{c}^{\nu *}, \text{ for some } \nu \in I .$$

We also construct a formula $\hat{\psi}_k$ such that

$$\mathfrak{M} \models \hat{\psi}_k(a, c) \quad \text{iff} \quad a = a^\nu \text{ and } c = c_k^{\nu n} \text{ for some } \nu \in I \text{ and } n \in N .$$

If ρ_k equals = then we can simply set

$$\hat{\psi}_k(x, z) := Ax \wedge C_k z \wedge \psi_k(z, x, \bar{d}) .$$

Suppose that $\rho_k \in \{\leq, \geq\}$. Defining

$$\chi(x, x') := Ax \wedge Ax' \wedge \forall z[Qz \wedge \psi_l(z, x, \bar{d}) \to \psi_l(z, x', \bar{d})] ,$$

where $Q := \{ c_l^{\nu \circ} \mid \nu \in I \}$, we obtain a formula such that

$$\mathfrak{M} \models \chi(a, a') \quad \text{iff} \quad a = a^u \text{ and } a' = a^\nu \text{ for some } u \; \rho_k \; \nu .$$

Hence, we can set

$$\hat{\psi}_k(x, z) := Ax \wedge C_k z \wedge \forall x'[Ax \to [\chi(x', x) \leftrightarrow \psi_k(z, x', \bar{d})]] .$$

Let $N^+ := \mathbb{Z} + N + \mathbb{Z}$ be the extension of the ordering N by two copies of $\mathbb{Z}$. By compactness, we can find extensions $(\bar{b}^n)_{n \in N^+}$ and $(\bar{c}^{\nu n})_{\nu \in I, n \in N^+}$ of $(\bar{b}^n)_n$ and $(\bar{c}^{\nu n})_{\nu, n}$ that behave in the same way with respect to the formulae ψ_k and φ. W.l.o.g. assume that $|\bar{b}^n|$ and $|\bar{c}^{\nu n}|$ are minimal. Then $(\bar{b}^n \bar{c}^{\nu n})_n$ forms a single φ-class and, by Theorem 4.9, there exists a formula

$$\eta\big(\bar{y}, \bar{z}, \bar{c}^{\nu *}, \bar{b}[\bar{m}], \bar{c}^\nu[\bar{m}], \bar{B}, \bar{C}^\nu\big)$$

with parameters $\bar{B}$, $\bar{C}^\nu$, $\bar{c}^{\nu *}$, $\bar{b}^{m_0}, \dots, \bar{b}^{m_l}$, and $\bar{c}^{\nu m_0}, \dots, \bar{c}^{\nu m_l}$, for $\bar{m} \subseteq N^+ \setminus N$, such that

$$\mathfrak{M} \models \eta\big(\bar{b}, \bar{c}, \bar{c}^{\nu *}, \bar{b}[\bar{m}], \bar{c}^\nu[\bar{m}], \bar{B}, \bar{C}^\nu\big)$$

iff $\quad \bar{b} = \bar{b}^n$ and $\bar{c} = \bar{c}^{\nu n}$, for some $n \in N$.

Set $P_k^m := \{ c_k^{\nu m} \mid \nu \in I \}$ and

$$\zeta_0(x, \bar{z}^*, \bar{u}) := Ax \wedge \vartheta^*(x, \bar{z}^*) \wedge \bigwedge_{k,i} [P_k^{m_i} u_k^i \wedge \hat{\psi}_k(x, u_k^i)] .$$

Then we have

$$\mathfrak{M} \models \zeta_0(a, \bar{c}^*, \bar{e}) \quad \text{iff} \quad a = a^\nu,\ \bar{c}^* = \bar{c}^{\nu *}, \text{ and } \bar{e} = \bar{c}^\nu[\bar{m}], \text{ for some } \nu \in I .$$

Let $\hat{\eta}(x, \bar{y}, \bar{z}, \bar{c}^{\nu *}, \bar{b}[\bar{m}], \bar{c}^\nu[\bar{m}], \bar{B}, \bar{C})$ be the formula obtained from η by replacing the parameter C_k^ν by the formula $\hat{\psi}_k$ and set

$$\zeta(x, \bar{y}, \bar{z}, \bar{z}^*, \bar{u}) := \zeta_0(x, \bar{z}^*, \bar{u}) \wedge \hat{\eta}(x, \bar{y}, \bar{z}, \bar{z}^*, \bar{b}[\bar{m}], \bar{u}, \bar{B}, \bar{C}) .$$

Then it follows that

$$\mathfrak{M} \models \zeta(a, \bar{b}, \bar{c}, \bar{c}^*, \bar{e}) \quad \text{iff} \quad a = a^\nu,\ \bar{b} = \bar{b}^n,\ \bar{c} = \bar{c}^{\nu n},\ \bar{c}^* = \bar{c}^{\nu *}, \text{ and } \bar{e} = \bar{c}^\nu[\bar{m}], \text{ for some } \nu \in I \text{ and } n \in N .$$

Consequently, we have

$$\mathfrak{M} \models \exists \bar{y}' \exists \bar{z}' \exists \bar{z}^* \exists \bar{u} \zeta(a, b\bar{y}', c\bar{z}', \bar{z}^*, \bar{u})$$

iff $\quad a = a^\nu$, $b = b_0^n$, and $c = c_0^{\nu n}$, for some $\nu \in I$ and $n \in N$,

and $\mathfrak{M}$ admits coding. □

Lemma 4.20. *Suppose that there are sequences $\bar{d}$, $(a^\nu)_{\nu \in I}$, $(\bar{b}^n)_{n < \omega}$, and $(c^{\nu n})_{\nu \in I, n < \omega}$ and a formula φ satisfying the following conditions:*

- *$(a^\nu (c^{\nu n})_n)_{\nu \in I}$ is indiscernible over $(\bar{b}^n)_n \cup \bar{d}$.*
- *I is dense and it has no least element and no greatest one.*
- *There is some $\rho \in \{=, \leq, \geq\}$ such that*

$$\mathfrak{M} \models \varphi(a^u, \bar{b}^n, c^{\nu n}, \bar{d}) \quad \text{iff} \quad u \; \rho \; \nu .$$

- *There are relations $\sigma_0 \in \{=, \leq, \geq\}$ and $\sigma_-, \sigma_+ \in \{\emptyset, I \times I, =, \neq, <, >, \leq, \geq\}$ such that*

$$\begin{aligned} &\mathfrak{M} \models \varphi(a^\nu, \bar{b}^k, c^{\nu n}, \bar{d}) && \text{iff} \quad k \; \sigma_0 \; n , \\ &\mathfrak{M} \models \varphi(a^u, \bar{b}^k, c^{\nu n}, \bar{d}) && \text{iff} \quad k \; \sigma_- \; n , \quad \textit{for } u < \nu , \\ &\mathfrak{M} \models \varphi(a^u, \bar{b}^k, c^{\nu n}, \bar{d}) && \text{iff} \quad k \; \sigma_+ \; n , \quad \textit{for } u > \nu . \end{aligned}$$

Then $\mathfrak{M}$ admits coding.

Proof. We start by constructing a formula ψ such that

$$\mathfrak{M} \vDash \psi(a^v, \bar{b}^n, c^{vn}) \quad \text{and} \quad \mathfrak{M} \nvDash \psi(a^u, \bar{b}^k, c^{vn}) \quad \text{for } u \neq v\,.$$

Let $A := \{\, a^v \mid v \in I \,\}$, $C^\circ := \{\, C^{v\circ} \mid v \in I \}$, and $C := \{\, c^{vn} \mid v \in I,\ n < \omega \,\}$. If ρ equals $=$ then we can set

$$\psi(x, \bar{y}, z) := \forall x'(Ax' \to (\varphi(x', \bar{y}, z, \bar{d}) \leftrightarrow x' = x))\,.$$

Clearly, we have $\mathfrak{M} \vDash \psi(a^v, \bar{b}^n, \bar{c}^{vn})$ and, by indiscernibility, it follows that $\mathfrak{M} \nvDash \psi(a^u, \bar{b}^k, \bar{c}^{vn})$, for all $u \neq v$.

For $\rho \in \{\leq, \geq\}$, we define

$$\chi(x, x') := Ax \wedge Ax' \wedge \forall z[C^\circ z \wedge \varphi(x', \bar{b}^\circ, z, \bar{d}) \to \varphi(x, \bar{b}^\circ, z, \bar{d})]\,.$$

This formula satisfies

$$\mathfrak{M} \vDash \vartheta(a, a') \quad \text{iff} \quad a = a^u \text{ and } a' = a^v \text{ for some } u \mathrel{\rho} v\,.$$

Hence, we can obtain the desired formula ψ by setting

$$\psi(x, \bar{y}, z) := \forall x'[Ax' \to (\varphi(x', \bar{y}, z, \bar{d}) \leftrightarrow \vartheta(x', x))]\,.$$

Again, by indiscernibility, we have $\mathfrak{M} \nvDash \psi(a^u, \bar{b}^k, \bar{c}^{vn})$, for all $u \neq v$.

If we can show that the constructed formula ψ satisfies

$$\mathfrak{M} \nvDash \psi(a^v, \bar{b}^k, c^{vn}) \quad \text{for all } k \neq n\,,$$

then it follows that

$$\mathfrak{M} \vDash \psi(a^u, \bar{b}^k, c^{vn}) \quad \text{iff} \quad u = v \text{ and } k = n\,,$$

and $\mathfrak{M}$ admits coding. Hence, suppose that

$$\mathfrak{M} \vDash \psi(a^v, \bar{b}^k, c^{vn}) \quad \text{for some } k < n\,.$$

Then $\sigma_0 = \leq$. Fix some $s \in I$. W.l.o.g. assume that $|\bar{b}^n|$ is minimal. Then we can use Theorem 4.9 to find a formula $\eta(\bar{y}, z)$ (with monadic parameters) such that

$$\mathfrak{M} \vDash \eta(\bar{b}, c) \quad \text{iff} \quad \bar{b} = \bar{b}^n \text{ and } c = c^{sn},\ \text{for some } n\,.$$

Defining

$$\vartheta(\bar{y}, \bar{y}') := \exists z \eta(\bar{y}, z) \wedge \exists z \eta(\bar{y}', z) \wedge \forall z(\eta(\bar{y}', z) \to \eta(\bar{y}, z))$$

we obtain a formula such that

$$\mathfrak{M} \vDash \vartheta(\bar{b}, \bar{b}') \quad \text{iff} \quad \bar{b} = \bar{b}^k \text{ and } \bar{b}' = \bar{b}^n,\ \text{for some } k \leq n\,.$$

If we define

$$\begin{aligned} \zeta_0(x, \bar{y}, z) &:= Ax \wedge Cz \wedge \exists z' \eta(\bar{y}, z') \wedge \psi(x, \bar{y}, z)\,, \\ \zeta(x, \bar{y}, z) &:= \zeta_0(x, \bar{y}, z) \wedge \forall \bar{y}'[\zeta_0(x, \bar{y}', z) \to \vartheta(\bar{y}', \bar{y})]\,, \end{aligned}$$

then we have

$$\begin{aligned} \mathfrak{M} \vDash \zeta(a, \bar{b}, c) \quad \text{iff} \quad & a = a^v,\ \bar{b} = \bar{b}^n,\ \text{and } c = c^{vn}, \\ & \text{for some } v \in I \text{ and } n < \omega\,. \end{aligned}$$

Again, $\mathfrak{M}$ admits coding.

The remaining case that $\mathfrak{M} \vDash \psi(a^v, \bar{b}^k, c^{vn})$, for some $k > n$, is handled symmetrically. ☐

Theorem 4.21. *Suppose that $(\bar{a}^v)_{v \in I}$ is a proper infinite indiscernible sequence over U and let Δ be a set of formulae (over $\varnothing$) such that $2^{|\Delta|} \leq \kappa$ where $\kappa := |\Sigma| + \aleph_0$ is the number of first-order formulae over the signature Σ. If there exist a U-class p, an index $v \in I$, and a number $n < \omega$ such that*

$$\mathrm{ti}^n_\Delta(\bar{a}^v|_p / U \cup \bar{a}[\neq v]) > \kappa$$

then $\mathfrak{M}$ admits coding.

Proof. By compactness, we may assume that I is dense without endpoints. Fix n-tuples $\bar{c}^{vi} \subseteq \bar{a}^v|_p$, for $i < \kappa^+$ such that

$$\mathrm{tp}_\Delta(\bar{c}^{vi} / U \cup \bar{a}[\neq v]) \neq \mathrm{tp}_\Delta(\bar{c}^{vk} / U \cup \bar{a}[\neq v])\,, \quad \text{for } i \neq k\,.$$

Choose some element $d^v \in \bar{a}^v|_p$ and indices $s < v < t$ in I. To simplify notation we set $W := U \cup \bar{a}[<s] \cup \bar{a}[>t]$. By indiscernibility, we have

$$\mathrm{tp}_\Delta(\bar{c}^{vi} / W) \neq \mathrm{tp}_\Delta(\bar{c}^{vk} / W)\,, \quad \text{for } i \neq k\,.$$

For every $s < u < t$, let $\alpha_u : I \to I$ be an order isomorphism such that $\alpha_u(v) = u$ and $\alpha_u(x) = x$, for $x < s$ or $x > t$. Let π_u be a U-automorphism such that $\pi_u(\bar{a}^x) = \bar{a}^{\alpha_u(x)}$, for all $x \in I$. For $s < u < t$, set $\bar{c}^{ui} := \pi_u(\bar{c}^{vi})$ and $d^u := \pi_u(d^v)$.

By Lemma 4.13, all indices in the U-class p are related via $\trianglelefteq_U$. Hence, we can find, for every $i < \kappa^+$ and all $k < n$, a formula $\psi^i_k(x, y, \bar{z})$, a tuple $\bar{e}^i_k \subseteq W$, and a relation $\rho^i_k \in \{=, \leq, \geq\}$ such that

$$\mathfrak{M} \vDash \psi^i_k(c^{ui}_k, d^v, \bar{e}^i_k) \quad \text{iff} \quad u \; \rho^i_k \; v \,.$$

By choice of κ there exists a subset $J \subseteq \kappa^+$ of size $|J| = \kappa^+$ such that $\psi^i_k = \psi^l_k$ and $\rho^i_k = \rho^l_k$, for all $i, l \in J$. We denote this formula by ψ_k and the corresponding relation by ρ_k.

We can use Lemma 2.7 to find an infinite subset $J_0 \subseteq J$, a formula $\varphi \in \Delta$, and parameters $\bar{b}^i \in W^m$, for $i \in J_0$, such that

$$\mathfrak{M} \vDash \varphi(\bar{b}^i, \bar{c}^{vi}) \leftrightarrow \neg\varphi(\bar{b}^i, \bar{c}^{vk}) \,, \quad \text{for } i < k \text{ in } J_0 \,.$$

By Ramsey's theorem, there exists an infinite subset $J_1 \subseteq J_0$ and a relation $\sigma \in \{=, \neq, \leq, >\}$ such that

$$\mathfrak{M} \vDash \varphi(\bar{b}^i, \bar{c}^{vk}) \quad \text{iff} \quad i \; \sigma \; k \,,$$

for $i, k \in J_1$. There is a φ-class $H \subseteq [m + n]$ of the sequence $(\bar{b}^i \bar{c}^{vi})_i$ containing indices j, l with $j < m$ and $m \leq l < m + n$. If we replace in $\bar{b}^i$ every component b^i_l with $l \in [m] \setminus H$ by b^0_l and we replace in $\bar{c}^{vi}$ every component c^{vi}_l with $m + l \in [m + n] \setminus H$ by c^{v0}_l then we obtain two sequences that still satisfy

$$\mathfrak{M} \vDash \varphi(\bar{b}^i, \bar{c}^{vk}) \quad \text{iff} \quad i \; \sigma \; k \,.$$

Therefore, we may assume that there are sequences $(\bar{b}^i)_{i \in J_1}$ and $(\bar{c}^{vi})_{i \in J_1}$ and tuples $\bar{b}_* \subseteq W$ and $\bar{c}^v_* \subseteq \bar{a}^v|_p$ such that

$$\mathfrak{M} \vDash \varphi(\bar{b}^i, \bar{b}_*, \bar{c}^{vk}, \bar{c}^v_*) \quad \text{iff} \quad i \; \sigma \; k$$

and the sequence $(\bar{b}^i \bar{c}^{vi})_i$ has a single φ-class. To show that $\mathfrak{M}$ admits coding we distinguish two cases.

First assume that, for every k, we can choose ψ_k and $\bar{e}^i_k$ such that $\bar{e}^i_k = \bar{e}^l_k$, for all $i, l < \omega$. Then the sequences $(\bar{b}^i)_{i \in J_1}$, $(\bar{c}^{vi})_{v \in I, i \in J_1}$, $(\bar{c}^v_*)_{v \in I}$, and $(d^v)_{v \in I}$, and the tuple $\bar{b}_* \bar{e}^i_0 \dots \bar{e}^i_{n-1}$ satisfy the conditions of Lemma 4.19. Consequently, $\mathfrak{M}$ admits coding.

It remains to consider the case that there is some k such that we cannot choose the $\bar{e}^i_k$ to be equal. Then we can find an infinite subset $J_2 \subseteq J_1$ and a relation $\rho \in \{=, \neq, <, >, \leq, \geq\}$ such that, for all $i, l \in J_2$, we have

$$\mathfrak{M} \vDash \psi_k(c^{vi}_k, d^v, \bar{e}^l_k) \quad \text{iff} \quad i \; \rho \; l \,.$$

The sequences $(c^{vi}_k)_{v \in I, i \in J_2}$, $(\bar{c}^v_*)_{v \in I}$, $(\bar{e}^i_k)_{i \in J_2}$, and $(d^v)_{v \in I}$ satisfy the conditions of Lemma 4.20. Hence, $\mathfrak{M}$ admits coding. □

5 Finite satisfiability

One way to extend the notion of a non-forking type to arbitrary theories consists in considering finitely satisfiable types. Of course, many properties of forking – like symmetry and locality – are lost in this transition. Fortunately, sufficiently many basic properties remain to make the notion useful. Except for a few minor lemmas and changes of presentation all of the definitions and results in this section are taken from [43, 44, 45]. We include some of the proofs for convenience.

Definition 5.1. (a) A type p is *finitely satisfiable* in a set A if, for every finite subset $p_0 \subseteq p$, there exists a tuple $\bar{a} \subseteq A$ satisfying p_0.

(b) Let $\mathfrak{u}$ be an ultrafilter over A^α and let $U \subseteq M$ be a set of parameters. The *average type* of $\mathfrak{u}$ over U is

$$\mathrm{Av}(\mathfrak{u}/U) := \big\{ \varphi(\bar{x}, \bar{c}) \bigm| \bar{c} \subseteq U,\ [\![\varphi(\bar{a}, \bar{c})]\!]_{\bar{a} \in A^\alpha} \in \mathfrak{u} \big\} \,.$$

Example. (a) Suppose that $\mathfrak{M} = (M, E)$ is a structure where E is an equivalence relation with infinitely many classes all of which are infinite. Let $U \subseteq V \subseteq M$ be sets and $a \in M \setminus V$ an element with E-class $[a]$. The type $\mathrm{tp}(a/V)$ is finitely satisfiable in U if and only if

- $[a] \cap V = \varnothing$ and U/E is infinite, or
- $[a] \cap V \neq \varnothing$ and $[a] \cap U$ is infinite.

(b) Let $\mathfrak{M} = (M, <)$ be a dense linear order, $U \subseteq V \subseteq M$ sets, and $a \in M \setminus V$. The type $\mathrm{tp}(a/V)$ is finitely satisfiable in U if and only if, for all $v, v' \in V$ with $v < a < v'$, there is some $u \in U$ with $v < u < v'$.

The connection between average types and types that are finitely satisfiable is given by the following lemma.

Lemma 5.2. (a) *$U \subseteq V$ implies* $\mathrm{Av}(\mathfrak{u}/U) \subseteq \mathrm{Av}(\mathfrak{u}/V)$.

(b) *Let $\mathfrak{u}$ be an ultrafilter over A^α and $U \subseteq M$ a set of parameters. Then $\mathrm{Av}(\mathfrak{u}/U)$ is a complete α-type over U which is finitely satisfiable in A.*

(c) *For every partial α-type p over U which is finitely satisfiable in A, there exists some ultrafilter $\mathfrak{u}$ over A^α such that $p \subseteq \mathrm{Av}(\mathfrak{u}/U)$.*

The next two lemmas summarise the basic properties of finitely satisfiable types that hold without any stability assumption.

Lemma 5.3. (a) *Every α-type p over B which is finitely satisfiable in A can be extended to a complete type $q \in S^\alpha(B)$ which is also finitely satisfiable in A.*

(b) *If $\mathrm{tp}_\Delta(C_0/A \cup B)$ is finitely satisfiable in A and $\mathrm{tp}_\Delta(C_1/A \cup B \cup C_0)$ is finitely satisfiable in $A \cup C_0$ then $\mathrm{tp}_\Delta(C_0 \cup C_1/A \cup B)$ is finitely satisfiable in A.*

According to the preceding lemma the extension and transitivity properties of non-forking types generalise to finitely satisfiable types. In general, finitely satisfiable extensions are not unique. In order to have a unique extension we need the additional requirement that in the set of parameters every type is realised. This is statement (a) of the following lemma in the special case that $B = \varnothing$. Statement (b) contains the dual transitivity property which, the notion of a finitely satisfiable type being non-symmetric, also only holds under additional assumptions.

Lemma 5.4. *Suppose that every type $q \in S^{<\omega}_\Delta(U)$ that is realised in $V \cup A$ is also realised in $V \cup B$.*

(a) *If the types $p_i := \mathrm{tp}_\Delta(B \cup \bar{c}_i/V \cup A)$, for $i < 2$, are finitely satisfiable in U and $\mathrm{tp}(\bar{c}_0/V \cup B) = \mathrm{tp}(\bar{c}_1/V \cup B)$, then $p_0 = p_1$.*

(b) *If $\mathrm{tp}_\Delta(C \cup B/V \cup A)$ and $\mathrm{tp}_\Delta(C/V \cup B)$ are finitely satisfiable in U then so is $\mathrm{tp}_\Delta(C/V \cup A \cup B)$.*

The following theorem is one of the main tools to construct finitely satisfiable types.

Theorem 5.5 (Shelah). *Let $U \subseteq V$ be sets such that every type over U is realised in V. If $\bar{a} \in \mathbb{M}^\alpha$ and $\bar{b} \in \mathbb{M}^\beta$ are tuples such that $\mathrm{tp}(\bar{a}/U)$ is finitely satisfiable in U and $\mathrm{tp}(\bar{b}/V)$ is finitely satisfiable in V then there are $\bar{a}', \bar{b}' \subseteq \mathbb{M}$ such that*

- $\mathrm{tp}_\Delta(\bar{a}'/U) = \mathrm{tp}_\Delta(\bar{a}/U)$,
- $\mathrm{tp}_\Delta(\bar{b}'/V) = \mathrm{tp}_\Delta(\bar{b}/V)$,
- $\mathrm{tp}_\Delta(\bar{a}'/V \cup \bar{b}')$ *is finitely satisfiable in U, and*
- $\mathrm{tp}_\Delta(\bar{b}'/V \cup \bar{a}')$ *is finitely satisfiable in V.*

The main focus of this section is on indiscernible sequences $(\bar{a}^\nu)_\nu$ such that, for every index ν, the type $\mathrm{tp}(\bar{a}^\nu/U \cup \bar{a}[<\nu])$ is finitely satisfiable in U. Such sequences can be thought of as an analogue of Morley sequences in the unstable context. Some of the following results are only implicit in [43] so we include their proofs.

Definition 5.6. Let $U \subseteq V$ be sets. A *fan* over U and V is an indiscernible sequence $(\bar{a}^\nu)_{\nu \in I}$ over V such that, for all $\nu \in I$, the type

$$\mathrm{tp}(\bar{a}^\nu/V \cup \bar{a}[<\nu])$$

is finitely satisfiable in U.

Example. Consider the set $\mathbb{Z} \times \mathbb{R}$ with two binary relations

$$E := \{ (\langle i,x\rangle, \langle i,y\rangle) \mid i \in \mathbb{Z},\ x,y \in \mathbb{R} \},$$
$$< := \{ (\langle i,x\rangle, \langle k,y\rangle) \mid x < y,\ i,k \in \mathbb{Z},\ x,y \in \mathbb{R} \}.$$

Set $U := \mathbb{Z} \times (0,1)$ and $V := \mathbb{Z} \times (-\infty, 1)$. For $\nu \in I := (1,\infty) \subseteq \mathbb{R}$, let $\bar{a}^\nu$ be an enumeration of $\mathbb{Z} \times \{\nu\}$. The sequence $(\bar{a}^\nu)_{\nu \in I}$ is a fan over U/V.

Lemma 5.7 (Shelah [43]). *Let $(\bar{a}^\nu)_{\nu \in I}$ be a sequence of α-tuples and V a set. If there exists an ultrafilter $\mathfrak{u}$ over U^α such that*

$$\mathrm{tp}(\bar{a}^\nu/V \cup \bar{a}[<\nu]) = \mathrm{Av}(\mathfrak{u}/V \cup \bar{a}[<\nu]), \quad \textit{for all } \nu \in I,$$

then $(\bar{a}^\nu)_\nu$ is indiscernible over V.

Proof. We prove by induction on n that

$$\mathrm{tp}(\bar{a}[\bar{s}]/V) = \mathrm{tp}(\bar{a}[\bar{t}]/V),$$

for all strictly increasing sequences $\bar{s}, \bar{t} \in I^n$. Let $\bar{s} = \bar{s}' s_{n-1}$, $\bar{t} = \bar{t}' t_{n-1}$, and $\bar{c} \subseteq V$. By induction hypotheses it follows that

$$\begin{aligned}
& \varphi(\bar{x}_0, \dots, \bar{x}_{n-1}; \bar{c}) \in \mathrm{tp}(\bar{a}[\bar{s}]/V) \\
\text{iff}\quad & \{ \bar{b} \in U^\alpha \mid \mathfrak{M} \models \varphi(\bar{a}[\bar{s}'], \bar{b}; \bar{c}) \} \in \mathfrak{u} \\
\text{iff}\quad & \{ \bar{b} \in U^\alpha \mid \mathfrak{M} \models \varphi(\bar{a}[\bar{t}'], \bar{b}; \bar{c}) \} \in \mathfrak{u} \\
\text{iff}\quad & \varphi(\bar{x}_0, \dots, \bar{x}_{n-1}; \bar{c}) \in \mathrm{tp}(\bar{a}[\bar{t}]/V).
\end{aligned}$$

□

A kind of converse to this lemma is given by the next result.

Lemma 5.8 (Shelah [43]). *Let $(\bar a^v)_{v\in I}$ be an infinite indiscernible sequence of α-tuples. We can find a model $\mathfrak N$ of size $|N| = |\Sigma| + |\alpha| + \aleph_0$, where Σ is the signature in question, such that N is disjoint from $\bar a[I]$ and, for every $v \in I$, the type* $\mathrm{tp}(\bar a^v/N \cup \bar a[<v])$ *is finitely satisfiable in N.*

Proof. Let $J := I \cup \{ u_n \mid n < \omega \}$ be a linear order extending I such that

$$v < \dots < u_n < \dots < u_2 < u_1 < u_0\,, \quad \text{for all } v \in I\,.$$

Extend $(\bar a^v)_{v\in I}$ to an indiscernible sequence $(\bar a^v)_{v\in J}$. Let $\mathfrak M$ be a model containing $(\bar a^v)_{v\in J}$ and let $\mathfrak M^+$ be an expansion of $\mathfrak M$ by Skolem functions. Since $(\bar a^v)_{v\in I}$ is an infinite indiscernible sequence over $N_0 := \bigcup_{n<\omega} \bar a^{u_n}$ we can choose the Skolem functions such that the Skolem hull of N_0 is disjoint from $\bar a[I]$. We claim that this Skolem hull induces the desired model $\mathfrak N$.

To show that $\mathrm{tp}(\bar a^s/N \cup \bar a[<s])$ is finitely satisfiable in N, let us suppose that

$$\mathfrak M^+ \vDash \varphi(\bar a^s, \bar a[\bar v], \bar c)$$

where $v_0 < \dots < v_{n-1} < s$ are indices in I and $\bar c \subseteq N$. Fix Skolem terms $\bar t$ such that $\bar c = \bar t(\bar a^{u_0}, \dots, \bar a^{u_k})$, for some k. Since $(\bar a^v)_{v\in J}$ is indiscernible it follows that

$$\mathfrak M^+ \vDash \varphi\big(\bar a^s, \bar a[\bar v], \bar t(\bar a^{u_0}, \dots, \bar a^{u_k})\big)$$

implies

$$\mathfrak M^+ \vDash \varphi\big(\bar a^{u_{k+1}}, \bar a[\bar v], \bar t(\bar a^{u_0}, \dots, \bar a^{u_k})\big)\,.$$

Since $\bar a^{u_{k+1}} \in N$ we are done. ☐

For every tuple $\bar a$ we can create a fan $(\bar a^v)_v$ containing $\bar a$.

Lemma 5.9 (Shelah [45]). *Let $U \subseteq V$ be sets and suppose that* $\mathrm{tp}(\bar a/U)$ *is finitely satisfiable in U. For every linear order I, there exists a fan $(\bar a^v)_{v\in I}$ over U/V such that* $\mathrm{tp}(\bar a^v/U) = \mathrm{tp}(\bar a/U)$, *for all v.*

Proof. By compactness, it is sufficient to consider the case that $I = \omega$. Let $\mathfrak u$ be the ultrafilter such that $\mathrm{tp}(\bar a/U) = \mathrm{Av}(\mathfrak u/U)$. By induction on n, we choose tuples $\bar a^n$ such that

$$\mathrm{tp}(\bar a^n/V \cup \bar a^0 \dots \bar a^{n-1}) = \mathrm{Av}(\mathfrak u/V \cup \bar a^0 \dots \bar a^{n-1})\,.$$

By Lemma 5.7 it follows that $(\bar a^n)_{n<\omega}$ is a fan over U/V. ☐

The following two observations seem to be new.

Lemma 5.10. *For all disjoint sets $A, U \subseteq \mathbb M$ of size $|U| = \kappa$ and $|A| > 2^{2^\kappa}$, there exists a set U_+ of size $|U_+| = \kappa$ and elements $a, b \in A \setminus U_+$ such that* $\mathrm{tp}(a/U_+ \cup \{b\})$ *is finitely satisfiable in U_+.*

Proof. Fix an enumeration $(a^i)_{i<\lambda}$ of A. By the Theorem of Erdős and Rado we have $(2^{2^\kappa}) \to ((2^\kappa)^+)^2_{2^\kappa}$. Since $\lambda \geq (2^{2^\kappa})^+$ and there are at most 2^κ 2-types over U, we can therefore find a subset $I \subseteq \lambda$ of size $|I| = (2^\kappa)^+$ such that,

$$\mathrm{tp}(a^i a^k/U) = \mathrm{tp}(a^j a^l/U)\,, \quad \text{for all } i < k \text{ and } j < l \text{ in } I\,.$$

Fix indices $s < t$ in I. By compactness there exists an indiscernible sequence $(b^i)_{i<\omega}$ over U such that

$$\mathrm{tp}(b^i b^k/U) = \mathrm{tp}(a^s a^t/U)\,, \quad \text{for all } i < k < \omega\,.$$

Using a suitable U-automorphism we may assume that $b^0 = a^s$ and $b^1 = a^t$. By Lemma 5.8 there exists a set $U_+ \subseteq U$ of size $|U_+| = |U|$ that is disjoint from $b[\omega]$ and such that $\mathrm{tp}(b^1/U_+ \cup \{b^0\})$ is finitely satisfiable in U_+. ☐

Lemma 5.11. *Let $(\bar a^v)_{v\in I}$ be a sequence of α-tuples and $U \subseteq V$ sets such that, for every $v \in I$,*

$$\mathrm{tp}(\bar a^v/V \cup \bar a[<v])$$

is finitely satisfiable in U. If $|I| > 2^{2^{|U^\alpha|}}$ then there exists a subset $J \subseteq I$ of size $|J| = |I|$ such that the subsequence $(\bar a^v)_{v\in J}$ is indiscernible over V.

Proof. By Lemma 5.2 (c), there exist ultrafilters $\mathfrak u_v$, for $v \in I$, such that

$$\mathrm{tp}(\bar a^v/V \cup \bar a[<v]) = \mathrm{Av}(\mathfrak u_v/V \cup \bar a[<v])\,.$$

Since there are only $2^{2^{|U^\alpha|}}$ ultrafilters on U^α it follows that there is a subset $J \subseteq I$ of size $|J| = |I|$ such that $\mathfrak u_u = \mathfrak u_v$, for all $u, v \in J$. By Lemma 5.7 it follows that $(\bar a^v)_{v\in J}$ is indiscernible over V. ☐

An important property of fans $(\bar a^v)_{v\in I}$ over U/V is the fact that, for every tuple $\bar b \subseteq \bar a[I]$, the type $\mathrm{tp}(\bar b/V)$ is determined by the types $\mathrm{tp}(\bar b \cap \bar a^v/V)$, for $v \in I$.

Lemma 5.12 (Shelah [43]). *Let $(\bar{a}^v)_{v \in I}$ be a fan over U/V. Suppose that every type over U is realised in V. Let $\bar{u}, \bar{v} \in I^n$ be finite strictly increasing tuples and $s, t \in I$ indices with $s \leq \bar{u}\bar{v} \leq t$.*

If $\bar{b}^i \subseteq \bar{a}^{u_i}$ and $\bar{c}^i \subseteq \bar{a}^{v_i}$, for $i < n$, are tuples with

$$\mathrm{tp}_\Delta(\bar{b}^i/V) = \mathrm{tp}_\Delta(\bar{c}^i/V) \qquad \textit{for all } i\,,$$

then

$$\begin{aligned}&\mathrm{tp}_\Delta(\bar{b}^0 \dots \bar{b}^{n-1}/V \cup \bar{a}[<s] \cup \bar{a}[>t]) \\ &= \mathrm{tp}_\Delta(\bar{c}^0 \dots \bar{c}^{n-1}/V \cup \bar{a}[<s] \cup \bar{a}[>t])\,.\end{aligned}$$

Proof. First, we prove by induction on k that

$$\mathrm{tp}_\Delta(\bar{b}^0 \dots \bar{b}^{k-1}/V) = \mathrm{tp}_\Delta(\bar{c}^0 \dots \bar{c}^{k-1}/V)\,.$$

By assumption, we have $\mathrm{tp}_\Delta(\bar{b}^0/V) = \mathrm{tp}_\Delta(\bar{c}^0/V)$. Suppose that we have already shown that $\mathrm{tp}_\Delta(\bar{b}^0 \dots \bar{b}^{k-1}/V) = \mathrm{tp}_\Delta(\bar{c}^0 \dots \bar{c}^{k-1}/V)$. By symmetry, we may assume that $v_{k-1} \leq u_{k-1}$. Hence, $u_i, v_i < u_k$, for all $i < k$. Since $\bar{b}^i \subseteq \bar{a}^{u_i}$ and $\bar{c}^i \subseteq \bar{a}^{v_i}$ it follows by indiscernibility that

$$\mathrm{tp}_\Delta(\bar{b}^k \bar{b}^0 \dots \bar{b}^{k-1}/V) = \mathrm{tp}_\Delta(\bar{b}^k \bar{c}^0 \dots \bar{c}^{k-1}/V)\,.$$

Furthermore, by Lemma 5.4 (a), the assumption $\mathrm{tp}_\Delta(\bar{b}^k/V) = \mathrm{tp}_\Delta(\bar{c}^k/V)$ implies that

$$\mathrm{tp}_\Delta(\bar{b}^k/V \cup \bar{c}^0 \dots \bar{c}^{k-1}) = \mathrm{tp}_\Delta(\bar{c}^k/V \cup \bar{c}^0 \dots \bar{c}^{k-1})\,.$$

Combining these two equations we have

$$\mathrm{tp}_\Delta(\bar{b}^0 \dots \bar{b}^{k-1}/V) = \mathrm{tp}_\Delta(\bar{c}^0 \dots \bar{c}^{k-1}/V)\,.$$

Having shown that $\mathrm{tp}_\Delta(\bar{b}^0 \dots \bar{b}^{n-1}/V) = \mathrm{tp}_\Delta(\bar{c}^0 \dots \bar{c}^{n-1}/V)$ we can apply Lemma 5.4 (a) one more time to conclude that

$$\begin{aligned}&\mathrm{tp}_\Delta(\bar{b}^0 \dots \bar{b}^{n-1}/V \cup \bar{a}[<s] \cup \bar{a}[>t]) \\ &= \mathrm{tp}_\Delta(\bar{c}^0 \dots \bar{c}^{n-1}/V \cup \bar{a}[<s] \cup \bar{a}[>t])\,.\end{aligned}$$ □

Corollary 5.13. *Let $(\bar{a}^v)_{v \in I}$ be a fan over U/V. Suppose that every type over U is realised in V. For every partition $I = I_0 + I_1 + I_2$ of I into three segments, we have*

$$\mathrm{ti}^n_{\mathrm{FO}}(\bar{a}[I_1]/V \cup \bar{a}[I_0 \cup I_2]) \leq 2^{|V|+|\Sigma|}.$$

Proof. If $\bar{a}, \bar{b} \subseteq \bar{a}[I_1]$ then $\mathrm{tp}(\bar{a}/V) = \mathrm{tp}(\bar{b}/V)$ implies

$$\mathrm{tp}(\bar{a}/V \cup \bar{a}[I_0 \cup I_2]) = \mathrm{tp}(\bar{b}/V \cup \bar{a}[I_0 \cup I_2])\,.$$

Since there are at most $2^{|V|+|\Sigma|}$ n-types over V the claim follows. □

The next lemma provides the connection between finite satisfiability and the relation $\trianglelefteq_U$ introduced in the previous section.

Lemma 5.14 (Shelah [43]). *Let $(\bar{a}^v)_{v \in I}$ be fan over U/V with $\alpha := |\bar{a}^v|$. Suppose that every type over U is realised in V and let $p, q \subseteq \alpha$ be sets of indices.*

Then $\mathrm{tp}(\bar{a}^v|_p/V \cup \bar{a}^v|_q)$ is finitely satisfiable in U if and only if $p \ntrianglelefteq_V q$.

Proof. ($\Leftarrow$) Suppose that $s < t$ are indices with

$$\mathrm{tp}(\bar{a}^s|_p \bar{a}^s|_q/V) = \mathrm{tp}(\bar{a}^t|_p \bar{a}^s|_q/V)$$

and let $\varphi(\bar{x}, \bar{a}^v|_q) \in \mathrm{tp}(\bar{a}^v|_p/V \cup \bar{a}^v|_q)$. Then $\varphi(\bar{x}, \bar{a}^s|_q) \in \mathrm{tp}(\bar{a}^t|_p/V \cup \bar{a}^s|_q)$. Since this type is finitely satisfiable in U we can find some tuple $\bar{b} \subseteq U$ such that $\mathfrak{M} \vDash \varphi(\bar{b}, \bar{a}^s|_q)$. Hence, $\mathrm{tp}(\bar{a}^s|_q/U) = \mathrm{tp}(\bar{a}^v|_q/U)$ implies that $\mathfrak{M} \vDash \varphi(\bar{b}, \bar{a}^v|_q)$.

($\Rightarrow$) If $\mathrm{tp}(\bar{a}^v|_p/V \cup \bar{a}^v|_q)$ is finitely satisfiable in U then, by indiscernibility, so is $\mathrm{tp}(\bar{a}^s|_p/V \cup \bar{a}^s|_q)$. By definition of a fan $\mathrm{tp}\big(\bar{a}^s|_p \bar{a}^s|_q/V \cup \bar{a}[<s]\big)$ is finitely satisfiable in U. It follows by Lemma 5.4 (b) that so is the type

$$\mathrm{tp}\big(\bar{a}^s|_p \big/ V \cup \bar{a}^s|_q \cup \bar{a}[<s]\big)\,.$$

Since, for $t > s$, $\mathrm{tp}\big(\bar{a}[>t] \big/ V \cup \bar{a}^s|_p \bar{a}^s|_q \cup \bar{a}[<s]\big)$ is also finitely satisfiable in U we can use Lemma 5.4 (b) again to show that so is

$$\mathrm{tp}\big(\bar{a}^s|_p \cup \bar{a}[>t] \big/ V \cup \bar{a}^s|_q \cup \bar{a}[<s]\big)\,.$$

On the other hand, we know that the type $\mathrm{tp}\big(\bar{a}^t|_p \cup \bar{a}[>t] \big/ V \cup \bar{a}^s|_q \cup \bar{a}[<s]\big)$ is finitely satisfiable in U, for all $t > s$. Therefore, Lemma 5.12 implies that

$$\mathrm{tp}\big(\bar{a}^s|_p \cup \bar{a}[>t] \big/ V \cup \bar{a}[<s]\big) = \mathrm{tp}\big(\bar{a}^t|_p \cup \bar{a}[>t] \big/ V \cup \bar{a}[<s]\big)\,.$$

Hence, it follows from Lemma 5.4 (a) that

$$\mathrm{tp}\big(\bar{a}^s|_p \cup \bar{a}[>t] \big/ V \cup \bar{a}^s|_q \cup \bar{a}[<s]\big) = \mathrm{tp}\big(\bar{a}^t|_p \cup \bar{a}[>t] \big/ V \cup \bar{a}^s|_q \cup \bar{a}[<s]\big)\,.$$

Consequently, we have

$$\mathrm{tp}\big(\bar{a}^s|_p \bar{a}^s|_q \big/ V \cup \bar{a}[<s] \cup \bar{a}[>t]\big) = \mathrm{tp}\big(\bar{a}^t|_p \bar{a}^s|_q \big/ V \cup \bar{a}[<s] \cup \bar{a}[>t]\big)\,.$$ □

We use fans as a technical tool to investigate the properties of finitely satisfiable types. The basic idea is as follows. Given some tuple $\bar{a}$ we construct a fan $(\bar{c}^v)_{v\in I}$ over U/V with $\bar{c}^\circ = \bar{a}$. By the preceding lemma, $\mathrm{tp}(\bar{a}|_p/V \cup \bar{a}|_q)$ is finitely satisfiable in U if and only if $p \ntrianglelefteq_V q$. In this way we can apply the results of Section 4 to study finitely satisfiable types.

Definition 5.15. For sets $A, B, U \subseteq \mathbb{M}$, we write

$$A \sqsubseteq_U B \quad :\text{iff} \quad \mathrm{tp}(A/U \cup B) \text{ is not finitely satisfiable in } U\,.$$

Theorem 5.16 (Shelah [43]). *If $\mathfrak{M}$ does not admit coding and $A, B \subseteq \mathbb{M}$, $c \in \mathbb{M}$ then $A \not\sqsubseteq_M B$ implies $A \cup \{c\} \not\sqsubseteq_M B$ or $A \not\sqsubseteq_M B \cup \{c\}$.*

Proof. Fix enumerations $\bar{a}$ of A and $\bar{b}$ of B. Let $\mathfrak{M}_+ \succ \mathfrak{M}$ be an elementary extension such that every type over M is realised in M_+. Since $\mathfrak{M}$ is a model the type $\mathrm{tp}(\bar{b}/M)$ is finitely satisfiable in M. Hence, we can use Lemma 5.3 (a) to choose a tuple $\bar{b}'$ realising $\mathrm{tp}(\bar{b}/M)$ such that $\mathrm{tp}(\bar{b}'/M_+)$ is finitely satisfiable in M. Let $\bar{a}'$ be a tuple such that $\mathrm{tp}(\bar{a}'\bar{b}'/M) = \mathrm{tp}(\bar{a}\bar{b}/M)$. We apply Lemma 5.3 (a) again to choose a tuple $\bar{a}''$ realising $\mathrm{tp}(\bar{a}'/M \cup \bar{b}')$ such that $\mathrm{tp}(\bar{a}''/M_+ \cup \bar{b}')$ is finitely satisfiable in M. By Lemma 5.3 (b), it follows that $\mathrm{tp}(\bar{a}''\bar{b}'/M_+)$ is finitely satisfiable in M. Finally, select an element c' such that $\mathrm{tp}(\bar{a}''\bar{b}'c'/M) = \mathrm{tp}(\bar{a}\bar{b}c/M)$.

Let $(\bar{d}^v)_{v\in I}$ be a fan over M/M_+ with $\bar{d}^\circ = \bar{a}''\bar{b}'c'$. By Lemma 5.14, we have $\bar{a}'' \ntrianglelefteq_M \bar{b}'$. Hence, it follows by Lemma 4.15 that $\bar{a}''c \ntrianglelefteq_M \bar{b}'$ or $\bar{a}'' \ntrianglelefteq_M \bar{b}'c$. By Lemma 5.14, this means that at least one of

$$\mathrm{tp}(\bar{a}''c'/M_+ \cup \bar{b}') \quad \text{and} \quad \mathrm{tp}(\bar{a}''/M_+ \cup \bar{b}'c')$$

is finitely satisfiable in M. Consequently, so is one of

$$\mathrm{tp}(\bar{a}''c'/M \cup \bar{b}') \quad \text{and} \quad \mathrm{tp}(\bar{a}''/M \cup \bar{b}'c')\,.$$

Since $\mathrm{tp}(\bar{a}\bar{b}c/M) = \mathrm{tp}(\bar{a}''\bar{b}'c'/M)$ it follows that one of $\mathrm{tp}(\bar{a}c/M \cup \bar{b})$ and $\mathrm{tp}(\bar{a}/M \cup \bar{b}c)$ is finitely satisfiable in M. $\square$

Lemma 5.17. *$\bar{a} \not\sqsubseteq_U \{b\}$ and $\bar{a}b \not\sqsubseteq_U \bar{c}$ implies $\bar{a} \not\sqsubseteq_U b\bar{c}$.*

Proof. Fix a set $V \supseteq U$ in which every type over U is realised. By Lemma 5.3 (a), we can find a tuple $\bar{a}'$ realising $\mathrm{tp}(\bar{a}/U \cup \{b\})$ such that the type $\mathrm{tp}(\bar{a}'/V \cup \{b\})$ is finitely satisfiable in U. In the same way we obtain a tuple $\bar{a}''b''$ realising $\mathrm{tp}(\bar{a}'b/V)$ such that $\mathrm{tp}(\bar{a}''b''/V \cup \bar{c})$ is finitely satisfiable in U. By Lemma 5.4 (b), it follows that $\mathrm{tp}(\bar{a}''/V \cup b''\bar{c})$ is finitely satisfiable in U. Since $\mathrm{tp}(\bar{a}b\bar{c}/U) \subseteq \mathrm{tp}(\bar{a}''b''\bar{c}/V)$ the result follows. $\square$

Corollary 5.18. *Suppose that $\mathfrak{M}$ does not admit coding.*

(a) *If $\bar{a} \sqsubseteq_M b \sqsubseteq_M \bar{c}$ then $\bar{a} \sqsubseteq_M \bar{c}$.*

(b) *If $\bar{a} \sqsubseteq_M \bar{b}$ then $a_i \sqsubseteq_M \bar{b}$, for some i.*

(c) *If $\bar{a} \sqsubseteq_M \bar{b}$ then $\bar{a} \sqsubseteq_M b_i$, for some i.*

Proof. (a) Suppose that $\bar{a} \not\sqsubseteq_M \bar{c}$. By Theorem 5.16, we have $\bar{a}b \not\sqsubseteq_M \bar{c}$ or $\bar{a} \not\sqsubseteq_M b\bar{c}$. It follows that $b \not\sqsubseteq_M \bar{c}$ or $\bar{a} \not\sqsubseteq_M b$.

(b) W.l.o.g. we may assume that $\bar{a}$ and $\bar{b}$ are finite tuples. We prove the claim by induction on $|\bar{a}|$. Suppose that $\bar{a}c \sqsubseteq_M \bar{b}$. As $\bar{a} \not\sqsubseteq_M \bar{b}c$ and $c \not\sqsubseteq_M \bar{b}$ would imply that $\bar{a}c \not\sqsubseteq_M \bar{b}$ it follows that we have $\bar{a} \sqsubseteq_M \bar{b}c$ or $c \sqsubseteq_M \bar{b}$. In the latter case we are done. Assume that $\bar{a} \sqsubseteq_M \bar{b}c$. Together with $\bar{a}c \sqsubseteq_M \bar{b}$ it follows from Theorem 5.16 that $\bar{a} \sqsubseteq_M \bar{b}$. By induction hypothesis, there is some $a_i \sqsubseteq_M \bar{b}$.

(c) W.l.o.g. we may assume that $\bar{a}$ and $\bar{b}$ are finite tuples. We prove the claim by induction on $|\bar{b}|$. Suppose that $\bar{a} \sqsubseteq_M \bar{b}c$. If $\bar{a} \sqsubseteq_M c$ then we are done. If $\bar{a}c \sqsubseteq_M \bar{b}$ then Theorem 5.16 implies $\bar{a} \sqsubseteq_M \bar{b}$ and, by induction hypothesis, there is some i with $\bar{a} \sqsubseteq_M b_i$. Hence, we may assume that $\bar{a}c \not\sqsubseteq_M \bar{b}$ and $\bar{a} \not\sqsubseteq_M c$. But, by Lemma 5.17, this implies that $\bar{a} \not\sqsubseteq_M \bar{b}c$. Contradiction. $\square$

Corollary 5.19. *If $\mathfrak{M}$ does not admit coding then $\sqsubseteq_M$ forms a preorder on $\mathbb{M} \setminus M$.*

Proof. The reflexivity of $\sqsubseteq_M$ follows immediately form the definition, and we have seen in Corollary 5.18 that it is transitive. $\square$

6 Linear decompositions

In this final section we prove our main result. We show that the partition width of any structure $\mathfrak{M}$ that does not admit coding is bounded by $2^{2^{\aleph_0}}$. If we could improve the bound to a finite partition width then this would solve Seese's conjecture. We will construct the desired partition refinement of $\mathfrak{M}$ inductively from *partial* partition refinements.

Definition 6.1. Let $\mathfrak{M}$ be a structure and $A, C \subseteq M$.

(a) A *partial partition refinement* of A is a system $(U_v)_{v\in T}$ of subsets $U_v \subseteq A$ indexed by a tree $T \subseteq 2^{<\alpha}$ with the following properties:

- $U_{\langle\rangle} = A$,
- $U_v = U_{v0} \cup U_{v1}$, for all $v \in T$ (where we set $U_w := \varnothing$, for $w \notin T$),
- $U_v = \bigcap_{u<v} U_u$ if $|v|$ is a limit ordinal.

(b) Let $(U_v)_{v\in T}$ be a partial partition refinement of A. The *n-width* of $(U_v)_v$ over C is the cardinal

$$w_n((U_v)_v/C) := \sup_{v\in T} \operatorname{eti}^n(U_v/C \cup (A \setminus U_v)) .$$

Lemma 6.2. *Suppose that $\mathfrak{M}$ is a structure with a finite signature that does not admit coding. Let κ be an infinite cardinal and $A \subseteq M$ a set of size $|A| > 2^{2^\kappa}$ such that*

$$\operatorname{ti}^n_\Delta(A/M \setminus A) \leq \kappa , \quad \textit{for all finite sets } \Delta \textit{ and all } n < \omega .$$

There exists a partial partition refinement $(U_v)_{v\in T}$ of A such that

- *$w_n((U_v)_v/M \setminus A) \leq 2^{2^\kappa}$, for all n,*
- *if v is a leaf of T then $U_v \subset A$ and $\operatorname{ti}^n_\Delta(U_v/M \setminus U_v) \leq \aleph_0$, for all finite sets Δ of formulae and every $n < \omega$.*

Proof. Fix an increasing sequence $(\Delta_i)_{i<\omega}$ of finite sets $\Delta_i \subseteq \mathrm{FO}$ with union $\bigcup_{i<\omega} \Delta_i = \mathrm{FO}$. By Lemma 2.6, we can fix sets $C_i \subseteq M \setminus A$, for $i < \omega$, of size $|C_i| = \kappa$ such that, for $\bar a, \bar b \subseteq A$,

$$\operatorname{tp}_{\Delta_i}(\bar a/C_i) = \operatorname{tp}_{\Delta_i}(\bar b/C_i) \quad \text{implies} \quad \operatorname{tp}_{\Delta_i}(\bar a/M \setminus A) = \operatorname{tp}_{\Delta_i}(\bar b/M \setminus A) .$$

Let $C_\omega := \bigcup_{i<\omega} C_i$ and choose a model $C_* \supseteq C_\omega$ of size $|C_*| = \kappa$. It follows that

$$\operatorname{tp}(\bar a/C_*) = \operatorname{tp}(\bar b/C_*) \quad \text{implies} \quad \operatorname{tp}(\bar a/M \setminus A) = \operatorname{tp}(\bar b/M \setminus A) .$$

By Lemma 5.10 we can find a set $C \supseteq C_*$ of size $|C| = \kappa$ and elements $a, b \in A \setminus C$ such that $\operatorname{tp}(a/C \cup \{b\})$ is finitely satisfiable in C. Let $D_0 \supseteq C$ be a set such that every type over C is realised in D_0. We can choose D_0 of size $|D_0| \leq 2^\kappa$. By Lemma 5.3 (a) there is an element a' realising $\operatorname{tp}(a/C \cup \{b\})$ such that $\operatorname{tp}(a'/D_0 \cup \{b\})$ is finitely satisfiable in C. Let π be a $(U \cup \{b\})$-automorphism with $\pi(a') = a$ and set $D := \pi[D_0]$. Then $\operatorname{tp}(a/D \cup \{b\})$ is finitely satisfiable in C and every type over C is realised in D.

Fix an enumeration $\bar a$ of A and an $|A|$-dense linear order I, i.e., a linear order I such that, for all subsets $X < Y$ of I of size $|X|, |Y| < |A|$, there is some element $i \in I$ with $X < i < Y$. We can use Lemma 5.9 to find a fan $(\bar a^v)_{v\in I}$ over C/D with $\operatorname{tp}(\bar a^v/C) = \operatorname{tp}(\bar a/C)$. By applying suitable automorphisms we may assume that $A \subseteq \bar a[I]$ and, for all $v \in I$, the set $A_v := \bar a^v \cap (A \setminus C)$ is either empty or it consists of a single strong C-class. By Corollary 5.13, we have

$$\operatorname{ti}^n(\textstyle\bigcup_{v\in H} A_v/D \cup \bigcup_{v\in I\setminus H} A_v) \leq 2^{|D|} \leq 2^{2^\kappa} ,$$

for every convex subset $H \subseteq I$. Furthermore, the fact that $\operatorname{tp}(a/D \cup \{b\})$ is finitely satisfiable in C implies that $a \in A_u$ and $b \in A_v$, for some $u \neq v$. Hence $A_u \subset A$, for all $v \in I$.

Let $\alpha := |I|^+$ and fix an antichain $J \subseteq 2^{<\alpha}$ such that $\langle I, \leq\rangle \cong \langle J, \leq_{\mathrm{lex}}\rangle$. Let $\eta : I \to J$ be the corresponding bijection and let $T \subseteq 2^{<\alpha}$ be the prefix closure of J. For $v \in T$, we set

$$U_v := \bigcup \{ A_v \mid v \leq \eta(u) \} .$$

Then $(U_v)_{v\in T}$ is a partial partition refinement of A such that

$$\operatorname{ti}^n(U_v/M \setminus U_v) = \operatorname{ti}^n\Big(\bigcup_{u\in H} A_u/C \cup \bigcup_{u\in I\setminus H} A_u\Big) \leq 2^{2^\kappa} ,$$

where $H := \{ u \in I \mid v \leq \eta(u) \}$. Furthermore, if $v \in T$ is a leaf then $v = \eta(u)$, for some $u \in I$, and Theorem 4.21 implies that

$$\operatorname{ti}^n_\Delta(U_v/M \setminus U_v) = \operatorname{ti}^n_\Delta(A_u/M \setminus A_u) \leq \aleph_0 ,$$

for all finite sets Δ of formulae and every $n < \omega$. ☐

Theorem 6.3. *Let $\mathfrak{M}$ be a structure with a finite signature. If $\mathfrak{M}$ does not admit coding then* $\operatorname{pwd}\mathfrak{M} \leq 2^{2^{\aleph_0}}$.

Proof. We construct a partition refinement $(U_v)_v$ of $\mathfrak{M}$ with $\operatorname{pwd}_n(U_v)_v \leq 2^{2^{\aleph_0}}$, for every n. If $|M| \leq 2^{2^{\aleph_0}}$ the claim is trivial. Therefore, we may assume that $|M| > 2^{2^{\aleph_0}}$. By Lemma 6.2, there exists a partial partition refinement $(U_v)_{v\in T_0}$ of M of the desired width. If $v \in T_0$ is a leaf then we have $\operatorname{ti}^n_\Delta(U_v/M \setminus U_v) \leq \aleph_0$, for all finite Δ and n, and we can use the lemma again to find a partial partition refinement of U_v of the desired width. This partial partition refinement can be inserted into the first one. We repeat this procedure until we obtain a partial partition refinement $(U_v)_v$ with $|U_v| \leq 2^{2^{\aleph_0}}$, for all leaves v. Then we can use arbitrary partition refinements of the leaves U_v to complete it to a partition refinement of $\mathfrak{M}$. ☐

In conjunction with Lemma 3.2 it follows that there exists a dichotomy between axiomatisable classes with a bounded partition width and those with an unbounded one.

Corollary 6.4. *Let T be a complete first-order theory over a finite signature. If T has a model $\mathfrak{M}$ with* $\operatorname{pwd}\mathfrak{M} > 2^{2^{\aleph_0}}$ *then* $\operatorname{pwd}\mathfrak{N}$ *is unbounded when $\mathfrak{N}$ ranges over all models of T.*

7 Conclusion

We have shown that there exists a dichotomy between structures with a definable pairing function and structures with small partition width. This can be seen as a weak form of Seese's conjecture. Unfortunately, the bound on the partition width we obtained in rather high.

Open Problem. *Try to improve the bound of Theorem 6.3 to* $\mathrm{pwd}\,\mathfrak{M} \le \aleph_0$.

Note that a lower bound is given by the grid $\mathfrak{G} := \langle \mathbb{Z} \times \mathbb{Z}, E \rangle$ where

$$E = \{ \langle \langle i,k \rangle, \langle j,l \rangle \rangle \mid |i-j| + |k-l| = 1 \}.$$

The graph $\mathfrak{G}$ does not admit coding and its partition width is $\aleph_0$.

This example shows that our methods are not sufficiently strong to prove the original form of Seese's conjecture. Note that in the above example there are no first-order definable pairing functions, but there is an MSO-definable one. Hence, to resolve the conjecture it seems to be necessary to modify the definition of admitting coding to include MSO-definable functions.

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