Technische Universität Darmstadt

Fachbereich Mathematik

Masterthesis

Bisimulation Invariant MSO over Classes of Finite Transition Systems

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Abstract A method identifying certain types of linear orders is developed and utilized to show that the bisimulation invariant fragment of MSO and L_μ are equally expressive with respect to a particular classe of finite models, the so called *lassos*. The results are then generalized to be applicable to broader classes of transition systems.

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Introduction

In the past both modal and temporal logics have proven themselves to be reliable tools for program verification, a discipline with the goal of assuring that programs satisfy certain requirements expected of them – requirements that could not necessarily be validated by means of mere simulation, for example certain behaviours over possibly infinite runtimes.

One of the earlier logics used for these means was *HML* – the so called *Hennessy-Milner-Logic* going back to Hennessy and Milner [8] – a dynamic logic used to specify properties of so called transition systems, structures similar to an automaton, that can be utilized to model programs. However, HML misses a certain level of expressiveness, as explained by Bradfield and Stirling.

> [...] HML is obviously inadequate to express many properties, as it has no means of saying *always in the future* or other temporal connectives – except by allowing infinitary conjunction [4, p.3].

As one can imagine, infinite formulae pose many problems for methods of automated verification or model checking. This problem was solved in the 80s by the computer scientist Dexter Kozen [11], who enriched HML with fixpoint operators. This gave birth to the modal μ-calculus as it is known today, sometimes denoted by L_μ.

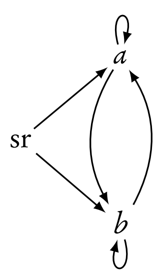
Despite the fact that many other modal logics were introduced, such as linear temporal logic (*LTL*) or computational tree logic (*CTL*), the possibility of recursive assertions made the modal μ-calculus L_μ the most expressive among these languages [1, p.431].

Another ingredient in the analysis of programs is automata theory – the study of abstract machines and automata, as well as the computational problems that can be solved using them – where the methods utilized today mainly go back to Büchi [6] and Rabin [13]. The intrinsic ability of automata to recognize certain patterns turned out to be mostly equivalent to the expressive properties of monadic second order logic, or *MSO*, which is the extension of first order logic enriched by the possibility of quantifying over set variables.

However, MSO is more expressive than L_μ, since MSO has the “built in” ability to check for equality and is thus able to “count” numbers of successors, where L_μ is only cabable of checking wheather a successor of a certain property exists.

Formally, this manifests in the property that the μ-calculus is not capable of distinguishing so called *bisimilar* models.

If seen as black boxes, bisimilar programs are those exhibiting the same behaviour; and thus cannot be distinguished by observation. This essentially just goes back to the fact that the same function can have multiple implementations. As an example, both transition systems



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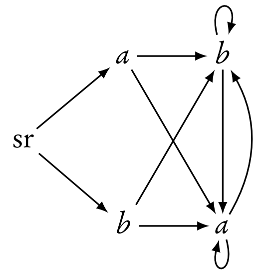


are bisimilar – the paths one can take in either graph are identical. While both of these transition systems would be equivalent for any L_μ-formula, MSO is capable of distinguishing both structures.

The question arose, wheather the modal and temporal logics had some "natural" relation to those long established logics like FO or MSO. For FO an answer was given by van Benthem [14].

> A modal formula can be translated into an equivalent bisimulation invariant first-order logic formula (over transition[systems]) with one free variable. [...] Van Benthem proved the converse: [A] bisimulation invariant first-order logic formula with one free variable is equivalent to a modal formula. Modal logic is the bisimulation invariant fragment of first-order logic. [4, p.23]

A similar result was given for MSO by Janin and Walukiewicz [10], which essentially states that the modal μ-calculus can be identified with the bisimulation invariant fragment of MSO. However, where van Benthems result for FO also holds if the class of models is restricted to finite transition systems, a similar result for MSO is not yet known, since the approaches utilizing automata seem to fail.

Instead of automata, this thesis will use the composition method to prove similar assertions. We will apply known results about the behaviour of MSO-formulae under interpretations and other operations, and then prove equivalence of bisimulation invariant MSO and L_μ on three subclasses of the class of finite transition systems.

In the first chapter the elementary notions of MSO, L_μ and results about bisimulation invariance will be discussed, up to the introduction of a property we will call the the *unravelling property*, which turns out to be equivalent to the equivalence of bisimulation invariant MSO and L_μ over classes of finite transition systems. The second chapter will then introduce the class of lassos and show the equivalence of bisimulation invariant MSO and L_μ on this class via the unravelling property. Utilizing the methods showcased in the second chapter, the third chapter will then apply similar techniques to extend the results to the classes of *hierarchical lassos* and *n-typegraphs*, which will be introduced as well.

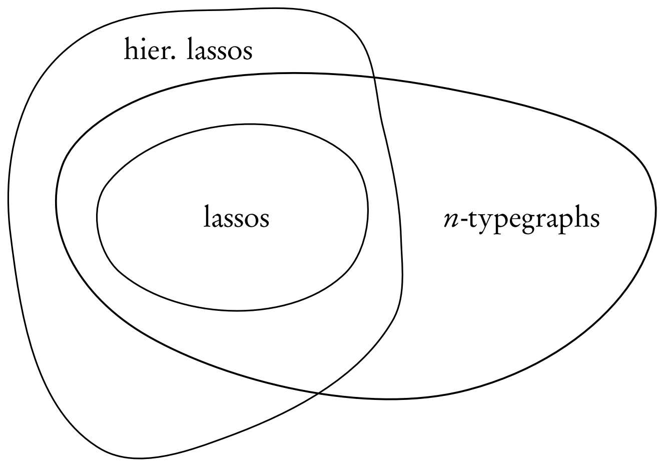


Figure 0.1: The relationship between the classes investigated for $n > 2$.

One should be aware that these classes are related. The class of lassos is a subclass of both n-typegraphs for $n > 2$ and hierarchical lassos. However, not all hierarchical lassos are n-typegraphs and vice versa.

Chapter 1

Context and Preliminaries

1.1 Transition Systems

We will now start by defining the notions introduced in the previous section formally, starting with the notion of transition systems.

Definition 1.1. Let Prop be a set of unary relation symbols. A tuple

$$\mathfrak{M} = \left\langle S^{\mathfrak{M}}, \mathrm{sr}^{\mathfrak{M}}, \left\{\mathrm{succ}^{\mathfrak{M}}\right\}, \left\{p^{\mathfrak{M}}\right\}_{p \in \mathrm{Prop}} \right\rangle$$

is called a *transition system* if $S^{\mathfrak{M}}$ is a nonempty set of states, $\mathrm{sr}^{\mathfrak{M}}$ is a unary relation with $\mathrm{sr}^{\mathfrak{M}} = \{x\}$ for some unique $x \in S^{\mathfrak{M}}$, $\mathrm{succ}^{\mathfrak{M}}$ is a binary relation on $S^{\mathfrak{M}}$ and each $p^{\mathfrak{M}}$ denotes a subset of $S^{\mathfrak{M}}$.

The *set of successors of a state* s is defined as

$$\mathrm{succ}^{\mathfrak{M}}(s) = \left\{s' \in S^{\mathfrak{M}} : (s, s') \in \mathrm{succ}^{\mathfrak{M}}\right\}.$$

If $s, t \in S$ and $(s, t) \in \mathrm{succ}^{\mathfrak{M}}$ then we say that *there is a transition from* s *to* t.

An alternative definition of transition systems, as the one given in [4], utilizes a ternary relation defined on $S^{\mathfrak{M}} \times \mathscr{L} \times S^{\mathfrak{M}}$, where $\mathscr{L}$ is a set of labels, instead of a single binary relation.[1] This amounts essentially to multiple different transition relations. For simplicity, we will assume any transition system to have only a single transition relation, namely succ.

Example 1.2. Any directed graph with a distinguished source and some colouring of the nodes can be seen as a transition system.

In computer science transition systems are utilized to model programs. Closely related to the idea behind the modelling is the notion of bisimilarity, a relation on transition systems, which can be used to describe states that *behave the same*, in the sense that the set of possible transitions is identical.

[1]These structures are often called *labelled transition systems*.

Definition 1.3. Two transition systems $\mathfrak{M}$ and $\mathfrak{N}$ are called *bisimilar* if there is a relation $R \subseteq S^{\mathfrak{M}} \times S^{\mathfrak{N}}$ such that $(\mathrm{sr}^{\mathfrak{M}}, \mathrm{sr}^{\mathfrak{N}}) \in R$ and for every $(s,t) \in R$ and $p \in \mathrm{Prop}$, it holds that

(prop) $s \in p^{\mathfrak{M}}$ if and only if $t \in p^{\mathfrak{N}}$,

(forth) If $(s,s') \in \mathrm{succ}^{\mathfrak{M}}$, then there exists a t' with $(t,t') \in \mathrm{succ}^{\mathfrak{N}}$ and $(s',t') \in R$,

(back) If $(t,t') \in \mathrm{succ}^{\mathfrak{N}}$, then there exists an s' with $(s,s') \in \mathrm{succ}^{\mathfrak{M}}$ and $(s',t') \in R$.

If Prop is a finite set we can assume without loss of generality that the unary predicates $p^{\mathfrak{M}}$ are disjoint, by replacing all possible combinations of assignments of unary relations to nodes with a single new unary relation representing said assignment, i.e., for any $I \subseteq \mathrm{Prop}$

$$p_I(s) := \bigwedge_{p_i \in I} p_i(x) \wedge \bigwedge_{p_i \notin I} \neg p_i(x).$$

We can translate back via

$$p_i(x) \equiv \bigvee_{I:\, p_i \in I} p_I(x).$$

Since by keeping track of the set of relations the process described above can clearly be reversed, we can assume any structure to have unique node colours.

1.2 MSO and Theories of Linear Orders

Now the logic MSO. The definition follows the one given by Blumensath [2]. However, the notation was modified slightly.

Definition 1.4. Let Σ be a signature consisting only of relational symbols R_i with an associated arity. Let t, s be terms build up from first-order variables, let R be an n-ary relation and let X, Y, Z be set-variables. The set of *MSO-formulae with respect to* Σ, or short MSO(Σ) is the smallest set containing all formulae of the form

$$\begin{aligned} &t = s, \\ &Zt, \\ &Rt_0, \ldots, t_{n-1}, \\ &X \subseteq Y, \end{aligned}$$

that is closed under disjunction, conjunction, negation and first- and second-order quantification. The semantics of those formulae is defined as follows.

Let $\mathfrak{S}$ be a relational Σ-structure with universe S, φ be an MSO(Σ) formula, Φ a finite set of MSO(Σ) formulae, Var be the set of variables and $\mathfrak{I}\colon \mathrm{Var} \to \mathscr{P} S$ be an interpretation of the variables. Then the semantics of MSO(Σ) is defined inductively by

$$\begin{aligned}
\mathfrak{S} &\models t_0 = t_1 &&:\Longleftrightarrow && \mathfrak{I}(t_0) = \mathfrak{I}(t_1),\\
\mathfrak{S} &\models R t_0 \dots t_{n-1} &&:\Longleftrightarrow && \left\langle \mathfrak{I}(t_0), \dots, \mathfrak{I}(t_{n-1}) \right\rangle \in R^{\mathfrak{S}},\\
\mathfrak{S} &\models X \subseteq Y &&:\Longleftrightarrow && \mathfrak{S} \models \mathfrak{I}(X) \subseteq \mathfrak{I}(Y),\\
\mathfrak{S} &\models \neg\varphi &&:\Longleftrightarrow && \mathfrak{S} \not\models \varphi,\\
\mathfrak{S} &\models \bigvee_{\varphi\in\Phi} \varphi &&:\Longleftrightarrow && \text{there is some } \varphi \in \Phi \text{ such that } \mathfrak{S} \models \varphi,\\
\mathfrak{S} &\models \bigwedge_{\varphi\in\Phi} \varphi &&:\Longleftrightarrow && \mathfrak{S} \models \varphi \text{ for all } \varphi \in \Phi,\\
\mathfrak{S} &\models \exists R \varphi &&:\Longleftrightarrow && \text{there is some set } R^{\mathfrak{S}}\\
& && && \text{such that } \left\langle \mathfrak{S}, R^{\mathfrak{S}} \right\rangle \models \varphi,
\end{aligned}$$

and

$$\mathfrak{S} \models \forall R \varphi \quad :\Longleftrightarrow \quad \left\langle \mathfrak{S}, R^{\mathfrak{S}} \right\rangle \models \varphi \text{ for all suitable sets } R^{\mathfrak{S}}.$$

MSO as above is defined in a general form. However, since only transition systems are considered, we mostly will have to deal with signatures consisting only of a transition relation, unary predicates and a unary relation sr.

When using MSO it is often convenient to quantify only over some specified subsets. To do so, one can define the relativization of a formula.

Definition 1.5. Let φ be a relational MSO-formula without the variable X. Then we define the *relativization* $\varphi'(X)$ of φ inductively by

$$\begin{aligned}
(x = y)' &:= (x = y), & (\psi_1 \vee \psi_2)' &:= \psi_1' \vee \psi_2',\\
(\mathrm{succ}(x,y))' &:= \mathrm{succ}(x,y), & (\exists Y.\varphi(Y))' &:= \exists Y.(Y \subseteq X \wedge \varphi'(Y)),\\
(\mathrm{sr}(x))' &:= \mathrm{sr}(x), & (\forall Y.\varphi(Y))' &:= \forall Y.(Y \subseteq X \to \varphi'(Y)),\\
(\neg\varphi)' &:= \neg\varphi', & (\exists y.\varphi(y))' &:= \exists y.(Xy \wedge \varphi'(y)),\\
(\psi_1 \wedge \psi_2)' &:= \psi_1' \wedge \psi_2', & (\forall y.\varphi(y))' &:= \forall y.(Xy \to \varphi'(y)),
\end{aligned}$$

such that $\varphi'(X)$ says "*φ holds on the set X*".

One of the main ideas of this thesis is to characterize linear orders by all their properties that can be recognized by an MSO-formula of a certain complexity. For this purpose some definitions need do be stated.

Definition 1.6. Let L be a logic with quantifiers, $\mathfrak{A}$ be a structure and n be a natural number. Then we define the set $\mathrm{Th}^n_L(\mathfrak{A})$ as the set of all L-formulae of quantifier rank of at most n that hold in $\mathfrak{A}$. The quantifier rank of a formula φ will be denoted by $\mathrm{qr}(\varphi)$. For two structures $\mathfrak{A}$ and $\mathfrak{B}$ we write $\mathfrak{A} \equiv_m \mathfrak{B}$ if for every L-formula φ with quantifier rank (at most) m

$$\mathfrak{A} \models \varphi \iff \mathfrak{B} \models \varphi$$

holds.

Remark 1.7. It is a known fact that even though there may be inifinitely many formulae of a certain quantifier rank, there are only finitely many formulae which differ up to logical equivalence. With the use of $\mathrm{Th}^n_L(\mathfrak{A})$ we will almost always refer to such a finite set of representatives.

As mentioned in the introduction we will use the composition method to show equivalence of bisimulation invariant MSO and L_μ on certain classes. To do so, we will give some formal definitions to explain how we can compose mutliple substructures to a desired one.

Definition 1.8. We define the disjoint union of two Σ-structures $\mathfrak{A}$ and $\mathfrak{B}$ as the $(\Sigma \cup \{\chi_{\mathfrak{A}}, \chi_{\mathfrak{B}}\})$-structure $\mathfrak{A} \oplus \mathfrak{B}$ with the universe $A \cup B$ and the relations

$$\begin{aligned} R^{\mathfrak{A}\oplus\mathfrak{B}} &:= R^{\mathfrak{A}} \cup R^{\mathfrak{B}}, \quad \text{for } R \in \Sigma, \\ \chi_{\mathfrak{A}} &:= A, \\ \chi_{\mathfrak{B}} &:= B. \end{aligned}$$

To combine disjoint unions of structures, we will frequently utilize so called MSO-*interpretations* as defined below.

Definition 1.9. Let Σ and Γ be a relational signatures. An MSO-interpretation from Σ to Γ is an operation which transforms Σ-structures to Γ-structures that is denoted by a list

$$(\delta(x), (\varphi_R)_{R\in\Gamma})$$

of MSO-formulae called a *definition scheme*. The interpretation maps a Σ-structure $\mathfrak{A}$ to the Γ-structure $\mathfrak{B} := (\delta^{\mathfrak{A}}, (\varphi_R^{\mathfrak{A}})_{R\in\Gamma})$, where

$$\delta^{\mathfrak{A}} := \{a \in A : \mathfrak{A} \models \delta(a)\}$$

is the universe of $\mathfrak{B}$ and

$$\varphi_R^{\mathfrak{A}} := \{\overline{a} : \mathfrak{A} \models \varphi_R(\overline{a})\} \qquad \text{for } R \in \Sigma$$

are the relations. The *quantifier rank of an interpretation* is given by the maximal quantifier rank of any formula within its definition scheme.

A central operation we will utilize is the linear sum, which is essentially a special case of the generalized sum as defined in [3].

Definition 1.10. Let $\mathfrak{A}$ and $\mathfrak{B}$ be linear orders. We denote by $\mathfrak{A}+\mathfrak{B}$ the linear order for which every element of $\mathfrak{A}$ comes before every element of $\mathfrak{B}$ and the first element of $\mathfrak{B}$ is the successor of the last element of $\mathfrak{A}$ if it exists. Let I be a linear order and let $\{\mathfrak{A}_i\}_{i\in I}$ be a familiy of linear orders. We will write $\sum_{i\in I}\mathfrak{A}_i$ for the linear order where for two elements $x\in\mathfrak{A}_k$ and $y\in\mathfrak{A}_j$ we find $x\leq y$ if $k<j$ or $k=j$ and $x\leq y$. If $\mathfrak{A}$ is a linear order we write $\mathfrak{A}^\omega$ for $\sum_\omega\mathfrak{A}$.

Since we will work with certain kinds of MSO-interpretations, some well known facts about them should be stated.

Proposition 1.11. *Let $\mathfrak{A}$ be a Σ-structure and τ be an interpretation of quantifier rank m. For every formula $\varphi\in\mathrm{MSO}$ with quantifier rank n there exists a formula φ^τ with quantifier rank $n+m$ such that*

$$\tau(\mathfrak{A})\models\varphi \iff \mathfrak{A}\models\varphi^\tau.$$

Corollary 1.12. *Let τ be an interpretation from Σ to Γ with quantifier rank m. For two Σ-structures $\mathfrak{A}$ and $\mathfrak{B}$ we find*

$$\mathfrak{A}\equiv_{k+m}\mathfrak{B} \implies \tau(\mathfrak{A})\equiv_k\tau(\mathfrak{B}).$$

Proposition 1.13. *If $\mathfrak{A}_1\equiv_m\mathfrak{A}_2$ and $\mathfrak{B}_1\equiv_m\mathfrak{B}_2$ hold, then*

$$\mathfrak{A}_1\oplus\mathfrak{B}_1\equiv_m\mathfrak{A}_2\oplus\mathfrak{B}_2$$

holds as well.

Proofs of the following results are taken from [3] or can be found there.[2]

Lemma 1.14. *Let I be a linear order and let $(\mathfrak{A}_i)_{i\in I}$ and $(\mathfrak{B}_i)_{i\in I}$ be two families of linear orders. If*

$$\mathfrak{A}_i\equiv_n\mathfrak{B}_i$$

holds for all $i\in I$, then

$$\sum_{i\in I}\mathfrak{A}_i\equiv_n\sum_{i\in I}\mathfrak{B}_i$$

holds as well.

[2]Blumensath proved these results utilizing a variant of MSO called MSO_0, which is essentially as expressive as the original.

With some facts about interpretations in mind we can now formulate what is meant by saying "*theories of linear orders*".

Corollary 1.15. *Let Σ be a signature consisting of $\{\leq\}$ and a finite set of colours* Prop. *Then the set of n-theories of linear orders*

$$\Theta_n(m) := \mathscr{P}\{\varphi\colon \varphi \in \mathrm{MSO}^n(\Sigma),\ qr(\varphi) \leq n, \textit{there ex. linear order } \mathfrak{A} \textit{ with } \mathfrak{A} \models \varphi\}$$

can be equipped with two operations $\cdot$ and $^\omega$ such that

$$\mathrm{Th}^n_{\mathrm{MSO}}(\mathfrak{A} + \mathfrak{B}) = \mathrm{Th}^n_{\mathrm{MSO}}(\mathfrak{A}) \cdot \mathrm{Th}^n_{\mathrm{MSO}}(\mathfrak{B})$$

and

$$\mathrm{Th}^n_{\mathrm{MSO}}\left(\sum_{i<\omega} \mathfrak{A}\right) = \mathrm{Th}^n_{\mathrm{MSO}}(\mathfrak{A})^\omega.$$

1.3 The μ-Calculus

Within this thesis mostly MSO will be utilized, however, since we will establish results concerning the μ-calculus, some familiarity with its definition seems appropriate. The definition given below follows the one stated by Bradfield and Stirling [4].

Definition 1.16. Let Prop be a set of propositions and Var be a set of variables. Then the *set of L_μ-formulae* is defined to be the smallest set closed under disjunction, conjunction and negation such that

- for all $P \in \mathrm{Prop}$ it holds that P is a formula,
- for all $Z \in \mathrm{Var}$ it holds that Z is a formula,
- if φ is a formula so is $\Diamond\varphi$,
- if φ is a formula where X occurs only under an even number of negations, then $\mu X.\varphi(X)$ is a formula.

Then one defines

$$\Box\varphi := \neg\Diamond\neg\varphi$$

as well as

$$\nu X.\varphi(X) := \neg\mu X.\neg\varphi(X).$$

The semantics of L_μ are defined as follows. As *L_μ-structures* only transition systems are considered. Given a structure $\mathfrak{S}$ and an interpretation $\mathfrak{I}\colon \mathrm{Var} \to \mathscr{P}S$ of the variables, one defines the set $||\varphi||_{\mathfrak{I}}^{\mathfrak{S}}$ of states satisfying the formula φ by induction via

$$\begin{aligned}
||P||_{\mathfrak{I}}^{\mathfrak{S}} &:= P^{\mathfrak{S}},\\
||Z||_{\mathfrak{I}}^{\mathfrak{S}} &:= \mathfrak{I}(Z),\\
||\neg\varphi||_{\mathfrak{I}}^{\mathfrak{S}} &:= S^{\mathfrak{S}} \setminus ||\varphi||_{\mathfrak{I}}^{\mathfrak{S}},\\
||\varphi_1 \wedge \varphi_2||_{\mathfrak{I}}^{\mathfrak{S}} &:= ||\varphi_1||_{\mathfrak{I}}^{\mathfrak{S}} \cap ||\varphi_2||_{\mathfrak{I}}^{\mathfrak{S}},\\
||\Diamond\varphi||_{\mathfrak{I}}^{\mathfrak{S}} &:= \left\{s : \exists t.\langle s,t\rangle \in \mathrm{succ}^{\mathfrak{S}} \wedge t \in ||\varphi||_{\mathfrak{I}}^{\mathfrak{S}}\right\},
\end{aligned}$$

as well as

$$||\mu X.\varphi(X)||_{\mathfrak{I}}^{\mathfrak{S}} := \bigcap \left\{s \subseteq S^{\mathfrak{S}} : s \supset ||\varphi||_{\mathfrak{I}[X:=s]}^{\mathfrak{S}}\right\}.$$

Consequently, the definitions of

$$\begin{aligned}
||\varphi_1 \vee \varphi_2||_{\mathfrak{I}}^{\mathfrak{S}} &:= ||\varphi_1||_{\mathfrak{I}}^{\mathfrak{S}} \cup ||\varphi_2||_{\mathfrak{I}}^{\mathfrak{S}},\\
||\Box\varphi||_{\mathfrak{I}}^{\mathfrak{S}} &:= \left\{s : \forall t.\langle s,t\rangle \in \mathrm{succ}^{\mathfrak{S}} \implies t \in ||\varphi||_{\mathfrak{I}}^{\mathfrak{S}}\right\}
\end{aligned}$$

and

$$||\nu Z.\varphi(Z)||_{\mathfrak{I}}^{\mathfrak{S}} := \bigcup \left\{s \subseteq S^{\mathfrak{S}} : s \subseteq ||\varphi||_{\mathfrak{I}[Z:=s]}^{\mathfrak{S}}\right\}$$

follow. For a sentence φ we now define

$$\mathfrak{S}, s \models \varphi :\iff s \in ||\varphi||_{\mathfrak{I}}^{\mathfrak{S}}.$$

Whereas the intuitive reading of an MSO-sentence comes with ease, since MSO is ultimately an extension of FO, the meaning of an L_μ-sentence is harder to grasp due to the fact that it utilizes assertions about fixed points. To get an understanding of the ν and μ operators, it helps to view μ as finite iteration, while ν can be interpreted as infinite iteration. The following canonical examples are taken from [4, p.9f].

Example 1.17. The formula

$$\mu Z.(P \vee \Box Z)$$

means as much as "*on all infinite length paths, P eventually holds*". A more strict form of the assertion above would be

$$\mu Z.(Q \vee (P \wedge \Diamond Z))$$

which would mean "*on some path, P holds until Q holds, and Q eventually holds*". An example of the ν operator would be the formula

$$\nu Z.(P \wedge \Box Z),$$

meaning "*P holds along every path*".

1.4 Bisimulation Invariance & the Unravelling Property

We earlier introduced the notion of bisimilarity of transition systems. Now, we will formalize what it means for a formula to be invariant with respect to bisimilar transition systems.

Definition 1.18. A formula φ with at most one free variable is called *bisimulation invariant* if

$$\mathfrak{M}, \mathrm{sr}^{\mathfrak{M}} \models \varphi \iff \mathfrak{N}, \mathrm{sr}^{\mathfrak{N}} \models \varphi$$

holds for any bisimilar $\mathfrak{N}$ and $\mathfrak{M}$.

If a formula φ cannot distinguish between bisimilar structures in some class $\mathscr{C}$, we say that φ is *bisimulation invariant over* $\mathscr{C}$.

With this definition in mind we can state the following, well known result about L_μ.

Lemma 1.19. *Every L_μ-formula is bisimulation invariant.*

If one wants to be precise about statements concerning bisimulation invariance, notation has a tendency to become bulky. For purposes of readability we will thus define the following expression.

Notation 1.20. We say that *"Over the class $\mathscr{C}$ bisimulation invariant* MSO *coincides with L_μ"* if every MSO-formula which is bisimulation invariant over $\mathscr{C}$ is equivalent to an L_μ-formula over $\mathscr{C}$.

Another central notion when dealing with transition systems (or directed graphs in a general sense) is the notion of the unravelling of said transition system.

Definition 1.21. Let $\mathfrak{N}$ be a transition system. The *unravelling* $\Omega(\mathfrak{N})$ *of* $\mathfrak{N}$ is the tree utilizing the same colouring such that

- The source of the transition system coressponds to the root of the tree and is coloured the same way.

- If a_{tr} is a node in the tree coressponding to a state a_{sys} in the transition system, then for every successor b_{sys} of a_{sys} there is a unique child b_{tr} of a_{tr} having the same colour as b_{sys}.

Intuitively this can be viewed as the tree where every branch is one of the different possible paths contained in the transition system that start with the source. With respect to bisimulation invariance the unravelling of a transition system has some useful properties.

Lemma 1.22. *Let $\mathfrak{M}$ be a transition system. Then $\mathfrak{M}$ is bisimilar to $\Omega(\mathfrak{M})$.*

Proof. Consider the canonical projection $h : \Omega(\mathfrak{M}) \to \mathfrak{M}$ which maps any element of the unravlling of $\mathfrak{M}$ to its preimage. Then one can easily check that the set $\{(m, h(m))\}_{m \in \Omega(\mathfrak{M})}$ is a bisimulation relation, since the colouring of states m and $h(m)$ coincides and by the definition of the unravelling the back and forth property are fulfilled as well. ■

By symmetry of the bisimilarity relation we get the following corollary.

Corollary 1.23. *If the unravelling of two transition systems is the same, these transition systems are bisimilar.*

As mentionend in the introduction, Janin and Walukiewicz [10] have shown that there is a somewhat "natural" relation between MSO and L_μ.

Theorem 1.24 (Janin and Walukiewicz)**.** *The μ-calculus is expressively equivalent to the bisimulation invariant fragment of* MSO*: An* MSO *formula $\varphi(x)$ is bisimulation invariant if and only if it is equivalent to a μ-calculus formula.*

This theorem has been proven by translating MSO formulae into automata and automata into L_μ-formulae. The assertion then follows via the well known relationship of MSO to automata and the following theorem.

Theorem 1.25. *Over the class of trees bisimulation invariant* MSO *coincides with L_μ.*

Hirsch [9, p.144] provided another assertion about the bisimulation invariant fragment of MSO.

Theorem 1.26 (Hirsch)**.** *Over the class of regular trees bisimulation invariant* MSO *coincides with L_μ.*

The previous theorem is of a particular interest because regular trees are exactly the unravellings of finite transition systems.

A similar result, which can be found in [5], makes clear that, for some logic to be able to distinguish bisimilar models, the capability of counting successors of the same colour plays an important role.

Theorem 1.27. *Over the class of deterministic trees bisimulation invariant* MSO *coincides with L_μ.*

Since linear orders are by default deterministic transition systems we can make the following conclusion about the expressiveness of L_μ and MSO with respect to linear orders of a certain type.

Corollary 1.28. *For any* MSO*-formula φ we can find an L_μ-formula ψ such that*

$$\mathfrak{A} \models \varphi \iff \mathfrak{A} \models \psi$$

holds for every infinite path $\mathfrak{A} = \langle \omega, (p_i)_{i \in I} \rangle$.

As mentioned before, the μ-calculus can not distinguish between bisimilar models.

Corollary 1.29. *For any formula φ of the μ-calculus we find*

$$\mathfrak{M} \models \varphi \iff \Omega(\mathfrak{M}) \models \varphi.$$

We will now give the most central definition of this thesis, a property relating the definability of a class of transition systems to its definability on a selected bisimilar representative, its unravelling.

Definition 1.30 (Unravelling Property)**.** Let $\mathscr{C}$ be a class of transition systems. We say that *$\mathscr{C}$ has the Unravelling Property* if, for every $\varphi \in$ MSO that is bisimulation invariant over $\mathscr{C}$, there exists a $\hat{\varphi} \in$ MSO that is bisimulation invariant over trees such that

$$\mathfrak{C} \models \varphi \iff \Omega(\mathfrak{C}) \models \hat{\varphi} \quad \text{for all } \mathfrak{C} \in \mathscr{C}.$$

Since unravellings of transition systems are trees, the property that the formulae we construct are bisimulation invariant over trees is essential, as will become clear with the subsequent theorem, to which most of our previous work leads up to.

The theorem essentially states that if we have some class $\mathscr{C}$ with the Unravelling Property, then on any class inbetween $\mathscr{C}$ and its closure under finite bisimilar models $\mathscr{C}^+$, i.e. any $\mathscr{C}'$ with $\mathscr{C} \subseteq \mathscr{C}' \subseteq \mathscr{C}^+$, including $\mathscr{C}$ and the closure themselves, bisimulation invariant MSO with respect to $\mathscr{C}'$ and L_μ will coincide.

Theorem 1.31 (Unravelling Theorem)**.** *Let $\mathscr{C}$ be a class of finite transition systems which has the Unravelling Property, let $\mathscr{C}^+$ be the class of finite transition systems bisimilar to one in $\mathscr{C}$ and let $\mathscr{C}'$ be some class with $\mathscr{C} \subseteq \mathscr{C}' \subseteq \mathscr{C}^+$.*

Then over $\mathscr{C}'$ bisimulation invariant MSO *coincides with L_μ.*

Proof. Let φ be an MSO-formula which is bisimulation invariant over $\mathscr{C}'$. By assumption, we find a formula $\hat{\varphi}$ that is bisimulation invariant over trees such that

$$\mathfrak{C} \models \varphi \iff \Omega(\mathfrak{C}) \models \hat{\varphi} \quad \text{for all } \mathfrak{C} \in \mathscr{C}.$$

By Theorem 1.25 we can find an $\psi \in L_\mu$ that is equivalent to $\hat{\varphi}$ over trees. Then ψ is equivalent to φ over $\mathscr{C}'$, since given some $\mathfrak{B} \in \mathscr{C}'$ we find a $\mathfrak{C} \in \mathscr{C}$ that is bisimilar to $\mathfrak{B}$ and

$$\begin{aligned} \mathfrak{B} \models \psi &\iff \mathfrak{C} \models \psi \\ &\iff \Omega(\mathfrak{C}) \models \psi \\ &\iff \Omega(\mathfrak{C}) \models \hat{\varphi} \\ &\iff \mathfrak{C} \models \varphi \\ &\iff \mathfrak{B} \models \varphi. \end{aligned}$$

■

A natural question to ask is wheather the reverse direction of the Unravelling Theorem holds as well, which will be answered in the following corollary.

Corollary 1.32. *Any class of transition systems has the Unravelling Property if and only if over said class bisimulation invariant* MSO *coincides with L_μ.*

Proof. The direction "⇒" follows from the Unravelling Theorem. For the reverse direction let $\mathscr{C}$ be a class of transition systems over which L_μ and bisimulation invariant MSO coincide.

Then for any MSO-formula φ we find an L_μ-formula ψ such that for all $\mathfrak{M} \in \mathscr{C}$ the equivalence

$$\mathfrak{M} \models \varphi \iff \mathfrak{M} \models \psi$$

holds. Since any μ-calculus formula is bisimulation invariant over all transition systems it follows that

$$\mathfrak{M} \models \psi \iff \Omega(\mathfrak{M}) \models \psi$$

holds as well. Since MSO is in general more expressive as L_μ we can now obtain an MSO-formula $\hat{\varphi}$ via translation of ψ such that

$$\mathfrak{T} \models \psi \iff \mathfrak{T} \models \hat{\varphi}, \qquad \text{for all trees } \mathfrak{T}$$

holds. We thus find

$$\mathfrak{M} \models \varphi \iff \Omega(\mathfrak{M}) \models \hat{\varphi} \qquad \text{for all } \mathfrak{M} \in \mathscr{C}.$$

Note however that φ will in general not be the same formula as $\hat{\varphi}$, since ψ is equivalent to an L_μ-formula and thus bisimulation invariant not only over $\mathscr{C}$. ■

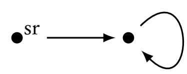
Chapter 2

About Types and Lassos

We are now interested in bisimulation invariant MSO with respect to a specific class of finite transition systems, called lassos.

Definition 2.1. We define a *lasso* to be a finite, connected transition system in which every node has a unique successor, whose source has no predecessor and in which there exists a path from the source to every node.

By the finiteness property and the existence of a successor for every node it follows that every lasso must contain a circle, and by the uniqueness of the successor it follows that it cannot contain more than one, meaning



would be a simple example of lassos.

Observation 2.2. Because any lasso $\mathfrak{M}$ must contain a circle $\Omega(\mathfrak{M})$ will be of the form ab^{ω}.

Each lasso is build up from two linear orders via an interpretation, which will be formalized in the following definition.

Definition 2.3. Let

$$\begin{aligned} \mathrm{end}(x) &:= \neg\exists y.\, \mathrm{succ}(x,y) \\ \mathrm{start}(x) &:= \neg\exists y.\, \mathrm{succ}(y,x) \wedge \chi_{\mathfrak{B}}(x). \end{aligned}$$

The *lasso interpretation* T over the relational signature $\mathrm{Prop} \cup \{\mathrm{sr}\}$ is the interpretation

$$(\delta(x), (\varphi_P)_{P \in \mathrm{Prop}}, \varphi_{\mathrm{sr}})$$

with quantifier rank one, where

$$\begin{aligned}
\delta(x) &:= \mathit{true}, \\
\varphi_P(x) &:= P(x), \text{ for } P \in \mathrm{Prop} \\
\varphi_{\mathrm{succ}}(x,y) &:= \mathrm{succ}(x,y) \vee (\mathrm{start}(y) \wedge \mathrm{end}(x)) \\
\varphi_{\mathrm{sr}}(x) &:= \mathrm{sr}(x).
\end{aligned}$$

Since any lasso $\mathfrak{M}$ can be built up from two paths by the lasso interpretation we can define the *type* of a lasso as the theories of the two linear orders.

Definition 2.4. The *m-type* of a lasso $\mathfrak{M} = T(\mathfrak{A} \oplus \mathfrak{B})$ is defined as the tuple (σ, τ) of $\mathrm{MSO}_m(\mathrm{Prop} \cup \{\mathrm{sr}\})$-theories τ and σ such that $\mathrm{Th}_m(\mathfrak{A}) = \sigma$ and $\mathrm{Th}_m(\mathfrak{B}) = \tau$. We denote the m-type of a lasso $\mathfrak{M}$ with $\mathrm{type}_m(\mathfrak{M})$.

Indeed, by a composition argument we find that the theory of a lasso depends only on the two linear orders utilized, i.e. the theories of the "tail" and the "loop" as will become clear with the following lemma.

Lemma 2.5. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be lassos. Then we find*

$$\mathrm{type}_{m+1}(\mathfrak{M}) = \mathrm{type}_{m+1}(\mathfrak{N}) \implies \mathrm{Th}_m(\mathfrak{M}) = \mathrm{Th}_m(\mathfrak{N}).$$

Proof. Since the lassos have the same type there exist linear orders $\mathfrak{A} \equiv_m \mathfrak{C}$ and $\mathfrak{B} \equiv_m \mathfrak{D}$ with $\mathfrak{M} = T(\mathfrak{A} \oplus \mathfrak{B})$ and $\mathfrak{N} = T(\mathfrak{C} \oplus \mathfrak{D})$. Then the equivalences

$$\begin{aligned}
& \varphi \in \mathrm{Th}_m(\mathfrak{M}) \\
\iff & \varphi \in \mathrm{Th}_m(T(\mathfrak{A} \oplus \mathfrak{B})) \\
\overset{*)}{\iff} & \varphi^\tau \in \mathrm{Th}_{m+1}(\mathfrak{A} \oplus \mathfrak{B}) \\
\overset{**)}{\iff} & \varphi^\tau \in \mathrm{Th}_{m+1}(\mathfrak{C} \oplus \mathfrak{D}) \\
\overset{*)}{\iff} & \varphi \in \mathrm{Th}_m(T(\mathfrak{C} \oplus \mathfrak{D})) \\
\iff & \varphi \in \mathrm{Th}_m(\mathfrak{N})
\end{aligned}$$

hold, where $**)$ follows since the theories coincide and $*)$ holds by Proposition 1.11. ■

2.1 Bisimilar Lassos

If one considers linear orders, equality of n-theories does not neccessarily imply that the linear orders are identical. This means that if we consider two lassos, one with type $(\sigma\tau, \rho\tau)$ and one with type $(\sigma, \tau\rho)$, we cannot make any assertions about wheather the unravellings of both lassos are identical.

However, we need some assurances about the existence of bisimilar lassos, i.e. if one has a lasso of type $(\sigma\tau, \rho\tau)$ we need to be sure about the existence of some lasso that is bisimilar to it and has type $(\sigma, \tau\rho)$.

Results of this kind will be established in this section, where we will not compare lassos directly, but make assertions linked to partitions of the unravelling, so called *factorizations*.

Definition 2.6. Let $\mathfrak{M}$ be an infinite path and let $\xi(a_1, a_2)$ denote the interval of $\mathfrak{M}$ from a_1 to the predecessor of a_2.

We say that $k_0 < k_1 < \dots$ *is a factorization of type* (σ, τ) *of* $\mathfrak{M}$ if there exist linear orders $\mathfrak{A} = \xi(0, k_0)$ and $\mathfrak{B}_i = \xi(k_{i-1}, k_i)$ for $0 < i < \omega$ with $\mathrm{Th}_m(\mathfrak{A}) = \sigma$ and $\mathrm{Th}_m(\mathfrak{B}_i) = \tau$ such that $\mathfrak{M} = \mathfrak{A} + \sum_{i<\omega} \mathfrak{B}_i$.

Lemma 2.7. *Let $\mathfrak{M}$ be a lasso such that $\Omega(\mathfrak{M})$ has a factorization of m-type (σ, τ) with $\sigma\tau = \sigma$ and $\tau^2 = \tau$. Then there exists a lasso $\mathfrak{N}$ of type (σ, τ) that is bisimilar to $\mathfrak{M}$.*

Proof. Let $\mathfrak{M}$ be of the form $T(\mathfrak{A} \oplus \mathfrak{B})$. Let $k_0 < k_1 < \dots$ be a factorization of $\Omega(\mathfrak{M})$ of type (σ, τ), i.e. $\mathrm{Th}_m(\xi(0, k_0)) = \sigma$ and $\mathrm{Th}_m(\xi(k_i, k_{i+1})) = \tau$ for $i \geq 0$.

Let $h : \Omega(\mathfrak{M}) \to \mathfrak{M}$ be the canonical homomorphism mapping an element of the unravelling to the corresponding vertex of $\mathfrak{M}$.

Since $\mathfrak{M}$ is finite there exists an infinite set $I \subseteq \omega$ such that $h(k_i) = h(k_j)$ for all $i, j \in I$. Let $w_0 < w_1 < \dots$ be an enumeration of I.

Then there exists a decomposition $\mathfrak{B} = \mathfrak{C} + \mathfrak{D}$ and numbers $n_i < \omega$ such that

$$\begin{aligned} \xi(0, w_o) &= \mathfrak{A} + \mathfrak{B}^{n_0} + \mathfrak{C} \\ \xi(w_{i-1}, w_i) &= \mathfrak{D} + \mathfrak{B}^{n_i} + \mathfrak{C} \qquad \text{for } i > 0. \end{aligned}$$

Choosing a suitable infinite subset of I we can assume that $n_i \geq 1$. By the assumptions that $\sigma\tau = \sigma$ and $\tau^2 = \tau$ it follows that $\mathfrak{B}^n \equiv_m \mathfrak{B}$. Hence we know that $\mathrm{Th}_m(\mathfrak{A} + \mathfrak{B} + \mathfrak{C}) = \sigma$ and $\mathrm{Th}_m(\mathfrak{D} + \mathfrak{B} + \mathfrak{C}) = \tau$.

We can now define $\mathfrak{N} := T((\mathfrak{A} + \mathfrak{B} + \mathfrak{C}) \oplus (\mathfrak{D} + \mathfrak{B} + \mathfrak{C}))$, meaning

$$\Omega(\mathfrak{N}) = \mathfrak{A} + \mathfrak{B}^\omega = \Omega(\mathfrak{M})$$

which shows that $\mathfrak{N}$ and $\mathfrak{M}$ are bisimilar. ■

Since n-theories of linear orders form a finite semigroup, the following result will be useful.

Lemma 2.8. *Any lasso is bisimilar to a lasso with type (σ, τ) such that $\sigma\tau = \sigma$ and $\tau^2 = \tau$.*

Proof. Let $\mathfrak{M} = T(\mathfrak{A} \oplus \mathfrak{B})$ be a lasso of m-type (σ', τ'). Since k-theories of linear orders form a finite semigroup each theory has an idempotent power. We choose an $n < \omega$ such that $\tau'^n \cdot \tau'^n = \tau'^n$ holds, i.e. we find $\mathrm{Th}_m(\mathfrak{B}^n + \mathfrak{B}^n) = \mathrm{Th}_m(\mathfrak{B}^n) = \tau'^n$. Now define $\sigma = \sigma' \tau'^n$ and $\tau = \tau'^n$.

Obviously we find $\sigma\tau = \sigma'\tau'^n\tau'^n = \sigma'\tau'^n = \sigma$ and $\tau^2 = \tau'^n\tau'^n = \tau$, which means that the lasso $\mathfrak{N} = T((\mathfrak{A} + \mathfrak{B}^n) \oplus \mathfrak{B}^n)$ fulfills the assertion since

$$\Omega(\mathfrak{M}) = \mathfrak{A} + \mathfrak{B}^\omega = \Omega(\mathfrak{N}). \qquad \blacksquare$$

2.2 Identifying Types of Lassos

Our goal is now to express that some linear order has a factorization of a certain type via an MSO-formula.

Definition 2.9. Let s and t be states of a transition system. Let $\mathrm{conn}(X)$ hold if X is connected. We define the *reachability relation* $\succ$ as

$$s \succ t \iff \text{there exists a path from } s \text{ to } t.$$

Clearly the reachability relation is MSO-definable. Via relativization of $\succ$ one can then define a formula

$$\mathrm{path}(X) := \forall t. \forall s. [(Xt \wedge Xs) \rightarrow (t \succ s \vee s \succ t)] \wedge \mathrm{conn}(X).$$

Lemma 2.10. *For every lasso-type (σ, τ) there exists an MSO-formula $\psi_{\sigma,\tau}$ that is bisimulation invariant over trees such that $\Omega(\mathfrak{M}) \models \psi_{\sigma,\tau}$ holds if and only if $\Omega(\mathfrak{M})$ has a factorization of type (σ, τ).*

Proof. We will construct $\psi_{\sigma,\tau}$ explicitly. First we define three formulae needed to partition $\Omega(\mathfrak{M})$ and will discuss their correctness. The first is

$$\begin{aligned} \mathrm{srTo}(X, Y) := & \forall x. [Xx \rightarrow \mathrm{sr}(x) \vee \exists y. (Xy \wedge \mathrm{succ}(y, x))] \\ & \wedge \forall y. [Yy \rightarrow \neg Xy] \\ & \wedge \exists x. [Yx \wedge (\mathrm{sr}(x) \vee \exists y. (\mathrm{succ}(y, x) \wedge Xy))] \\ & \wedge \mathrm{path}(X). \end{aligned}$$

The formula holds if and only if X is a simple path from the source to the first element of Y that occurs.

The second formula is

$$\begin{aligned} \mathrm{fromTo}(Z, Y) := & \exists x. (Zx \wedge Yx \wedge \neg \exists y. (\mathrm{succ}(y, x) \wedge Zy)) \\ & \wedge \forall x \forall y [(Zx \wedge \mathrm{succ}(x, y)) \rightarrow ((Zy \wedge \neg Yy) \vee (Yy \wedge \neg Zy))] \\ & \wedge \mathrm{path}(Z) \end{aligned}$$

and expresses that Z is a connected subset starting with an element of Y which ends with the predecessor of the next element in Y.

The third formula is

$$\begin{aligned}\inf(Y) :=& \forall X.[\exists x.Xx \wedge \mathrm{path}(X) \wedge (\forall x \exists y.[Xx \rightarrow \mathrm{succ}(x,y) \wedge Xy])] \\ & \rightarrow \exists z.[Yz \wedge Xz])\end{aligned}$$

essentially stating that "*every branch contains infinitely many elements of Y*" by stating that every X which is nonempty and closed under sucessors must contain an element of Y, thus essentially stating that Y is infinite.

Then $\psi_{\sigma,\tau}$ is given by

$$\begin{aligned}\psi_{\sigma,\tau} :=& \exists Y.[\inf(Y) \\ & \wedge \forall X.(\mathrm{srTo}(X,Y) \rightarrow \varphi'_\sigma(X)) \\ & \wedge \forall Z.(\mathrm{fromTo}(Z,Y) \rightarrow \varphi'_\tau(Z))],\end{aligned}$$

where $\varphi'_\sigma(X)$ and $\varphi'_\tau(X)$ are the relativizations of φ_σ and φ_τ respectively. The formula $\psi_{\sigma,\tau}$ thus expresses that every branch has an infinite partition into linear pieces, such that the theory σ applies to the path up to the first element of Y and on all paths starting with an element of Y up to the predecessor of the next element of Y the theory τ holds.

It remains to argue that the constructed formula is bisimulation invariant over trees.

Consider the subformulae $\inf(X)$, $\mathrm{srTo}(X,Y)$, $\mathrm{fromTo}(Z,Y)$ and relativizations of the form $\varphi'_\tau(Z)$ as constructed. Clearly the first three act bisimulation invariant over trees. The formula inf only consideres the occurence of nodes reached via succ, and trees do not contain circles; srTo characterizes paths starting with the source leading to the first elements contained in Y on each branch of the tree and fromTo characterizes arbitrary paths between elements in Y.

This means that the relativizations of σ and τ act in any case only on linear paths, not on branching subtrees, which means the formula $\psi_{\sigma,\tau}$ acts bisimulation invariant over trees, since by the bisimilarity condition one can not have paths of a different type between the source and the elements of Y as well as between the elements of Y. ■

With a formula which can recognize factorizations of linear orders we can now provide the main result of this section.

Theorem 2.11. *The class of lassos has the Unravelling Property.*

Proof. Let φ be of quantifier rank m and bisimulation invariant over the class of lassos.

Let $J \subseteq \{(\sigma, \tau) \colon \sigma, \tau \text{ theories with } \sigma\tau = \sigma \text{ and } \tau^2 = \tau\}$ be the finite set of $(m+1)$-types of lassos such that

$$\mathrm{Th}_{m+1}(\mathfrak{A}) = \sigma \text{ and } \mathrm{Th}_{m+1}(\mathfrak{B}) = \tau \quad \Longrightarrow \quad T(\mathfrak{A} \oplus \mathfrak{B}) \models \varphi$$

holds for all $(\sigma, \tau) \in J$. Then, for any two finite linear orders $\mathfrak{A}$ and $\mathfrak{B}$ over $\mathrm{Prop} \cup \{\mathrm{sr}\}$ and $\mathfrak{M} = T(\mathfrak{A} \oplus \mathfrak{B})$ we claim that the following statements are equivalent:

(i) $\mathfrak{M} \models \varphi$,

(ii) $\mathrm{type}_m(\mathfrak{M}) \in J$,

(iii) $\Omega(\mathfrak{M}) \models \bigvee_{(\sigma,\tau)\in J} \psi_{\sigma,\tau}$,

where $\psi_{\sigma,\tau}$ is the formula constructed in Lemma 2.10.

We will show "(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i)".

"(i) $\Rightarrow$ (ii)" follows by construction of J and"(ii) $\Rightarrow$ (iii)" is clear by construction of the formula $\psi_{\sigma,\tau}$.

To show "(iii) $\Rightarrow$ (i)" let $\Omega(\mathfrak{M})$ have a factorization of type $(\sigma', \tau') \in J$. We then know by Lemma 2.7 that there exists a lasso $\mathfrak{N}$ bisimilar to $\mathfrak{M}$ that is of type (σ', τ'). Since $(\sigma', \tau') \in J$ implies $\mathfrak{N} \models \varphi$ the assertion now follows by bisimulation invariance of φ.

The constructed formula is indeed bisimulation invariant over trees, since every subformula of the form $\psi_{\sigma,\tau}$ is. ■

Utilizing this result and some results cited in the previous chapter, we thus can provide the following corollary.

Corollary 2.12. *Let $\mathscr{L}$ denote the class of lassos. Then over any class $\mathscr{L}'$ with $\mathscr{L} \subseteq \mathscr{L}' \subseteq \mathscr{L}^+$ bisimulation invariant* MSO *and the μ-calculus coincide.*

Proof. This follows from the Unravelling Theorem. ■

Chapter 3

Similar Results for other Classes

Having proven the desired result for lassos, the established theoretical framework will now be applied to broader classes of transition systems, starting with hierarchical lassos.

3.1 Hierarchical Lassos

Definition 3.1. A *hierarchical lasso of level* 1 is a simple lasso as introduced in the previous chapter. A *hierarchical lasso of level* $n+1$ is a simple lasso, the so called *head lasso*, with hierarchical lassos of level n attached to it by identifying their source with a node of the head lasso. We call the attached lassos *sublassos*. The hierarchical lasso of level n on which a last layer of sublassos is attached is called *core lasso*, and the simple lassos attached to the core lasso are called *outer lassos*.

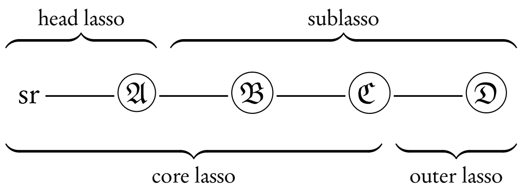


Figure 3.1: The parts of a hierarchical lasso.

Definition 3.2. Let $\mathfrak{M}$ be an hierarchical lasso of finite level and let Θ be the set of all m-types of simple lassos. The *lasso reduction* $\mathscr{R}_m(\mathfrak{M})$ yields the core lasso of $\mathfrak{M}$ on which every node to which an outer lasso of type $\theta \in \Theta$ is attached gets the colour T_θ as an additional colour.

Definition 3.3. We call a node x of a hierarchical lasso an *m-branching node*, if there exist two paths X and Y starting at x such that there exists no m-type θ such that both X and Y have a factorization of type θ.

Definition 3.4. We call a hierarchical lasso *proper* if the loop of every lasso – except the outer lassos – has branching nodes. Otherwise we call the hierarchical lasso *improper*.

At this point one can mention that the closure under bisimulation of the class of lassos contains improper hierarchical lassos, so the interesting cases for this section will be in particular the proper structures.

Sketches for the proofs of Lemma 3.9 and Lemma 3.10 which utilize the Muchnik iteration and related results as found in [3] were given by Blumensath.

Definition 3.5 (Muchnik Iteration)**.** Let $\mathfrak{A} = \langle A, R \rangle$ be a relational structure. The *Muchnik iteration* of $\mathfrak{A}$ is given by the structure

$$(\mathfrak{A})^* := \langle A^*, R^*, \leq, \mathrm{cl} \rangle$$

where

$$\begin{aligned} R_i^* &:= \{(wa_1, \dots, wa_n) : w \in A^*, (a_1, \dots, a_n) \in R\}, \\ \mathrm{cl} &:= \{waa : w \in A^*, a \in A\}, \end{aligned}$$

A^* denotes the set of finite words over A and the relation $\leq$ is the prefix order on A^*.

A known result about the Muchnik Iteration is that it is in fact an MSO compatible operation.

Theorem 3.6 (Muchnik)**.** *Let Σ be a finite relational signature. For every formula $\varphi \in \mathrm{MSO}[\Sigma \cup \{\leq, \mathrm{cl}\}]$ one can find a formula $\varphi^* \in \mathrm{MSO}[\Sigma]$ for which we find*

$$\mathfrak{A}^* \models \varphi \iff \mathfrak{A} \models \varphi^* \qquad \textit{for all } \Sigma\textit{-structures } \mathfrak{A}.$$

We will now define what we will call the *lasso iteration*, an operation on structures which attaches a layer of outer lassos to a hierarchical lasso in accordance to a given colouring.

Definition 3.7 (Lasso Iteration)**.** Let

$$\begin{aligned} (|x| = 1) &:= \exists y. [\forall z [z \leq x \leftrightarrow (z = x \vee z = y)] \wedge (x \neq y)] \\ (|x| = 2) &:= \exists y_1. \exists y_2 \big[\forall z [z \leq x \leftrightarrow (z = x \vee z = y_1 \vee z = y_2)] \\ &\qquad \wedge y_1 \neq y_2 \wedge y_1 \neq x \wedge y_2 \neq x \big] \end{aligned}$$

For $x \in A^*$ let $\mathrm{pr}_i(x) \in A$ denote the prefix of length i of x.

Let Θ be the set of $(m+1)$-types of simple lassos, and for each $\theta \in \Theta$ let $\mathfrak{M}_\theta$ be a fixed lasso of type θ called the *θ-representative*.

For some lasso $\mathfrak{M}$ utilizing the sets of colours $(P_i)_{\in I}$ and $(T_\theta)_{\theta\in\Theta}$ we define the operation

$$\mathscr{E}_m(\mathfrak{M}) := \tau\left(\left(\mathfrak{M} \oplus \bigoplus_{\theta\in\Theta} \mathfrak{M}_\theta\right)^*\right)$$

where $-^*$ denotes the Muchnik iteration and τ is the interpretation given by

$$\begin{aligned}
\delta(x) &:= (|x| = 1 \wedge \chi_{\mathfrak{M}}(x)) \\
&\quad \vee \bigvee_{\theta\in\Theta} (|x| = 2 \wedge \chi_{\mathfrak{M}}(\mathrm{pr}_1(x)) \wedge T_\theta(\mathrm{pr}_1(x)) \wedge \chi_{\mathfrak{M}_\theta}(x)) \\
\varphi_{P_i} &:= P_i(x) \qquad \text{for every colour } P_i, \\
\varphi_{\mathrm{succ}}(x,y) &:= (|x| = 1 \wedge |y| = 1 \wedge \mathrm{succ}(x,y)) \\
&\quad \vee \Big[(|x| = 2 \wedge |y| = 2 \wedge \mathrm{succ}(x,y) \Big] \\
&\quad \vee \Big[\, |x| = 1 \wedge |y| = 2 \wedge \mathrm{pr}_1(y) = x \\
&\quad \wedge \bigvee_{\theta\in\Theta} T_\theta(x) \wedge \chi_{\mathfrak{M}_\theta}(y) \wedge \exists z.[\mathrm{sr}(z) \wedge \mathrm{succ}(z,y)] \Big].
\end{aligned}$$

If $\mathfrak{M}$ is a hierarchical lasso, then $\mathscr{E}_m(\mathscr{R}_m(\mathfrak{M}))$ is the hierarchical lasso where each outer lasso $\mathfrak{N}_i$ of $\mathfrak{M}$ is replaced by a $\mathrm{type}_m(\mathfrak{N}_i)$-representative.

Indeed, both operations $\mathscr{E}_n$ and $\mathscr{R}_n$ have some nice properties as will be shown in the following proposition.

Proposition 3.8. *Let $\mathfrak{M}$, $\mathfrak{M}'$, $\mathfrak{N}$ and $\mathfrak{N}'$ be hierarchical lassos.*

1. *There exists a function f such that if $\mathfrak{M} \equiv_{f(m)} \mathfrak{M}'$ then $\mathscr{E}_m(\mathfrak{M}) \equiv_m \mathscr{E}_m(\mathfrak{M}')$.*
2. *If $\mathfrak{N}$ is bisimilar to $\mathfrak{N}'$ then $\mathscr{E}_m(\mathfrak{N})$ is bisimilar to $\mathscr{E}_m(\mathfrak{N}')$.*
3. *It holds that $\mathfrak{M} \equiv_m \mathscr{E}_m(\mathscr{R}_m(\mathfrak{M}))$.*
4. *It holds that $\mathscr{R}_m(\mathscr{E}_m(\mathfrak{N})) = \mathfrak{N}$.*

Proof. We will provide an argument for each claim.

1. This follows by Corollary 1.12, Proposition 1.13 and the Muchnik Theorem, since $\mathscr{E}_m$ is a composition of disjoint union, Muchnik iteration and an interpretation.
2. The second claim follows since the operation $\mathscr{E}_m$ attaches identical representatives to nodes which correspond to the same nodes in the unravellings of $\mathfrak{N}$ and $\mathfrak{N}'$, meaning every bisimulation of $\mathfrak{N}'$ and $\mathfrak{N}$ can be extended to one of $\mathscr{E}_m(\mathfrak{N}')$ and $\mathscr{E}_m(\mathfrak{N})$ via the identity relation on subtrees corresponding to the outer lassos.

3. This follows by a composition argument. If one considers hierarchical lassos as interpretations of the form $H(\mathfrak{N} \oplus \bigoplus_{i \leq k} \mathfrak{N}_k)$, where $\mathfrak{N}$ is the core lasso and each $\mathfrak{N}_k$ is one of the outer lassos and H attaches the $\mathfrak{N}_i$ to previously marked nodes in $\mathfrak{N}$, then we find by Corollary 1.12, Proposition 1.13, and Lemma 2.5 that

$$\mathfrak{M} = H\left(\mathfrak{N} \oplus \bigoplus_{i \leq k} \mathfrak{N}_k\right) \equiv_m H\left(\mathfrak{N} \oplus \bigoplus_{i \leq k} \mathfrak{M}_{\theta_i}\right) = \mathscr{E}_m(\mathscr{R}_m(\mathfrak{M}))$$

where $\mathfrak{M}_{\theta_i}$ is a copy of the type($\mathfrak{N}_i$)-representatives. ■

The proof that the class of hierarchical lassos has the Unravelling Property will not be shown as before by construction of an explicit formula, but by induction. For this, the following two lemmas are essential.

Lemma 3.9. *Let φ be an* MSO*-formula of quantifier rank m that is bisimulation invariant over the class of hierarchical lassos. There exists an* MSO*-formula $\hat{\varphi}$ that is bisimulation invariant over hierarchical lassos such that*

$$\mathfrak{M} \models \varphi \iff \mathscr{R}_m(\mathfrak{M}) \models \hat{\varphi}.$$

Proof. By Proposition 3.8 we know that $\mathscr{R}_m(\mathfrak{M}) \equiv_{f(m)} \mathscr{R}_m(\mathfrak{M}')$ implies that

$$\mathfrak{M} \equiv_m \mathscr{E}_m(\mathscr{R}_m(\mathfrak{M})) \equiv_m \mathscr{E}_m(\mathscr{R}_m(\mathfrak{M}')) \equiv_m \mathfrak{M}'.$$

This means there exists some function g such that

$$\mathrm{Th}_m(\mathfrak{M}) = g(\mathrm{Th}_{f(m)}(\mathscr{R}_m(\mathfrak{M}))).$$

We set

$$\hat{\varphi} := \bigvee \{\theta : \theta \text{ is an } m\text{-theory with } \varphi \in g(\theta)\}.$$

Then we find that

$$\begin{aligned} &\mathscr{R}_m(\mathfrak{M}) \models \hat{\varphi} \\ \iff& \varphi \in g(\mathrm{Th}_{f(m)}(\mathscr{R}_m(\mathfrak{M}))) = \mathrm{Th}_m(\mathfrak{M}) \\ \iff& \mathfrak{M} \models \varphi. \end{aligned}$$

It thus remains to show that the constructed formula is indeed bisimulation invariant over lassos. Let $\mathfrak{M}$ and $\mathfrak{N}$ be bisimilar hierarchical lassos. Then by Proposition 3.8

we find that

$$\begin{aligned}
&\mathfrak{M} \models \hat{\varphi} \\
\overset{\text{i)}}{\Longrightarrow}\ &\mathcal{R}_m(\mathcal{E}_m(\mathfrak{M})) \models \hat{\varphi} \\
\overset{\text{ii)}}{\Longrightarrow}\ &\mathcal{E}_m(\mathfrak{M}) \models \varphi \\
\overset{\text{iii)}}{\Longrightarrow}\ &\mathcal{E}_m(\mathfrak{N}) \models \varphi \\
\overset{\text{ii)}}{\Longrightarrow}\ &\mathcal{R}_m(\mathcal{E}_m(\mathfrak{N})) \models \hat{\varphi} \\
\overset{\text{i)}}{\Longrightarrow}\ &\mathfrak{N} \models \hat{\varphi},
\end{aligned}$$

where i) follows from Proposition 3.8 (4), ii) follows since we have shown that $\mathcal{R}_m(\mathfrak{M}) \models \hat{\varphi}$ iff $\mathfrak{M} \models \varphi$ and iii) follows from Proposition 3.8 (2) and bisimulation invariance of φ. ∎

Lemma 3.10. *For every* MSO*-formula φ that is bisimulation invariant over trees there exists an* MSO*-formula $\hat{\varphi}$ that is bisimulation invariant over trees such that*

$$\Omega(\mathcal{R}_m(\mathfrak{M})) \models \varphi \iff \Omega(\mathfrak{M}) \models \hat{\varphi}$$

holds for every proper hierarchical lasso $\mathfrak{M}$.

Proof. We will start by introducing subformulas needed to characterize $\hat{\varphi}$.

- Via the formula obtained in Lemma 2.10 one can construct for every m-type θ a formula $\Psi_\theta(x)$ stating that there exists an infinite path at x that has a factorization of type θ.

- Since the set of m-types is finite one can construct a formula BranchNode(x) that states that a node is a branching node by satisfying $\Psi_\theta(x) \wedge \Psi_{\theta'}(x)$ for two types θ and θ' such that no linear order with a factorization of type θ has a factorization of type θ'.

- One can now define a formula IsCore(X) stating that X is a subtree such that every node in X has at least one successor, such that every infinite path in X contains infinitly many nodes for which BranchNode(X) holds, and that no node not in X is a branching node.

- For a given φ one can obtain a formula $\tilde{\varphi}(X)$ via relativizing φ to X and replacing every atom $T_\theta(x)$ by $\Psi_\theta(x)$.

Then one can set

$$\hat{\varphi} := \exists X.[\mathrm{IsCore}(X) \wedge \tilde{\varphi}(X)].$$

Since X can be guessed as the set of all nodes of the core lasso of some proper $\mathfrak{M}$ we find

$$\Omega(\mathscr{R}_m(\mathfrak{M})) \models \varphi \implies \Omega(\mathfrak{M}) \models \hat{\varphi},$$

since every infinite path of the core lasso of a proper hierarchical lasso must contain infinitely many branching nodes.

For the reverse direction assume $\Omega(\mathfrak{M}) \models \hat{\varphi}$. The set chosen as X must contain the core lasso, since the subtree containing all branching nodes corresponds to the unravelling of the core lasso. The claim now follows by construction of $\hat{\varphi}$. ■

Theorem 3.11. *The class of proper hierarchical lassos has the Unravelling Property.*

Proof. We will prove this claim by induction on the level of the lassos where the induction start for a hierarchical lasso of level one follows from Theorem 2.11.

For the induction step let $\mathfrak{M}$ be of level $n+1$ and assume the assertion to hold for a hierarchical lasso of level n. By Lemma 3.9 there exists a formula ψ that is bisimulation invariant over hierarchical lassos such that

$$\mathfrak{M} \models \varphi \iff \mathscr{R}_m(\mathfrak{M}) \models \psi$$

holds. By the induction hypothesis we can now find a formula $\hat{\psi}$ such that

$$\mathscr{R}_m(\mathfrak{M}) \models \psi \iff \Omega(\mathscr{R}_m(\mathfrak{M})) \models \hat{\psi}$$

is fulfilled, since $\mathscr{R}_m(\mathfrak{M})$ is of level n. By Lemma 3.10 we now obtain the formula $\hat{\varphi}$ such that

$$\Omega(\mathscr{R}_m(\mathfrak{M})) \models \hat{\psi} \iff \Omega(\mathfrak{M}) \models \hat{\varphi}$$

holds. ■

As before, we can now apply the Unravelling Theorem to show equality of bisimulation invariant MSO and L_μ over the class of proper hierarchical lassos. Via the bisimulation closure property we thus know that the same result applies to the class of all hierarchical lassos.

Corollary 3.12. *Let $\mathscr{L}_H(m)$ denote the class of proper hierarchical lassos of level m. Then over any class $\mathscr{L}_H(m)'$ with $\mathscr{L}_H(m) \subseteq \mathscr{L}_H(m)' \subseteq \mathscr{L}_H(m)^+$ bisimulation invariant* MSO *and the μ-calculus coincide.*

Proof. This follows from the Unravelling Theorem. ■

3.2 n-Typegraphs

We will now apply the findings of the previous chapters so they can be utilized for arbitrary finite transition systems of bounded size.

Definition 3.13. Let $\mathfrak{M}$ be a finite transition system.
Then we define the *m-typegraph* $\mathfrak{G} = \langle V^{\mathfrak{G}}, E^{\mathfrak{G}} \rangle$ of $\mathfrak{M}$ as the (multi-)graph such that

- The set of nodes $V^{\mathfrak{G}}$ is given by all nodes of $\mathfrak{M}$ with no or more than one successor, where each node is colored by an additional colour $(P_v)_{v \in V^{\mathfrak{G}}}$ with $P_v = \{v\}$. We will call this additional colouring the *correspondence marking*.
- For all $x, y \in V^{\mathfrak{G}}$ and every path between x and y in $\mathfrak{M}$ that contains no other node of $V^{\mathfrak{G}}$ there is a unique edge between x and y that is labelled by the m-theory of the corresponding path (including x and y).

We then say that $\mathfrak{M}$ is a $\mathfrak{G}$-system, and we say $\mathfrak{G}$ is of size m if there are m nodes in $V^{\mathfrak{G}}$.

Remark 3.14. Note that every finite transition system $\mathfrak{M}$ has a unique typegraph $\mathfrak{G}$. However, multiple different transition systems can have the same typegraph, and the typegraphs of bisimilar models may differ.

Example 3.15. Consider the transition system

$$\begin{tikzcd}
 & \bullet \arrow[r] & \bullet \arrow[dr, bend left] & \\
1 \arrow[ur, bend left] \arrow[r] \arrow[dr, bend right] & \bullet \arrow[r] & \bullet \arrow[r] & 2 \\
 & \bullet \arrow[r] & \bullet \arrow[r] & 3.
\end{tikzcd}$$

Then its m-typegraph is given by

$$\begin{tikzcd}
1 \arrow[r, bend left, "\sigma_1"] \arrow[r, bend right, "\sigma_2"'] \arrow[dr, bend right, "\sigma_3"] & 2 \\
 & 3 ,
\end{tikzcd}$$

where the numbers denote the correspondence between the nodes and σ_1, σ_2 and σ_3 correspond to the m-theories of the respective paths.

Again we will utilize the existence of an MSO-interpretation to relate unravellings of n-typegraphs to their (bisimilar) preimages under Ω.

Lemma 3.16. *Let $\mathfrak{G}$ by a typegraph of size m. Then there exists an interpretation G such that, for all $\mathfrak{G}$-systems $\mathfrak{M}$,*

$$G\left(\mathfrak{G} \oplus \sum_{i\in I} \mathfrak{A}_i\right) = \mathfrak{M}$$

holds for suitable linear orders $\mathfrak{A}_i$.

Proof. We can choose the $\mathfrak{A}_i$ as the paths between the states corresponding to the nodes of $\mathfrak{G}$ (including those states), adding the colour P_v of the correspondence marking to each end and starting node of the $\mathfrak{A}_i$ if the node corresponds to $v \in V^{\mathfrak{G}}$.

Let $\chi_{\mathfrak{A}} := \bigvee_{i\in I} \chi_{\mathfrak{A}_i}$. Then the desired interpretation is given by

$$\begin{aligned}
\delta(x) &:= \chi_{\mathfrak{G}}(x) \vee \exists y.\exists z.(\mathrm{succ}(x,z) \wedge \mathrm{succ}(y,x))\\
\varphi_{\mathrm{sr}}(x) &:= \mathrm{sr}(x) \wedge \chi_{\mathfrak{G}}(x)\\
\varphi_P(x) &:= P(x) \qquad \text{for each colour } P \text{ utilized by } \mathfrak{M}
\end{aligned}$$

$$\begin{aligned}
\varphi_{\mathrm{succ}}(x,y) := &\left[\chi_{\mathfrak{G}}(x) \wedge \bigvee_{s\in V} \exists w.(\mathrm{succ}(w,y) \wedge P_s(w) \wedge \chi_{\mathfrak{A}}(y) \wedge P_s(x))\right] &\text{(i)}\\
&\vee \left[\chi_{\mathfrak{G}}(y) \wedge \bigvee_{s\in V} \exists w.(\mathrm{succ}(x,w) \wedge P_s(w) \wedge \chi_{\mathfrak{A}}(x) \wedge P_s(y))\right] &\text{(ii)}\\
&\vee \Bigg[\chi_{\mathfrak{G}}(x) \wedge \chi_{\mathfrak{G}}(y)\\
&\quad \wedge \exists u.\exists q. \bigvee_{v,s\in V} \Bigg[P_v(x) \wedge P_v(u) \wedge P_s(y) \wedge P_s(q)\\
&\qquad \wedge \bigvee_{i\in I} \left(\chi_{\mathfrak{A}_i}(u) \wedge \chi_{\mathfrak{A}_i}(q) \wedge \mathrm{succ}(u,q)\right)\Bigg]\Bigg] &\text{(iii)}\\
&\vee (\chi_{\mathfrak{A}}(x) \wedge \chi_{\mathfrak{A}}(y) \wedge \mathrm{succ}(x,y)). &\text{(iv)}
\end{aligned}$$

The subformula ending in line (i) identifies a node of $\mathfrak{G}$ and attaches the corresponding node that succeeds the node with the colour used for identification, i.e. the nodes with colour P_v get “choped off”. The second subformula acts like the first in opposite direction: A node of some $\mathfrak{A}_i$ gets attached to a node in $\mathfrak{G}$.

Subfomula (iii) treats the special case in which two nodes of $\mathfrak{G}$ are successors, i.e. where the path used to connect x and y only contains two elements which both get “chopped off”.

The last line treats the successor relation within the $\mathfrak{A}_i$. ■

Note that we encode a lot of knowledge about the topology of a transition system into its typegraph. Because of that and the existence of an MSO-interpretation as in the previous lemma we can now relate information about the type to information about the theory of the transition system in consideration.

Lemma 3.17. *There exists a function g over the natural numbers such that if the $g(n)$-typegraphs of any two transition systems $\mathfrak{M}$ and $\mathfrak{N}$ are isomorphic, then the n-theories of $\mathfrak{M}$ and $\mathfrak{N}$ are equal as well.*

Proof. Since a $\mathfrak{G}$-system can be seen as an interpretation of a typegraph and a disjoint sum of linear orders the assertion follows from Corollary 1.12 and Proposition 1.13 analogously to Lemma 2.5. ■

Since we will again construct a formula to show the Unravelling Property for the class of transtition systems with typegraphs bounded by n, we need the existence of some finite set of representatives that suffices for the construction of said formula.

Proposition 3.18. *Up to bisimilarity there are only finitely many different n-typegraphs of a size bounded by m.*

Proof. Since there are only finitely many n-theories there are only finitely many n-typegraphs of size bounded by m that have no multiple edges, and every n-typegraph of a size bounded by m is bisimilar to one of this form. ■

Theorem 3.19. *The class of transition systems with a typegraph bounded by n has the Unravelling Property.*

Proof. Given an MSO-formula φ we will construct a fromula $\hat{\varphi}$ which fulfills the Unravelling Property

$$\mathfrak{M} \models \varphi \iff \Omega(\mathfrak{M}) \models \hat{\varphi}.$$

Let $m = g(\mathrm{qr}(\varphi))$ where g is the function from Lemma 3.17. We start by constructing subformulae $\theta_{\mathfrak{G}}$ for each typegraph $\mathfrak{G}$ that will hold if $\mathfrak{M}$ has the m-typegraph $\mathfrak{G}$. The formula will achieve this by doing three things.

- It will guess disjoint subsets $Z_1, ..., Z_k$ whose elements correspond to the nodes of the typegraph.
- It will ensure that the paths from some vertex in Z_i to some vertex in Z_j will have the types declared by the typegraph and no others (however, it will not count *how many* paths of each type exist).
- It will ensure that there are no other paths than the ones already considered.
- It will ensure that every path will eventually reach some element of some of the Z_i, and since the source is in one of the Z_i the formula does not ignore some subgraph.

Now let $\mathfrak{G}$ be an m-typegraph with k nodes. Let $\mathfrak{I}_{i,j}$ be the set of types between the i-th and the j-th node of the typegraph. For any $\sigma \in \mathfrak{I}_{i,j}$ we denote by $\sigma(X)$ the relativization of $\bigwedge \sigma$ to X.

To construct a formula having the above properties, we define the subformula

$$\mathrm{closed}(X) = \forall x.[\exists y_1.(Xx \wedge \mathrm{succ}(x,y_1)) \rightarrow (\exists y_2.\,\mathrm{succ}(x,y_2) \wedge Xy_2)]$$

stating that if there exists a successor of an element x of X then x has a successor in X,

$$\begin{aligned}\mathrm{NoZinMid}(X,Z_1,\dots,Z_k) = \forall y.\Big(&\exists x.\exists z.(Xx \wedge Xy \wedge Xz\\ &\wedge \mathrm{succ}(x,y) \wedge \mathrm{succ}(y,z)) \rightarrow \bigwedge_{n=1}^{k} \neg Z_n y\Big),\end{aligned}$$

as well as

$$\begin{aligned}\mathrm{HasType}_{i,j}(X,Z_1,\dots,Z_k) = {} & \mathrm{path}(X) \wedge \mathrm{NoZinMid}(X,Z_1,\dots,Z_k)\\ & \wedge \exists x.(Xx \wedge \neg\exists y.(Xy \wedge \mathrm{succ}(y,x)) \wedge Z_i x)\\ & \wedge \exists x.(Xx \wedge \neg\exists y.(Xy \wedge \mathrm{succ}(x,y)) \wedge Z_j x),\end{aligned}$$

stating that X is a connected path from some element of Z_i to some element of Z_j containing no element of any of the $Z_1, \dots, Z_k$ in between. Then we define the formula

$$\begin{aligned}\theta_{\mathfrak{G}} := \exists Z_1.\dots\exists Z_k.\Bigg[&\bigwedge_{i,j\le k}\ \bigwedge_{\sigma\in\mathfrak{I}_{i,j}} \exists X.\big(\mathrm{HasType}_{i,j}(X,Z_1,\dots,Z_k) \wedge \sigma(X)\big)\\ &\wedge \bigwedge_{i,j\le k} \forall X.\Big(\mathrm{HasType}_{i,j}(X,Z_1,\dots,Z_k) \rightarrow \bigvee_{\sigma\in\mathfrak{I}_{i,j}} \sigma(X)\Big)\\ &\wedge \bigwedge_{j\ne i,\ i,j\le k} Z_i \cap Z_j = \emptyset\\ &\wedge \forall x.(\mathrm{sr}(x) \rightarrow Z_1 x)\\ &\wedge \neg\,\exists X.\Big(\mathrm{closed}(X) \wedge \neg\exists z.\bigvee_{i\le k}(Z_i z \wedge Xz)\Big)\Bigg]\end{aligned}$$

having the properties mentioned above. By Proposition 3.18 we know that up to bisimilation there exist only finitely many m-typegraphs. Let Γ be a finite set of representatives and let Γ_0 be the subset of all those that imply φ. Then we can define

$$\hat{\varphi} := \bigvee_{\mathfrak{G}\in\Gamma_0} \theta_{\mathfrak{G}}.$$

We now will argue that the constructed formula works as desired.

First we consider "$\Rightarrow$". Let $\mathfrak{M}$ have the m-typegraph $\mathfrak{G}$ with $\mathfrak{M} \models \varphi$. Then there exists some $\mathfrak{M}'$ that that is bisimilar to $\mathfrak{M}$ and has some typegraph $\mathfrak{G}' \in \Gamma$ bisimilar to $\mathfrak{G}$. By the bisimulation invariance of φ we find that $\mathfrak{M}' \models \varphi$, and by Lemma 3.17 transition systems with the same $g(\mathrm{qr}(\varphi))$-typegraphs have the same $\mathrm{qr}(\varphi)$-theory, thus it follows that $\mathfrak{M} \models \varphi$. Hence, by definition of Γ_0, we find $\mathfrak{G}' \in \Gamma_0$. It now follows that $\Omega(\mathfrak{M}) \models \theta_{\mathfrak{G}'}$ and thus $\Omega(\mathfrak{M}) \models \hat{\varphi}$.

For the reverse direction let $\Omega(\mathfrak{M}) \models \hat{\varphi}$. This means there exists some $\mathfrak{G}$ such that $\Omega(\mathfrak{M}) \models \theta_{\mathfrak{G}}$. Hence, $\mathfrak{M}$ has a typegraph $\mathfrak{G}'$ that is bisimilar to $\mathfrak{G}$. Then there exists some $\mathfrak{M}'$ with type $\mathfrak{G}$ that is bisimilar to $\mathfrak{M}$. Since $\mathfrak{G} \in \Gamma_0$ implies that $\mathfrak{M}' \models \varphi$ the assertion $\mathfrak{M} \models \varphi$ now follows from the bisimulation invariance of φ.

The constructed formula is bisimulation invariant over trees, since it only checks wheather linear paths of a certain type (and no others) exist and ignores multiple occurences. ■

Corollary 3.20. *Let $\mathcal{T}(n)$ denote the class of transition systems with a typegraph bounded by n. Then over any class $\mathcal{T}(n)'$ with $\mathcal{T}(n) \subseteq \mathcal{T}(n)' \subseteq \mathcal{T}(n)^+$ bisimulation invariant* MSO *and the μ-calculus coincide.*

Proof. The assertion follows from the previous theorem and Theorem 1.31. ■

Conclusion

To show equivalence of bisimulation invariant MSO and the μ-calculus on three classes of transition systmes, the class $\mathscr{L}$ of *lassos*, the class $\mathscr{L}_H(n)$ of *proper hierarchical lassos of level n* and the class $\mathscr{T}(n)$ of *systems with a typegraph bounded by n*, we utilized a result which shows that if a class has the Unravelling Property as introduced in Definition 1.30, on any class between said class and its (finite) closure under bisimulation, bisimulation invariant MSO and L_μ coincide.

We applied the composition method to prove the existence of formulae as needed for the Unravelling Property and thus obtained the following results.

Proposition. *If $\mathscr{C}$ is a class of transition systems let $\mathscr{C}^+$ denote the class of finite transition systems bisimilar to one in $\mathscr{C}$.*

- *Over any class $\mathscr{L}'$ with $\mathscr{L} \subseteq \mathscr{L}' \subseteq \mathscr{L}^+$ bisimulation invariant monadic second order logic and the μ-calculus coincide.*
- *Over any class $\mathscr{L}_H(n)'$ with $\mathscr{L}_H(n) \subseteq \mathscr{L}_H(n)' \subseteq \mathscr{L}_H(n)^+$ bisimulation invariant monadic second order logic and the μ-calculus coincide.*
- *Over any class $\mathscr{T}(n)'$ with $\mathscr{T}(n) \subseteq \mathscr{T}(n)' \subseteq \mathscr{T}(n)^+$ bisimulation invariant monadic second order logic and the μ-calculus coincide.*

Proof. Proves of the assertions can be found in Corollaries 2.12, 3.12, and 3.20. ■

It thus turns out that the composition method is viable to show equivalence of bisimulation invariant MSO and L_μ on specific classes of transition systems for which one can define a notion of *type*, i.e. a notion that links the theory of any transition systems to some coding of theories of linear orders which can be utilized to obtain said transition system by an MSO-compatible operation.

However, since the notion of a type of a transition system needs to encode a lot of information about its topology, it is not clear wheather one can obtain a genereal result – as the one for FO given by van Benthem – for monadic second order logic by application of the methods utilized to obtain the results above.

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