Simple Monadic Theories and Indiscernibles

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1 Introduction

Over the last two decades the beginnings of a model theory for monadic second-order logic have emerged. Since this logic is more expressive than first-order logic it is unsurprising that most structures possess an extremely complicated monadic second-order theory. Fortunately, there remain structures where the theory is simple enough to develop a structure theory.

The prime example of such a structure is the infinite binary tree which, according to Rabin's theorem, has a decidable monadic theory. It follows that any structure interpretable in this tree also has a simple theory. While the monadic theories of arbitrary trees can become highly undecidable, we can nevertheless develop a kind of structure theory for structures interpretable in them (see [4, 3]).

On the other extreme, every structure in which one can define arbitrarily large grids has a very complex monadic theory since we can reduce arithmetic to it. In particular, there is no hope for a structure theory for such structures.

There is a conjecture of Seese [8] stating that these two cases form a dichotomy: either a structure is interpretable in some tree or we can define arbitrarily large grids. For graphs (or structures with relations of arity at most 2) a variant of this conjecture has been solved by Courcelle and Oum [6]. But the general case of arbitrary structures is still open.

In the present article we approach this conjecture by considering a weaker statement about *first-order* theories and applying standard tools from first-order model theory. Instead of grids we consider first-order definable pairing functions

and we investigate the class of all structures without such a pairing function. We say that the theory of such a structure *does not admit coding.* Our focus lies on indiscernible sequences in structures without coding. We will prove several structure results for indiscernible sequences. Our main result is Theorem 4.13 which states that every indiscernible sequence can be extended (both in 'length' and 'width') to cover any given additional element. These technical results will be used in a forthcoming article [2] (see also [5]) to prove that every structure that does not admit coding looks, in a very general sense, like a tree.

Recently there has been renewed interest in first-order theories without the independence property [11, 13, 12, 7]. The simplest case studied in this context consists of the so-called *dp-minimal* theories introduced in [7]. One can show that theories without coding are dp-minimal but not every dp-minimal theory does not admit coding. Hence, the class of theories we investigate in the present article serves as a simple example of dp-minimal theories. But note that the structure results we obtain in Section 4 do not hold for arbitrary dp-minimal theories.

2 Dependent sequences

In this section we consider an indiscernible sequence $(\bar a^\nu)_{\nu\in I}$ of α-tuples, and we try to find a formula $\chi(\bar x)$ which defines the relation $\{\, \bar a^\nu \mid \nu \in I \,\}$. Of course, in general this is not possible. But if we allow monadic parameters there is a partial solution to this question. The combinatorial techniques used by the following lemmas are based on results by Shelah [9].

Let us recall some basic definitions and fix our notation. We define $[n] := \{0, \dots, n-1\}$. We tacitly identify tuples $\bar a = a_0 \dots a_{n-1} \in A^n$ with functions $[n] \to A$ and frequently we write $\bar a$ for the set $\{a_0, \dots, a_{n-1}\}$. This allows us to write $\bar a \subseteq A$ or $\bar a|_I$ for $I \subseteq [n]$. We use the words 'tuple' and 'sequence' synonymously. In particular, tuples may be infinite.

$2^{<\alpha}$ denotes the set of all binary sequences of length less than α and $\preceq$ is the prefix ordering on such sequences

$$x \preceq y \quad :\text{iff} \quad y = xz \text{ for some } z\,.$$

The empty sequence is denoted by $\langle\rangle$.

Definition 2.1. Let $(\bar a^\nu)_{\nu\in I}$ be a sequence of α-tuples indexed by a linear order I.

(a) We denote the *order type* of $\bar v \in I^m$ by $\operatorname{ord}(\bar v)$ and its *equality type* by $\operatorname{equ}(\bar v)$. For sets $C, D \subseteq I$, we write $C < D$ if $c < d$, for all $c \in C$ and $d \in D$. Analogously, we define $\bar u < \bar v$ for tuples $\bar u, \bar v \subseteq I$.

(b) For $\bar{v} \in I^m$, we set

$$\bar{a}[\bar{v}] := (\bar{a}^{v_0}, \dots, \bar{a}^{v_{m-1}}).$$

For $J \subseteq I$ and $s \in I$ we define

$$\bar{a}[J] := (\bar{a}^v)_{v \in J} \quad \text{and} \quad \bar{a}[<s] := (\bar{a}^v)_{v<s}.$$

The terms $\bar{a}[>s]$, $\bar{a}[\leq s]$, and so on, are defined analogously.

(c) For $\bar{v} \in I^\alpha$, we set

$$\bar{a}\langle\bar{v}\rangle := (a_i^{v_i})_{i<\alpha}.$$

Before turning to the general case below let us show how to define a bijection $\bar{a}^v \mapsto \bar{b}^v$ between two sequences $(\bar{a}^v)_{v \in I}$ and $(\bar{b}^v)_{v \in I}$.

Lemma 2.2. *Let $(\bar{a}^v)_{v \in I}$ and $(\bar{b}^v)_{v \in I}$ be two sequences indexed by the same linear order I. If there exists a formula $\varphi(\bar{x}, \bar{y})$ (possibly with monadic parameters) and a relation $\sigma \in \{=, \neq, \leq, \geq, <, >\}$ such that*

$$\mathfrak{M} \vDash \varphi(\bar{c}, \bar{d}) \quad \text{iff} \quad \bar{c} = \bar{a}^u \text{ and } \bar{d} = \bar{b}^v \text{ for some } u \, \sigma \, v,$$

then we can construct a formula $\psi(\bar{x}, \bar{y})$ such that

$$\mathfrak{M} \vDash \psi(\bar{c}, \bar{d}) \quad \text{iff} \quad \bar{c} = \bar{a}^v \text{ and } \bar{d} = \bar{b}^v \text{ for some } v \in I.$$

Proof. If $\sigma \in \{=, \neq\}$ then we can set $\psi := \varphi$ or $\psi := \neg\varphi$. By symmetry it therefore remains to consider the case that $\sigma = \{\leq\}$. We can construct a formula ϑ such that

$$\mathfrak{M} \vDash \vartheta(\bar{c}, \bar{d}) \quad \text{iff} \quad \bar{c} = \bar{a}^u \text{ and } \bar{d} = \bar{a}^v \text{ for some } u \leq v,$$

by setting

$$\vartheta(\bar{x}, \bar{x}') := \forall \bar{y}[\varphi(\bar{x}', \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y})].$$

Consequently, we obtain the desired formula ψ by

$$\psi(\bar{x}, \bar{y}) := \forall \bar{x}'[\varphi(\bar{x}', \bar{y}) \rightarrow \vartheta(\bar{x}', \bar{x})]. \qquad \square$$

The next lemmas provide a method to find sequences satisfying the preceding lemma.

Definition 2.3. (a) For a set Δ of formulae, we denote the Δ-type of $\bar{a}$ over U by $\mathrm{tp}_\Delta(\bar{a}/U)$.

(b) The *type index* $\mathrm{ti}_\Delta^n(A/U)$ of a set A over U is the maximal cardinal κ such that there exists a sequence $(\bar{a}^i)_{i<\kappa}$ of n-tuples $\bar{a}^i \in A^n$ with

$$\mathrm{tp}_\Delta(\bar{a}^i/U) \neq \mathrm{tp}_\Delta(\bar{a}^k/U), \quad \text{for } i \neq k.$$

Lemma 2.4. *Let Δ be a finite set of formulae, $B \subseteq M$ a set, and $(\bar{a}^u)_{u<\omega}$ an infinite sequence of tuples such that*

$$\mathrm{tp}_\Delta(\bar{a}^u/B) \neq \mathrm{tp}_\Delta(\bar{a}^v/B), \quad \textit{for all } u \neq v.$$

There exist an infinite subset $I \subseteq \omega$, a formula $\varphi \in \Delta$, a relation $\sigma \in \{=, \neq, \leq, >\}$, a number $m < \omega$, and tuples $\bar{b}^v \in B^m$, for $v \in I$, such that

$$\mathfrak{M} \vDash \varphi(\bar{a}^u, \bar{b}^v) \quad \text{iff} \quad u \, \sigma \, v, \quad \textit{for all } u, v \in I.$$

Proof. We adapt the proof of Ramsey's theorem. For indices $u \neq v$, fix some formula $\varphi_{uv}(\bar{x}, \bar{y}) \in \Delta$ and a tuple $\bar{c}_{uv} \subseteq B$ with

$$\mathfrak{M} \vDash \varphi_{uv}(\bar{a}^u, \bar{c}_{uv}) \leftrightarrow \neg\varphi_{uv}(\bar{a}^v, \bar{c}_{uv}).$$

We assume that $\bar{c}_{uv} = \bar{c}_{vu}$ and $\varphi_{uv} = \varphi_{vu}$, for all $u, v < \omega$.

We define two infinite increasing sequences $u_0 < u_1 < \dots < \omega$ and $v_0 < v_1 < \dots < \omega$ of indices and a decreasing sequence $\omega = I_0 \supset I_1 \supset \dots$ of infinite sets such that, for every $i < \omega$, we have $u_i, v_i \in I_i$ and

$$\mathfrak{M} \vDash \varphi_{u_i v_i}(\bar{a}^{u_i}, \bar{c}_{u_i v_i}) \leftrightarrow \neg\varphi_{u_i v_i}(\bar{a}^w, \bar{c}_{u_i v_i}), \quad \text{for all } w \in I_{i+1}.$$

Note that, in particular, this implies that

$$\mathfrak{M} \vDash \varphi_{u_i v_i}(\bar{a}^{u_i}, \bar{c}_{u_i v_i}) \leftrightarrow \neg\varphi_{u_i v_i}(\bar{a}^{u_k}, \bar{c}_{u_i v_i}), \quad \text{for } i < k.$$

We start with $I_0 := \omega$. For the induction step, suppose that I_i has already been defined. Fix arbitrary elements $u, v \in I_i$ with $u \neq v$. By symmetry, we may assume that

$$\mathfrak{M} \vDash \varphi_{uv}(\bar{a}^u, \bar{c}_{uv}) \wedge \neg\varphi_{uv}(\bar{a}^v, \bar{c}_{uv}).$$

Let $\quad J_0 := \{ w \in I_i \mid \mathfrak{M} \vDash \neg\varphi_{uv}(\bar{a}^w, \bar{c}_{uv}) \},$

$\qquad J_1 := \{ w \in I_i \mid \mathfrak{M} \vDash \varphi_{uv}(\bar{a}^w, \bar{c}_{uv}) \}.$

If J_0 is infinite then we set $u_i := u$, $v_i := v$, and $I_{i+1} := J_0$. Otherwise, we choose $u_i := v$, $v_i := u$, and $I_{i+1} := J_1$.

Set $\bar{b}^i := \bar{c}_{u_i v_i}$. We record for every pair $i < k$ of indices which of the following relations hold:

$$\mathfrak{M} \vDash \varphi_{u_i v_i}(\bar{a}^{u_i}, \bar{b}^i),$$
$$\mathfrak{M} \vDash \varphi_{u_i v_i}(\bar{a}^{u_i}, \bar{b}^k),$$
$$\mathfrak{M} \vDash \varphi_{u_i v_i}(\bar{a}^{u_k}, \bar{b}^i).$$

By Ramsey's Theorem, there exists an infinite subset $S \subseteq \omega$ such that, for all indices $i < k$ and $l < m$ in S,

- $\varphi_{u_i v_i} = \varphi_{u_k v_k}$,
- $\mathfrak{M} \vDash \varphi_{u_i v_i}(\bar{a}^{u_i}, \bar{b}^i) \leftrightarrow \varphi_{u_k v_k}(\bar{a}^{u_k}, \bar{b}^k)$,
- $\mathfrak{M} \vDash \varphi_{u_k v_k}(\bar{a}^{u_i}, \bar{b}^k) \leftrightarrow \varphi_{u_m v_m}(\bar{a}^{u_l}, \bar{b}^m)$.

Setting $\varphi := \varphi_{u_i v_i}$ it follows that, for $i < k$ in S,

$$\mathfrak{M} \vDash \varphi(\bar{a}^{v_i}, \bar{b}^i) \leftrightarrow \neg\varphi(\bar{a}^{v_k}, \bar{b}^i).$$

Consequently, we have

$$\mathfrak{M} \vDash \varphi(\bar{a}^{v_i}, \bar{b}^k) \quad \text{iff} \quad i \,\sigma\, k,$$

where $\sigma \in \{=, \neq, \leq, >\}$. □

For uncountable cardinals the proof is more involved.

Lemma 2.5. *Let κ be an infinite cardinal, Δ a set of formulae of size $|\Delta| \leq \kappa$, and $A, B \subseteq M$ sets. If $\mathrm{ti}^n_\Delta(A/B) > 2^\kappa$ then there exist a formula $\varphi(\bar{x}, \bar{y}) \in \Delta$, a number $m < \omega$, and tuples $\bar{a}^v \in A^n$ and $\bar{b}^v \in B^m$, for $v < \kappa^+$, such that*

$$\mathfrak{M} \vDash \varphi(\bar{a}^u, \bar{b}^u) \leftrightarrow \neg\varphi(\bar{a}^v, \bar{b}^u), \quad \textit{for all } u < v.$$

Proof. Let $\lambda := (2^\kappa)^+$. Fix a sequence $(\bar{a}^v)_{v<\lambda}$ of tuples $\bar{a}^v \in A^n$ such that,

$$\mathrm{tp}_\Delta(\bar{a}^u/B) \neq \mathrm{tp}_\Delta(\bar{a}^v/B), \quad \text{for } u \neq v.$$

We construct a family of sets $S_z \subseteq \lambda$, for $z \in 2^{<\lambda^+}$, such that

- $S_{\langle\rangle} = \lambda$,
- $S_z = S_{z0} \cup S_{z1}$,
- $S_x \supseteq S_y$, for $x \leq y$,
- $S_x \cap S_y = \varnothing$, for $x \nleq y$ and $y \nleq x$, and
- if $|S_z| > 1$ then $S_{z0}, S_{z1} \neq \varnothing$.

For each z, we will choose a formula $\varphi_z(\bar{x}, \bar{y})$ and parameters $\bar{b}^z \subseteq B$, and we set

$$S_z := \{\, u < \lambda \mid \text{for all } y < z \text{ we have } \mathfrak{M} \vDash \varphi_y(\bar{a}^u, \bar{b}^y) \text{ iff } y1 \leq z \,\}.$$

We define φ_z inductively. Suppose that φ_x and $\bar{b}^x$ have already been defined, for all $x < z$. Then we also know S_z. If $|S_z| \leq 1$ then we choose an arbitrary sequence $y < z$ and set $\varphi_z := \varphi_y$ and $\bar{b}^z := \bar{b}^y$. Otherwise, choose distinct elements $u, v \in S_z$. Since

$$\mathrm{tp}_\Delta(\bar{a}^u/B) \neq \mathrm{tp}_\Delta(\bar{a}^v/B)$$

we can find a formula $\varphi_z \in \Delta$ and parameters $\bar{b}^z \subseteq B$ such that

$$\mathfrak{M} \vDash \varphi_z(\bar{a}^u, \bar{b}^z) \leftrightarrow \neg\varphi_z(\bar{a}^v, \bar{b}^z).$$

Having defined $(S_z)_z$ we consider the sets

$$T := \{\, z \in 2^{<\lambda^+} \mid |S_z| > 1 \,\} \quad \text{and} \quad F := \{\, z \notin T \mid y \in T \text{ for all } y < z \,\}.$$

Then $|S_z| \leq 1$, for all $z \in F$ and $\lambda = \bigcup_{z\in F} S_z$. Consequently, we have $|F| \geq \lambda$.

Let α be the minimal ordinal such that $T \subseteq 2^{<\alpha}$. Then $|F| \leq 2^{|\alpha|}$ implies that $\lambda \leq 2^{|\alpha|}$. Since $2^\kappa < \lambda$ it follows that $\alpha \geq \kappa^+$. Hence, there exists some $\eta \in F$ with $|\eta| \geq \kappa^+$. For $i \leq \kappa^+$, let $z_i < \eta$ be the prefix of η of length $|z_i| = i$, and let $c_i < 2$ be the number such that $z_i c_i \nleq \eta$. For every i, choose some element $u_i \in S_{z_i c_i}$. Since $u_k \notin S_{z_i c_i}$, for $k > i$, it follows that

$$\mathfrak{M} \vDash \varphi_{z_i c_i}(\bar{a}^{u_i}, \bar{b}^{z_i c_i}) \leftrightarrow \neg\varphi_{z_i c_i}(\bar{a}^{u_k}, \bar{b}^{z_i c_i}), \quad \text{for } i < k.$$

By the Pigeon Hole Principle, there exists a subset $I \subseteq \kappa^+$ such that $\varphi_{z_i c_i} = \varphi_{z_k c_k}$, for all $i, k \in I$. Hence, $(\bar{a}^{u_i})_{i\in I}$ and $(\bar{b}^{z_i c_i})_{i\in I}$ are the desired sequences. □

Corollary 2.6. *Let κ be an infinite cardinal, Δ a set of formulae of size $|\Delta| \leq \kappa$, and $A, B \subseteq M$ sets. If $\mathrm{ti}^n_\Delta(A/B) > 2^{2^\kappa}$ then there exist a formula $\varphi(\bar{x}, \bar{y}) \in \Delta$, a relation $\sigma \in \{=, \neq, \leq, >\}$, a number $m < \omega$, and tuples $\bar{a}^v \in A^n$ and $\bar{b}^v \in B^m$, for $v < \kappa^+$, such that*

$$\mathfrak{M} \vDash \varphi(\bar{a}^u, \bar{b}^v) \quad \text{iff} \quad u \,\sigma\, v.$$

Proof. By Lemma 2.5, there exist a formula φ and sequences $(\bar{a}^i)_{i<(2^\kappa)^+}$ and $(\bar{b}^i)_{i<(2^\kappa)^+}$ such that

$$\mathfrak{M} \vDash \varphi(\bar{a}^i, \bar{b}^i) \leftrightarrow \neg\varphi(\bar{a}^k, \bar{b}^i), \quad \text{for } i < k.$$

By the Erdős-Rado Theorem, we have $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$. Hence, we can can find a subsequence $I \subseteq (2^\kappa)^+$ of size $|I| \geq \kappa^+$ such that

$$\mathfrak{M} \vDash \varphi(\bar{a}^i, \bar{b}^k) \leftrightarrow \varphi(\bar{a}^j, \bar{b}^l), \quad \text{for all indices } i, j, k, l \in I \text{ with } \operatorname{ord}(ik) = \operatorname{ord}(jl).$$

It follows that there is some relation $\sigma \in \{=, \neq, \leq, >\}$ such that, for all $i, k \in I$,

$$\mathfrak{M} \vDash \varphi(\bar{a}^k, \bar{b}^i) \quad \text{iff} \quad k \,\sigma\, i. \qquad \square$$

To generalise Lemma 2.2 we look at the fine structure of an indiscernible sequence. In [9] Shelah defines an equivalence relation on the indices of a certain sequence $(\bar{a}^\nu)_{\nu\in I}$ of α-tuples (actually enumerations of models) by calling two indices $i, k < \alpha$ equivalent if the bijection $a^\nu_i \mapsto a^\nu_k$, $\nu \in I$, is MSO-definable. Shelah's main result concering this equivalence relation is a characterisation via indiscernibility. Inspired by this work we consider the case of arbitrary indiscernible sequences. Taking the characterisation in terms of indiscernibility as the definition we show that this equivalence relation gives rise to definable bijections $a^\nu_i \mapsto a^\nu_k$, $\nu \in I$. The main ideas of the proof of this fact in Theorem 2.18 below are already contained in [9]. Our contribution consists in streamlining the presentation, showing that the result holds without the special assumptions of Shelah, and obtaining more precise information about the formulae defining the bijections.

Definition 2.7. (a) Let $\varphi(\bar{x}^0, \dots, \bar{x}^{k-1})$ be a formula where each $\bar{x}^i$ is an α-tuple of variables. A sequence $(\bar{a}^\nu)_{\nu\in I}$ of α-tuples is *φ-indiscernible* if, for all indices $\bar{u}^i, \bar{v}^i \in I^\alpha$, $i < k$, with $\operatorname{ord}(\bar{u}^0 \dots \bar{u}^{k-1}) = \operatorname{ord}(\bar{v}^0 \dots \bar{v}^{k-1})$, we have

$$\mathfrak{M} \vDash \varphi\big(\bar{a}\langle\bar{u}^0\rangle, \dots, \bar{a}\langle u^{k-1}\rangle\big) \leftrightarrow \varphi\big(\bar{a}\langle\bar{v}^0\rangle, \dots, \bar{a}\langle v^{k-1}\rangle\big).$$

(b) Let Δ be a set of such formulae. $(\bar{a}^\nu)_{\nu\in I}$ is *Δ-indiscernible* if it is φ-indiscernible, for every $\varphi \in \Delta$. If Δ is the set of all formulae over a set U of parameters we say that $(\bar{a}^\nu)_{\nu\in I}$ is indiscernible over U.

Example. A sequence $(\bar{a}^i)_i$ of 4-tuples satisfying

$$\mathfrak{M} \vDash \varphi(a^i_0, a^k_1, a^l_2, a^m_3) \quad \text{iff} \quad i = k \text{ or } (i < k \text{ and } l = m)$$

is φ-indiscernible.

The relation $\{\, \bar{a}^\nu \mid \nu \in I \,\}$ is usually not definable. Instead, we define relations $\{\, \bar{a}^\nu|_p \mid \nu \in I \,\}$ for certain subsets $p \subseteq \alpha$. The main part of this section consists in the proof that the sets p where this is possible form a partition of α.

Definition 2.8. (a) A *partition* of a set X is a set $P \subseteq \wp(X)$ such that $X = \bigcup P$ and $p \cap q = \varnothing$, for distinct $p, q \in P$.

(b) Every partition P on X induces the equivalence relation

$$x \approx_P y \quad :\text{iff} \quad \text{there is some } p \in P \text{ with } x, y \in p.$$

(c) The *refinement order* on partitions P and Q of X is defined by

$$P \sqsubseteq Q \quad :\text{iff} \quad \approx_P \subseteq \approx_Q,$$

and, for a family F of partitions of X, we define their *common refinement* by

$$\textstyle\bigsqcap F := X/{\approx} \quad \text{where} \quad \approx := \bigcap_{P\in F} \approx_P.$$

Definition 2.9. Let $(\bar{a}^\nu)_{\nu\in I}$ be a sequence of α-tuples and $\varphi(\bar{x}^0, \dots, \bar{x}^k)$ a formula where each $\bar{x}^i$ is an α-tuple of variables. A *φ-partition* of $(\bar{a}^\nu)_{\nu\in I}$ is a partition P of α such that

$$\mathfrak{M} \vDash \varphi\big(\bar{a}\langle\bar{u}^0\rangle, \dots, \bar{a}\langle\bar{u}^k\rangle\big) \leftrightarrow \varphi\big(\bar{a}\langle\bar{v}^0\rangle, \dots, \bar{a}\langle\bar{v}^k\rangle\big),$$

for all indices $\bar{u}^i, \bar{v}^i \in I^\alpha$, $i \leq k$, such that

$$\operatorname{ord}(\bar{u}^0|_p \dots \bar{u}^k|_p) = \operatorname{ord}(\bar{v}^0|_p \dots \bar{v}^k|_p), \quad \text{for every } p \in P.$$

Let Δ be a set of formulae. A *Δ-partition* is a partition P that is a φ-partition, for every $\varphi \in \Delta$.

Equivalently, P is a Δ-partition of $(\bar{a}^i)_i$ if, for every $p \in P$, the 'band' $(\bar{a}^i|_p)_i$ is indiscernible over its complement $(\bar{a}^i|_{\alpha\setminus p})_i$.

Example. Let $(\bar{a}^i)_i$ be an indiscernible sequence of 4-tuples and suppose that $\varphi(x_0x_1x_2x_3)$ is a formula such that

$$\mathfrak{M} \vDash \varphi(a_0^i, a_1^k, a_2^l, a_3^m) \quad \text{iff} \quad i = k \text{ or } (i < k \text{ and } l = m)\,.$$

There are two φ-partitions of $[4]$. The trivial partition with just one class and the partition with classes $\{0,1\}$ and $\{2,3\}$.

We will show that there is a unique minimal Δ-partition. We start by pointing out that there exists at least one Δ-partition. Then we show that the class of these partitions is closed under intersections.

Lemma 2.10. *If $(\bar{a}^\nu)_{\nu\in I}$ is a Δ-indiscernible sequence of α-tuples then $\{\alpha\}$ is a Δ-partition.*

Lemma 2.11. *If $(P_i)_{i<\kappa}$ is a decreasing sequence of Δ-partitions then $\bigsqcap_{i<\kappa} P_i$ is a Δ-partition.*

Proof. If κ is finite then we have $\bigsqcap_{i<\kappa} P_i = P_{\kappa-1}$, which is a Δ-partition. For infinite κ the claim follows from the fact that every formula $\varphi \in \Delta$ contains only finitely many variables. $\square$

Lemma 2.12. *Let $(\bar{a}^\nu)_{\nu\in I}$ be an infinite sequence of α-tuples. If P and Q are Δ-partitions then so is $P \sqcap Q$.*

Proof. It is sufficient to prove the claim for $\Delta = \{\varphi\}$. Since φ contains only finitely many variables we may assume w.l.o.g. that α is finite and that

$$P = \{p_0, \dots, p_{n-1}\} \quad \text{and} \quad Q = \{q_0, \dots, q_{m-1}\}\,.$$

For $i < m$, let $q_i' := \alpha \setminus q_i$. Since

$$P \sqcap Q = P \sqcap \{q_0, q_0'\} \sqcap \dots \sqcap \{q_{m-1}, q_{m-1}'\}$$

it is sufficient to prove the claim for $Q = \{q, q'\}$.

Let us introduce some shorthand. For $\bar{u}_i \in I^{p_i \cap q}$ and $\bar{v}_i \in I^{p_i \cap q'}$, we set

$$\bar{a}[\bar{u}_0, \dots, \bar{u}_{n-1}, \bar{v}_0, \dots, \bar{v}_{n-1}] := \bar{a}\langle\bar{x}\rangle\,,$$

where x_i is

- the l-th element of $\bar{u}_k$, if i is the l-th element of $p_k \cap q$,
- the l-th element of $\bar{v}_k$, if i is the l-th element of $p_k \cap q'$.

Suppose that $\varphi = \varphi(\bar{x}^0, \dots, \bar{x}^k)$. For $\bar{u}_i^l \in I^{p_i \cap q}$ and $\bar{v}_i^l \in I^{p_i \cap q'}$, we define

$$\begin{aligned}\varphi[\bar{u}_0^0 \dots \bar{u}_0^k,\ \dots,\ \bar{u}_{n-1}^0 \dots \bar{u}_{n-1}^k,\ \bar{v}_0^0 \dots \bar{v}_0^k,\ \dots,\ \bar{v}_{n-1}^0 \dots \bar{v}_{n-1}^k] :=&\\ \varphi\big(\bar{a}[\bar{u}_0^0, \dots, \bar{u}_{n-1}^0, \bar{v}_0^0, \dots, \bar{v}_{n-1}^0], \dots, \bar{a}[\bar{u}_0^k, \dots, \bar{u}_{n-1}^k, \bar{v}_0^k, \dots, \bar{v}_{n-1}^k]\big)\,.&\end{aligned}$$

We have to show that

$$\mathfrak{M} \vDash \varphi[\bar{u}_0, \dots, \bar{u}_{n-1}, \bar{v}_0, \dots, \bar{v}_{n-1}] \leftrightarrow \varphi[\bar{s}_0, \dots, \bar{s}_{n-1}, \bar{t}_0, \dots, \bar{t}_{n-1}]\,,$$

whenever $\mathrm{ord}(\bar{u}_i) = \mathrm{ord}(\bar{s}_i)$ and $\mathrm{ord}(\bar{v}_i) = \mathrm{ord}(\bar{t}_i)$. If we prove the following special case then the general one will follow by symmetry (w.r.t. permutations of P and Q) and induction.

Claim. *If $\mathrm{ord}(\bar{u}_0) = \mathrm{ord}(\bar{w}_0)$ then*

$$\begin{aligned}\mathfrak{M} \vDash\ &\varphi[\bar{u}_0, \bar{u}_1, \dots, \bar{u}_{n-1}, \bar{v}_0, \dots, \bar{v}_{n-1}]\\ &\leftrightarrow \varphi[\bar{w}_0, \bar{u}_1, \dots, \bar{u}_{n-1}, \bar{v}_0, \dots, \bar{v}_{n-1}]\,.\end{aligned}$$

Let $\bar{u}_* := \bar{u}_1 \dots \bar{u}_{n-1}$ and $\bar{v}_* := \bar{v}_1 \dots \bar{v}_{n-1}$. Since I is infinite we can find indices $\bar{s}_0, \bar{t}_0, \bar{s}_*, \bar{t}_* \subseteq I$ such that

$$\mathrm{ord}(\bar{s}_0\bar{s}_*) = \mathrm{ord}(\bar{u}_0\bar{u}_*)\,, \quad \mathrm{ord}(\bar{t}_0\bar{t}_*) = \mathrm{ord}(\bar{v}_0\bar{v}_*)\,, \quad \bar{s}_0, \bar{s}_* < \bar{t}_0, \bar{t}_*\,.$$

Since Q is a φ-partition we have

$$\mathfrak{M} \vDash \varphi[\bar{u}_0, \bar{u}_*, \bar{v}_0, \bar{v}_*] \leftrightarrow \varphi[\bar{s}_0, \bar{s}_*, \bar{t}_0, \bar{t}_*]\,.$$

Fix indices $\bar{s}_0', \bar{t}_0'$ such that

$$\mathrm{ord}(\bar{s}_0'\bar{t}_0') = \mathrm{ord}(\bar{s}_0\bar{t}_0) \quad \text{and} \quad \bar{s}_0', \bar{t}_0' < \bar{s}_0, \bar{s}_*\,.$$

Since P is a φ-partition we have

$$\mathfrak{M} \vDash \varphi[\bar{s}_0, \bar{s}_*, \bar{t}_0, \bar{t}_*] \leftrightarrow \varphi[\bar{s}_0', \bar{s}_*, \bar{t}_0', \bar{t}_*]\,.$$

Choose $\bar{s}_0''$ such that $\mathrm{ord}(\bar{s}_0''\bar{t}_0') = \mathrm{ord}(\bar{w}_0\bar{v}_0)$. Since $\mathrm{ord}(\bar{s}_0''\bar{s}_*) = \mathrm{ord}(\bar{s}_0'\bar{s}_*)$ and Q is a φ-partition it follows that

$$\mathfrak{M} \vDash \varphi[\bar{s}_0', \bar{s}_*, \bar{t}_0', \bar{t}_*] \leftrightarrow \varphi[\bar{s}_0'', \bar{s}_*, \bar{t}_0', \bar{t}_*]\,.$$

Finally, let $\bar{s}'_*, \bar{t}'_* \subseteq I$ be indices such that

$$\begin{aligned}
\operatorname{ord}(\bar{s}''_0\bar{s}'_*) &= \operatorname{ord}(\bar{s}''_0\bar{s}_*)\,,\\
\operatorname{ord}(\bar{t}'_0\bar{t}'_*) &= \operatorname{ord}(\bar{t}'_0\bar{t}_*)\,,\\
\operatorname{ord}(\bar{s}'_*\bar{t}'_*) &= \operatorname{ord}(\bar{u}_*\bar{v}_*)\,.
\end{aligned}$$

As Q is a φ-partition we have

$$\mathfrak{M} \vDash \varphi[\bar{s}''_0, \bar{s}_*, \bar{t}'_0, \bar{t}_*] \leftrightarrow \varphi[\bar{s}''_0, \bar{s}'_*, \bar{t}'_0, \bar{t}'_*]\,.$$

Furthermore, $\operatorname{ord}(\bar{s}''_0\bar{t}'_0) = \operatorname{ord}(\bar{w}_0\bar{v}_0)$ and $\operatorname{ord}(\bar{s}'_*\bar{t}'_*) = \operatorname{ord}(\bar{u}_*\bar{v}_*)$ implies that

$$\mathfrak{M} \vDash \varphi[\bar{s}''_0, \bar{s}'_*, \bar{t}'_0, \bar{t}'_*] \leftrightarrow \varphi[\bar{w}_0, \bar{u}_*, \bar{v}_0, \bar{v}_*]\,,$$

because P is a φ-partition. □

Combining the preceding lemmas we obtain the following result.

Theorem 2.13. *For every infinite Δ-indiscernible sequence $(\bar{a}^\nu)_{\nu\in I}$, there exists a unique minimal Δ-partition P.*

Definition 2.14. Let $(\bar{a}^\nu)_{\nu\in I}$ be an infinite Δ-indiscernible sequence of α-tuples and let P be the minimal Δ-partition of α corresponding to $(\bar{a}^\nu)_\nu$.

(a) The elements of P are called *Δ-classes.* Two indices i and k are *Δ-dependent* if $i \approx_P k$. Otherwise, they are *Δ-independent.*

(b) If Δ is the set of all first-order formulae over U we also also speak of *U-partitions, U-classes,* and *U-independent* indices.

Remark. Note that, if $i < \alpha$ is an index such that no variable x^l_i appears in Δ then $\{i\}$ is a Δ-class. Hence, if Δ is finite then every Δ-class is finite.

Remark. Let $(\bar{a}^\nu)_{\nu\in I}$ be an infinite indiscernible sequence over U. For every U-class p, the sequence $(\bar{a}^\nu|_p)_{\nu\in I}$ is indiscernible over $U \cup \bar{a}|_{\alpha\smallsetminus p}[I]$.

We adopt the usual convention of working in a sufficiently saturated monster model $\mathbb{M}$ into which we can embed every model $\mathfrak{M}$ under consideration. All elements and sets are tacitly assumed to be contained in $\mathbb{M}$. By an *U-automorphism,* we mean an automorphism π of $\mathbb{M}$ with $\pi|_U = \mathrm{id}_U$. We will frequently use the following standard facts from model theory.

Lemma 2.15. *Let $(\bar{a}^\nu)_{\nu\in I}$ be an infinite indiscernible sequence over U and let P be its minimal U-partition. For every family $(\beta_p)_{p\in P}$ of strictly increasing maps $\beta_p : I \to I$, there exists a U-automorphism π such that*

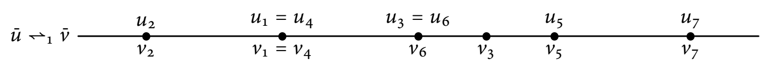
$$\pi(\bar{a}^\nu|_p) = \bar{a}^{\beta_p(\nu)}|_p\,, \quad \text{for all } p \in P \text{ and } \nu \in \operatorname{dom}\beta_p\,.$$

Lemma 2.16. *Let $(\bar{a}^\nu)_{\nu\in I}$ be an indiscernible sequence over U. For every order embedding $\alpha : I \to J$ there exists an indiscernible sequence $(\bar{b}^\nu)_{\nu\in J}$ over U such that $\bar{b}^{\alpha(\nu)} = \bar{a}^\nu$, for $\nu \in I$.*

An argument we will frequently employ below is worth singling out. Suppose we are given a sequence $x_0, \dots, x_n$ where x_0 has some property P while x_n does not. Then there is some index i with $x_i \in P$ and $x_{i+1} \notin P$. For instance, if $x_0 = u_0 \dots u_{m-1}$ and $x_n = v_0 \dots v_{m-1}$ are tuples then we can use the sequence $x_i := v_0 \dots v_{i-1}u_i \dots u_{m-1}$ to conclude that there are tuples $\bar{u}' \in P$ and $\bar{v}' \notin P$ that differ in exactly one component. A more involved example that appears in the proof of the next theorem is the following one. For an ordered index set I, indices $\bar{u}, \bar{v} \in I^n$, and a number $m < n$, we define $\bar{u} \rightharpoondown_m \bar{v}$ iff there exists some $k < n$ such that

- $u_k \neq v_k$ and $u_i = v_i$, for $i \neq k$, and
- either there are exactly m indices $i \neq k$ with $u_i = u_k$ and there is no i with

  $$u_k < u_i \leq v_k \quad \text{or} \quad v_k \leq u_i < u_k\,,$$

  or there are exactly m indices $i \neq k$ with $v_i = v_k$ and there is no i with

  $$v_k < v_i \leq u_k \quad \text{or} \quad u_k \leq v_i < v_k\,.$$

Let $\rightharpoondown_{<m} := \rightharpoondown_0 \cup \dots \cup \rightharpoondown_{m-1}$.

Lemma 2.17. *If I is densely ordered then any two tuples $\bar{u}, \bar{v} \in I^n$ are connected by a $\rightharpoondown_{<n}$-path.*

Proof. For a contradiction, suppose that $\bar{u}$ and $\bar{v}$ are not connected. As explained above we may assume that $\bar{u}$ and $\bar{v}$ differ in exactly one component. Say $\bar{u} = x\bar{z}$ and $\bar{v} = y\bar{z}$. Since the definition of $\rightharpoondown_m$ is invariant under permutations of the tuples we may assume that $\bar{z}$ is increasing and

$$z_0 \leq \dots \leq z_{k-1} \leq x \leq z_k \leq \dots \leq z_{l-1} \leq y \leq z_l \leq \dots \leq z_{n-2}\,.$$

We choose k and l such that $x < z_k$ and $z_{l-1} < y$. We derive a contradiction by induction on $l - k$. If $k = l$ then we have

$$x\bar{z} \rightleftharpoons_{<n} y\bar{z}\,.$$

Contradiction. Suppose that $k < l$. We claim that

$$x\bar{z} \rightleftharpoons_{<n} \cdots \rightleftharpoons_{<n} z_k\bar{z}\,.$$

Hence, the result follows by induction hypothesis. If $z_{k-1} = x < z_k$ then we can take any element $z_{k-1} < x' < z_k$ and it follows that

$$x\bar{z} \rightleftharpoons_{<n} x'\bar{z} \rightleftharpoons_{<n} z_k\bar{z}\,.$$

If $z_{k-1} < x < z_k$ then we immediately have

$$x\bar{z} \rightleftharpoons_{<n} z_k\bar{z}\,. \qquad \square$$

After these preparations we can finally prove that, for every Δ-class p, we can define the relation $\{\, \bar{a}^v|_p \mid v \in I \,\}$ with the help of monadic parameters. In the constructions below this will allow us to replace sequences $(\bar{a}^v)_v$ of tuples by sequences $(a_0^v)_v$ of singletons.

Theorem 2.18. *Suppose that $(\bar{a}^v)_{v\in I}$ is an infinite φ-indiscernible sequence of α-tuples where φ has r free variables. For each φ-class p and every finite subset $q \subseteq p$, there exists a formula $\chi_q(\bar{x}; \bar{y}, \bar{z}, \bar{Z})$ with the following property.*

If $\bar{s}, \bar{t} \in I^r$ are strictly increasing r-tuples with $\bar{s} < \bar{t}$ and

$$A_i := \{\, a_i^v \mid v \in I\,,\ \bar{s} < v < \bar{t} \,\}\,, \quad \text{for } i \in p\,,$$

then we have

$$\mathfrak{M} \vDash \chi_q(\bar{c}; \bar{a}[\bar{s}], \bar{a}[\bar{t}], \bar{A}) \quad \text{iff} \quad \bar{c} = \bar{a}^v|_q \text{ for some } v \in I \text{ with } \bar{s} < v < \bar{t}\,.$$

Proof. The proof is based on [9, Fact II.1.5]. We prove the claim by induction on $n := |q|$. For $q = \{i\}$, we can set

$$\chi_q(x) := A_i x\,.$$

Furthermore, if q and q' are sets such that $q \cap q' \neq \varnothing$ and χ_q and $\chi_{q'}$ exist, then we can define

$$\chi_{q\cup q'}(\bar{x}\bar{y}\bar{z}) := \chi_q(\bar{x}\bar{y}) \wedge \chi_{q'}(\bar{y}\bar{z})\,,$$

where the variables $\bar{x}$ correspond to the elements of $q \setminus q'$, $\bar{y}$ to $q \cap q'$, and $\bar{z}$ to $q' \setminus q$.

Consequently, there exists a partition $p = q_0 \cup \cdots \cup q_{n-1}$ such that each q_i is a maximal subset of p with the property that χ_{q_i} exists. We have to show that $n = 1$ and $q_0 = p$. Let $\bar{b}_* := \bar{a}^v|_{\alpha\setminus p}$, for an arbitrary index v. For $\bar{v} \in I^n$, we define

$$\varphi[\bar{v}] := \varphi(\bar{a}^{v_0}|_{q_0}, \dots, \bar{a}^{v_{n-1}}|_{q_{n-1}}, \bar{b}_*)\,.$$

We will show that

$$\mathfrak{M} \vDash \varphi[\bar{u}] \leftrightarrow \varphi[\bar{v}]\,, \quad \text{for all } \bar{u}, \bar{v} \in I^n.$$

It follows that each q_i is a φ-class which implies that $q_i = p$.

By Lemma 2.17 and the remarks preceding it, it is sufficient to prove that

$$\bar{u} \rightleftharpoons_m \bar{v} \quad \text{implies} \quad \mathfrak{M} \vDash \varphi[\bar{u}] \leftrightarrow \varphi[\bar{v}]\,.$$

We prove this claim by induction on m. Let k be the index witnessing the fact that $\bar{u} \rightleftharpoons_m \bar{v}$. By symmetry, we may assume that $\bar{u}$ is increasing, that $u_k < v_k$, and that $u_k \in \{\, u_i \mid i \neq k \,\}$. Hence, we have

$$u_0 \leq \cdots \leq u_{k-m-1} < u_{k-m} = \cdots = u_k < v_k < u_{k+1} \leq \cdots \leq u_{n-1}\,.$$

Define

$$\bar{s} := u_0 \dots u_{k-m-1}\,, \quad u := u_k\,, \quad v := v_k\,, \quad \bar{t} := u_{k+1} \dots u_{n-1}\,,$$

and set $\bar{b}_- := \bar{a}^{u_0}|_{q_0} \dots \bar{a}^{u_{k-m-1}}|_{q_{k-m-1}}$ and $\bar{b}_+ := \bar{a}^{u_{k+1}}|_{q_{k+1}} \dots \bar{a}^{u_{n-1}}|_{q_{n-1}}$.

For $m = 0$, the claim follows immediately by indiscernibility of $(\bar{a}^v)_v$. Suppose that $m = 1$ and that

$$\mathfrak{M} \vDash \varphi[\bar{s}, u, u, \bar{t}] \wedge \neg\varphi[\bar{s}, u, v, \bar{t}]\,.$$

If $\mathfrak{M} \vDash \neg\varphi[\bar{s}, v, u, \bar{t}]$ then we have

$$\mathfrak{M} \vDash \varphi[\bar{s}, x, y, \bar{t}] \quad \text{iff} \quad x = y\,,$$

and we can define

$$\chi_{q_{k-1}\cup q_k}(\bar{x}, \bar{y}) := \chi_{q_{k-1}}(\bar{x}) \wedge \chi_{q_k}(\bar{y}) \wedge \varphi(\bar{b}_-, \bar{x}, \bar{y}, \bar{b}_+, \bar{b}_*)\,,$$

in contradiction to our choice of q_k.

Thus, we have $\mathfrak{M} \vDash \varphi[\bar{s}, v, u, \bar{t}]$. This implies that

$$\mathfrak{M} \vDash \varphi[\bar{s}, x, y, \bar{t}] \quad \text{iff} \quad x \geq y\,.$$

As in Lemma 2.2, we obtain a formula

$$\vartheta(\bar{x}, \bar{x}') := \forall \bar{y}[\chi_{q_k}(\bar{y}) \wedge \varphi(\bar{b}_-, \bar{x}, \bar{y}, \bar{b}_+, \bar{b}_*) \rightarrow \varphi(\bar{b}_-, \bar{x}', \bar{y}, \bar{b}_+, \bar{b}_*)]$$

such that

$$\mathfrak{M} \vDash \vartheta(\bar{a}^x|_{q_{k-1}}, \bar{a}^y|_{q_{k-1}}) \quad \text{iff} \quad x \leq y\,,$$

and we can set

$$\begin{aligned}\chi_{q_{k-1} \cup q_k}(\bar{x}, \bar{y}) := {} & \chi_{q_{k-1}}(\bar{x}) \wedge \chi_{q_k}(\bar{y}) \\ & \wedge \forall \bar{x}'[\chi_{q_{k-1}}(\bar{x}') \wedge \varphi(\bar{b}_-, \bar{x}', \bar{y}, \bar{b}_+, \bar{b}_+) \rightarrow \vartheta(\bar{x}', \bar{x})]\,.\end{aligned}$$

Contradiction.

It remains to consider the case that $m > 1$. Again, assume that

$$\mathfrak{M} \vDash \varphi[\bar{s}, u \dots u, u, \bar{t}] \wedge \neg\varphi[\bar{s}, u \dots u, v, \bar{t}]\,.$$

By indiscernibility, the former implies that

$$\mathfrak{M} \vDash \varphi[\bar{s}, w \dots w, \bar{t}]\,, \qquad \text{for all } w \in I \text{ with } \bar{s} < w < \bar{t}\,.$$

On the other hand, if $\bar{w} \in I^{m+1}$ is a tuple such that $\bar{s} < \bar{w} < \bar{t}$ and $|\mathrm{rng}\, w| > 1$ then $\bar{s}\bar{w}\bar{t} \leftharpoondown_{<m} \bar{s}u \dots uv\bar{t}$. Hence, by induction hypothesis, we have

$$\mathfrak{M} \vDash \neg\varphi[\bar{s}, \bar{w}, \bar{t}]\,, \qquad \text{for all such } \bar{w}\,.$$

Consequently, we have

$$\mathfrak{M} \vDash \varphi[\bar{s}, \bar{w}, \bar{t}] \quad \text{iff} \quad w_0 = \dots = w_m\,,$$

and we can define

$$\begin{aligned}\chi_{q_{k-m} \cup \dots \cup q_k}(\bar{x}_0, \dots, \bar{x}_{k-m}) := {} & \chi_{q_{k-m}}(\bar{x}_0) \wedge \dots \wedge \chi_{q_k}(\bar{x}_{k-m}) \\ & \wedge \varphi(\bar{b}_-, \bar{x}_0, \dots, \bar{x}_{k-m}, \bar{b}_+, \bar{b}_*)\,,\end{aligned}$$

in contradiction to our choice of q_k. □

3 Pairing functions and coding

In [1] Baldwin and Shelah argue that the monadic theories of structures are hopelessly complicated if they *admit coding,* i.e., if they contain a first-order definable pairing function. Then they proceed by classifying the remaining structures by their first-order theories. Baldwin and Shelah show that, if the first-order theory is stable then structures that do not admit coding have a tree-like decomposition with countable components. The unstable case is considered in [9] but the resulting theory remains fragmentary.

In a forthcoming article [2] (see also [5]) we will complete the picture by proving that every structure that does not admit coding is tree-like in the sense that it has a so-called 'partition refinement' of bounded (though infinite) width. This also gives an alternative proof of the already known results on stable structures.

In the present article we develop the structure theory needed for this result. We start by collecting conditions that imply the definability of a pairing function. Special emphasis is placed on indiscernible sequences. In this section we present the needed definitions and results from [1], together with some simple consequences. The next section contains mostly new results.

Definition 3.1. A structure $\mathfrak{M}$ *admits coding* if there exist an elementary extension $\mathfrak{N} \succeq \mathfrak{M}$, unary predicates $\bar{P}$, and infinite sets $A, B, C \subseteq N$ such that in the structure $(\mathfrak{N}, \bar{P})$ there exists a first-order definable bijection $A \times B \rightarrow C$.

An alternative characterisation of coding is based on the existence of two equivalence relations.

Lemma 3.2. *Suppose that $\varphi(x, y)$ and $\psi(x, y)$ are formulae (with monadic parameters) and $(c^{uv})_{u,v<\omega}$ are elements such that*

$$\mathfrak{M} \vDash \varphi(c^{uv}, c^{st}) \quad \text{iff} \quad u = s\,,$$
$$\mathfrak{M} \vDash \psi(c^{uv}, c^{st}) \quad \text{iff} \quad v = t\,.$$

Then $\mathfrak{M}$ admits coding.

Proof. The formula $\chi(x, y, z) := \varphi(x, z) \wedge \psi(y, z)$ defines the bijection

$$\{ c^{u0} \mid u < \omega \} \times \{ c^{0v} \mid v < \omega \} \rightarrow \{ c^{uv} \mid u, v < \omega \}$$

sending the pair (c^{u0}, c^{0v}) to c^{uv}. □

A first simple criterion for coding is the independence property.

Definition 3.3. Let T be a first-order theory. A formula $\varphi(\bar{x}, \bar{y})$ has the *independence property* (w.r.t. T) if there exists a model $\mathfrak{M}$ of T containing sequences $(\bar{a}_X)_{X \subseteq \omega}$ and $(\bar{b}_i)_{i<\omega}$ such that

$$\mathfrak{M} \vDash \varphi(\bar{a}_X, \bar{b}_i) \quad \text{iff} \quad i \in X\,.$$

We say that a structure $\mathfrak{M}$ has the independence property if there exists a formula φ that has the independence property w.r.t. $\mathrm{Th}(\mathfrak{M})$. If $\bar{a}_X$ and $\bar{b}_i$ are singletons we say that $\mathfrak{M}$ has the *independence property on singletons.*

Lemma 3.4. *If $\mathfrak{M}$ has the independence property on singletons then it admits coding.*

Proof. Fix sequences $(a_X)_{X \subseteq \omega}$ and $(b_i)_{i \in \omega}$ and a formula $\varphi(x, y)$ such that

$$\mathbb{M} \vDash \varphi(a_X, b_i) \quad \text{iff} \quad i \in X\,.$$

Fix disjoint infinite sets $U, V \subseteq \{\, b_i \mid i < \omega \,\}$ and define $f : U \times V \to M$ by $f(b_i, b_k) := a_{\{i,k\}}$. Then we have

$$f(x, y) = z \quad \text{iff} \quad \mathbb{M} \vDash \varphi(z, x) \wedge \varphi(z, y)\,,$$

for $x \in U$, $y \in V$, and $z \in f(U, V)$. □

In [1] it is shown that the independence property and the independence property on singletons coincide if we allow unary predicates.

Lemma 3.5 (Baldwin, Shelah). *Suppose that $\mathfrak{M}$ has the independence property. There exists an elementary extension $\mathfrak{N} \succeq \mathfrak{M}$ and unary predicates $\bar{P}$ such that $(\mathfrak{N}, \bar{P})$ has the independence property on singletons.*

Consequently, the independence property implies coding.

Corollary 3.6 (Baldwin, Shelah). *If $\mathfrak{M}$ has the independence property then it admits coding.*

Related to the notion of coding is the notion of dp-minimality. Intuitively, a first-order theory is dp-minimal if there is no pairing function that is first-order definable *without monadic parameters.* In particular, every theory that does not admit coding is dp-minimal. The precise definition of dp-minimality is as follows (see [7]).

Definition 3.7. A first-order theory T is *not dp-minimal* if we can find first-order formulae $\varphi(x, \bar{y})$ and $\psi(x, \bar{y})$ (without parameters) and some model $\mathfrak{M}$ of T containing two indiscernible sequences $(\bar{a}^n)_{n<\omega}$ and $(\bar{b}^n)_{n<\omega}$ and a family $(c^{ik})_{i,k<\omega}$ such that

$$\begin{aligned} & \mathfrak{M} \vDash \varphi(c^{ik}, \bar{a}^n) \quad \text{iff} \quad n = i\,, \\ \text{and} \quad & \mathfrak{M} \vDash \psi(c^{ik}, \bar{b}^n) \quad \text{iff} \quad n = k\,. \end{aligned}$$

It will follow from the results of the next section that every theory that does not admit coding is dp-minimal. But there are dp-minimal theories that do admit coding. For instance, one can show that every o-minimal theory is dp-minimal. Hence, the theory of ordered abelian groups and the theory of real closed fields are dp-minimal, while they do admit coding.

4 Indiscernibles

This final section is concerned with the following question. Given an indiscernible sequence $(\bar{a}^v)_{v \in I}$ and an arbitrary element c what is their relationship? Is the sequence also indiscernible over c or can one distinguish intervals of I with the help of c? (We use the term 'interval' for arbitrary convex subsets. We do not require the existence of a supremum or infimum.) As an example we give a characterisation of the independence property in these terms, which is basically due to Shelah (see [10] and [11].)

Definition 4.1. Let $\varphi(\bar{x})$ be a formula and $(\bar{a}^v)_{v \in I}$ a sequence. We define

$$[\![\varphi(\bar{a}^v)]\!]_{v \in I} := \{\, v \in I \mid \mathbb{M} \vDash \varphi(\bar{a}^v) \,\}\,.$$

Lemma 4.2 (Shelah). *A formula $\varphi(\bar{x}, \bar{y})$ has the independence property if and only if there exists an indiscernible sequence $(\bar{a}^v)_{v \in I}$ and a tuple $\bar{c}$ such that the set $[\![\varphi(\bar{c}, \bar{a}^v)]\!]_{v \in I}$ cannot be written as union of finitely many intervals.*

Proof. $(\Rightarrow)$ Let $(\bar{a}_i)_{i<\omega}$ and $(\bar{b}_X)_{X \subseteq \omega}$ be sequences such that

$$\mathbb{M} \vDash \varphi(\bar{b}_X, \bar{a}_i) \quad \text{iff} \quad i \in X\,.$$

By compactness, we may assume that $(\bar{a}_i)_{i<\omega}$ is indiscernible. Take the set $X := \{\, 2i \mid i < \omega \,\}$ of even numbers and set $\bar{c} := \bar{b}_X$. Then $[\![\varphi(\bar{c}, \bar{a}_i)]\!]_i$ has the desired property.

(⇐) Fix a strictly increasing or a strictly decreasing subsequence $(u_i)_{i<\omega}$ of I such that every interval (u_i, u_{i+1}) contains elements of both $[\![\varphi(\bar{c}, \bar{a}^v)]\!]_v$ and $[\![\neg\varphi(\bar{c}, \bar{a}^v)]\!]_v$. Let $J := \{ u_i \mid i < \omega \}$ and set $\bar{b}_i := \bar{a}^{u_i}$. For every set $X \subseteq \omega$, we can fix a strictly increasing function $\alpha_X : J \to I$ such that

$$\alpha_X(u_i) \in [\![\varphi(\bar{c}, \bar{a}^v)]\!]_v \quad \text{iff} \quad i \in X .$$

Let π_X be an automorphism such that $\pi(\bar{a}^{\alpha_X(v)}) = \bar{a}^v$, for $v \in J$, and set $\bar{c}_X := \pi(\bar{c})$. Then it follows that

$$\mathbb{M} \vDash \varphi(\bar{c}_X, \bar{b}_i) \quad \text{iff} \quad i \in X .$$

Consequently, φ has the independence property. □

Corollary 4.3. *Let $(\bar{a}^v)_{v \in I}$ be an infinite indiscernible sequence and $\bar{c}$ a tuple such that the sets*

$$[\![\varphi(\bar{c}, \bar{a}^v)]\!]_{v \in I} \quad \textit{and} \quad [\![\neg\varphi(\bar{c}, \bar{a}^v)]\!]_{v \in I}$$

are both infinite. If $(\bar{a}^v)_v$ is totally indiscernible then $\mathfrak{M}$ admits coding.

Proof. By taking a suitable subsequence we may assume that I is countable. We can choose a bijection $\alpha : \mathbb{Q} \to I$ such that the sets

$$[\![\varphi(\bar{c}, \bar{a}^{\alpha(v)})]\!]_{v \in \mathbb{Q}} \quad \text{and} \quad [\![\neg\varphi(\bar{c}, \bar{a}^{\alpha(v)})]\!]_{v \in \mathbb{Q}}$$

are dense in $\mathbb{Q}$. If $(\bar{a}^v)_{v \in I}$ is totally indiscernible then so is the rearranged sequence $(\bar{a}^{\alpha(v)})_{v \in \mathbb{Q}}$. By the preceding lemma it follows that φ has the independence property. □

In order to develop a structure theory for structures that do not admit coding we investigate indiscernible sequences. In the following we derive a sequence of lemmas containing more and more strict conditions on definable intervals of indiscernible sequence. We will prove that the U-classes of such an indiscernible sequence are not affected if we add a new element c to U, i.e., every U-class is also a $(U \cup \{c\})$-class. The main result of this section states that, if the structure in question does not admit coding, then we can extend each indiscernible sequence $(\bar{a}^v)_{v \in I}$ to cover every given set, i.e., we can find an indiscernible sequence $(\bar{b}^v)_{v \in J}$ with $I \subseteq J$ and $\bar{b}^v \supseteq \bar{a}^v$, for $v \in I$, such that $\bar{b}[J]$ contains the given set. As a consequence it follows that every structure without coding has a basically linear structure.

Note that the result obviously fails for arbitrary structures. For instance, if $(\bar{a}_X)_{X \subseteq \omega}$ and $(\bar{b}_i)_{i<\omega}$ witness the independence property, then we cannot extend $(\bar{b}_i)_i$ to include the element $\bar{a}_{\{1,5\}}$.

Let us start with a simple example that illuminates the general structure of the more involved arguments below. Given two indiscernible sequences $(a^v)_{v \in I}$ and $(b^v)_{v \in I}$ with certain additional properties, we construct a family $(c^{uv})_{u,v \in I}$ and a definable bijection $(a_u, b_v) \mapsto c^{uv}$.

Lemma 4.4. *Let $(a^v)_{v \in \mathbb{Z}}$ and $(b^v)_{v \in \mathbb{Z}}$ be sequences such that $(a^v)_v$ is indiscernible over $U \cup b[\mathbb{Z}]$ and $(b^v)_v$ is indiscernible over $U \cup a[\mathbb{Z}]$. If there exist an element c, formulae $\varphi(x, y)$ and $\psi(x, y)$ over U, and relations $\rho, \sigma \in \{=, \neq, \leq, \geq, <, >\}$ such that*

$$\begin{aligned} &\mathfrak{M} \vDash \varphi(a^v, c) \quad \text{iff} \quad v \,\rho\, 0 \\ \textit{and} \quad &\mathfrak{M} \vDash \psi(b^v, c) \quad \text{iff} \quad v \,\sigma\, 0 , \end{aligned}$$

then $\mathfrak{M}$ admits coding.

Proof. Let π_{st} be an U-automorphism such that

$$\pi_{st}(a^v) = a^{v+s} \quad \text{and} \quad \pi_{st}(b^v) = b^{v+t},$$

and set $c^{st} := \pi_{st}(c)$. It follows that

$$\mathfrak{M} \vDash \varphi(a^v, c^{st}) \quad \text{iff} \quad \mathfrak{M} \vDash \varphi(a^{v-s}, c) \quad \text{iff} \quad v - s \,\rho\, 0 \quad \text{iff} \quad v \,\rho\, s ,$$

and similarly

$$\mathfrak{M} \vDash \psi(b^v, c^{st}) \quad \text{iff} \quad v \,\sigma\, t .$$

Let $A := \{ a^v \mid v \in I \}$ and $B := \{ b^v \mid v \in I \}$. We can construct formulae $\chi(x, y)$ and $\vartheta(x, y)$ such that

$$\begin{aligned} \mathfrak{M} &\vDash \chi(c^{uv}, c^{st}) \quad \text{iff} \quad u = s , \\ \mathfrak{M} &\vDash \vartheta(c^{uv}, c^{st}) \quad \text{iff} \quad v = t , \end{aligned}$$

by setting

$$\begin{aligned} \chi(x, y) &:= (\forall z.Az)[\varphi(z, x) \leftrightarrow \varphi(z, y)] , \\ \vartheta(x, y) &:= (\forall z.Bz)[\psi(z, x) \leftrightarrow \psi(z, y)] . \end{aligned}$$

By Lemma 3.2 it follows that $\mathfrak{M}$ admits coding. □

The following criterion for coding appears in [9].

Lemma 4.5 (Shelah). *Let $(\bar a^v)_{v\in I}$ be an infinite indiscernible sequence over U. Suppose that there exists a U-class p, an element $c \in \mathbb{M}$, a formula ψ over U, and indices $s < t$ such that*

- $\mathfrak{M} \vDash \psi(c, \bar a^s|_p, \bar a^t|_p)$,
- $\mathfrak{M} \vDash \neg\psi(c, \bar a^s|_p, \bar a^v|_p)$ *for infinitely many* $v > t$,
- $\mathfrak{M} \vDash \neg\psi(c, \bar a^v|_p, \bar a^t|_p)$ *for infinitely many* $v < s$.

Then $\mathfrak{M}$ admits coding.

In the preceding lemma we have considered the case that the truth value of φ changes if we move the index v outside the interval $[s, t]$. The next lemma states a dual version of this result where we consider instead indices $v \in (s, t)$.

Lemma 4.6. *Let $(\bar a^v)_{v\in I}$ be an infinite indiscernible sequence over U. If there exist an element c, a U-class p, a formula φ, and indices $s < t$ such that*

$$\mathfrak{M} \vDash \varphi(c, \bar a^s|_p) \wedge \varphi(c, \bar a^t|_p),$$
$$\mathfrak{M} \vDash \neg\varphi(c, \bar a^v|_p), \quad \text{for infinitely many } s < v < t,$$

then $\mathfrak{M}$ admits coding.

Proof. W.l.o.g. assume that $\bar a^v|_p = \bar a^v$. By Ramsey's theorem and compactness, we may assume that $I = \mathbb{R}$ and

$$\begin{aligned} &\mathfrak{M} \vDash \neg\varphi(c, \bar a^v), && \text{for all } s < v < t,\\ &\mathfrak{M} \vDash \varphi(c, \bar a^u) \leftrightarrow \varphi(c, \bar a^v), && \text{for all } u, v < s,\\ &\mathfrak{M} \vDash \varphi(c, \bar a^u) \leftrightarrow \varphi(c, \bar a^v), && \text{for all } u, v > t. \end{aligned}$$

For $u < v$, fix an order isomorphism $\alpha_{uv} : I \to I$ with $\alpha(s) = u$ and $\alpha(t) = v$ and let π_{uv} be an U-automorphism such that $\pi_{uv}(\bar a^x) = \bar a^{\alpha(x)}$. We set $c^{uv} := \pi_{uv}(c)$. Fix a partition $I = I_0 \cup I_1$ into infinite sets I_0 and I_1 with $I_0 < I_1$, $s \in I_0$ and $t \in I_1$.

First, consider the case that $\mathfrak{M} \vDash \varphi(c, \bar a^v)$, for all $v < s$. We can define the order of $(\bar a^v)_{v\in I_0}$ by

$$\vartheta(\bar x, \bar y) := (\forall z.Cz)[\varphi(z, \bar x) \to \varphi(z, \bar y)],$$

where $C := \{\, c^{uv} \mid u \in I_0,\ v \in I_1 \,\}$. Let $\chi(\bar x)$ be a formula with monadic parameters such that

$$\mathfrak{M} \vDash \chi(\bar b) \quad \text{implies} \quad \bar b = \bar a^v, \text{ for some } v \in I_0.$$

For the formula

$$\psi(z, \bar x) := \varphi(z, \bar x) \wedge \forall \bar y[\chi(\bar y) \wedge \vartheta(\bar x, \bar y) \to \neg\varphi(z, \bar y)]$$

we have

$$[\![\psi(c^{st}, \bar a^v)]\!]_{v\in I} = [\![\varphi(c^{st}, \bar a^v)]\!]_{v\in I} \smallsetminus (-\infty, s), \quad \text{for } s \in I_0 \text{ and } t \in I_1.$$

Similarly, if $\mathfrak{M} \vDash \varphi(c^{st}, \bar a^v)$, for all $v > t$, then we can construct a formula ψ such that

$$[\![\psi(c^{st}, \bar a^v)]\!]_{v\in I} = [\![\varphi(c^{st}, \bar a^v)]\!]_{v\in I} \smallsetminus (t, \infty).$$

Consequently, we can assume that

$$[\![\varphi(c^{st}, \bar a^v)]\!]_v = \{s, t\}, \quad \text{for all } s \in I_0 \text{ and } t \in I_1.$$

For all $s, u \in I_0$ and $t, v \in I_1$, it follows that

$$\mathfrak{M} \vDash \varphi(c^{st}, \bar a^u) \wedge \varphi(c^{st}, \bar a^v) \quad \text{iff} \quad u = s \text{ and } v = t.$$

Let $\chi(\bar x)$ be a formula with monadic parameters such that

$$\mathfrak{M} \vDash \chi(\bar b) \quad \text{implies} \quad \bar b = \bar a^v, \text{ for some } v \in I_0.$$

It follows that the formula

$$\psi(x, y, z) := \exists \bar x' \exists \bar y'[\chi(x\bar x') \wedge \chi(y\bar y') \wedge \varphi(z, x\bar x') \wedge \varphi(z, y\bar y')].$$

defines the bijection $(a_0^u, a_0^v) \mapsto c^{uv}$, for $u \in I_0$ and $v \in I_1$. □

For sequences $(\bar a^v)_v$ with a single U-class, it follows that, in the absence of coding, the structure of sets of the form $[\![\varphi(c, \bar a^v)]\!]_v$ is quite simple.

Corollary 4.7. *Suppose that $\mathfrak{M}$ does not admit coding and let $(\bar a^v)_{v\in I}$ be an indiscernible sequence over U where the order I has no minimal and no maximal element.*

For every U-class p, each element c, and all formulae $\varphi(x, \bar y)$ over U, one of the following cases holds:

- $\big|[\![\varphi(c, \bar a^v|_p)]\!]_v\big| \leq 1$
- $\big|[\![\neg\varphi(c, \bar a^v|_p)]\!]_v\big| \leq 1$

- $[\![\varphi(c, \bar{a}^v|_p)]\!]_v$ *is an initial segment of* I.
- $[\![\varphi(c, \bar{a}^v|_p)]\!]_v$ *is a final segment of* I.

Proof. We simplify notation by setting $[\![\varphi]\!] := [\![\varphi(c, \bar{a}^v|_p)]\!]_v$ and similarly for $[\![\neg\varphi]\!]$. Suppose that $[\![\varphi]\!]$ and $[\![\neg\varphi]\!]$ both contain at least two elements. We consider three cases.

(a) Suppose that, for every $v \in I$ there are elements $u, u' \in [\![\varphi]\!]$ with $u < v < u'$. We fix indices $s, t \in [\![\neg\varphi]\!]$ with $s < t$. The formula

$$\psi(z, \bar{x}, \bar{y}) := \neg\varphi(z, \bar{x}) \wedge \neg\varphi(z, \bar{y})$$

and the indices $s < t$ satisfy the conditions of Lemma 4.5. Hence, $\mathfrak{M}$ admits coding. A contradiction.

(b) If, for every $v \in I$, there are elements $u, u' \in [\![\neg\varphi]\!]$ with $u < v < u'$ then we obtain a contradiction as in (a) by exchanging φ and $\neg\varphi$.

(c) It follows that there are indices $s \leq t$ such that either

$$(-\infty, s) \subseteq [\![\varphi]\!] \quad \text{and} \quad (t, \infty) \subseteq [\![\neg\varphi]\!],$$

or

$$(-\infty, s) \subseteq [\![\neg\varphi]\!] \quad \text{and} \quad (t, \infty) \subseteq [\![\varphi]\!].$$

By symmetry, we may assume the former. If $s = t$ then we are done.

For a contradiction, suppose that there are elements $s \leq u < v \leq t$ with $u \in [\![\neg\varphi]\!]$ and $v \in [\![\varphi]\!]$. By indiscernibility and compactness, we may assume that I is dense. If $(u, v) \cap [\![\varphi]\!]$ is infinite then $\neg\varphi$ and the pair $u < t$ satisfy the conditions of Lemma 4.6. Otherwise, $(u, v) \cap [\![\neg\varphi]\!]$ is infinite and φ and the pair $s < v$ satisfy these conditions. In both cases it follows that $\mathfrak{M}$ admits coding. Contradiction. □

Remark. If the order I in the corollary is (Dedekind) complete then we can rephrase the statement as follows: there exists an index $s \in I$ and a relation $\sigma \in \{\varnothing, I \times I, =, \neq, \leq, \geq, <, >\}$ such that

$$\mathfrak{M} \vDash \varphi(c, \bar{a}^v|_p) \quad \text{iff} \quad v \,\sigma\, s.$$

In the remainder of this section we generalise this result. We start by considering formulae $\varphi(c, \bar{a}[\bar{v}])$ talking about several elements of the sequence. Then we generalise the results to the case of several U-classes.

Lemma 4.8. *Suppose that* $\mathfrak{M}$ *does not admit coding. Let* $(\bar{a}^v)_{v \in I}$ *be an indiscernible sequence over* U *and* p *a* U*-class. For every element* $c \in \mathbb{M}$*, there exists a linear order* $J \supseteq I$*, an element* $s \in J$*, and an indiscernible sequence* $(\bar{b}^v)_{v \in J}$ *over* U *such that* $\bar{b}^v = \bar{a}^v|_p$*, for* $v \in I$*, and*

$$\mathfrak{M} \vDash \varphi(c, \bar{b}[\bar{u}]) \leftrightarrow \varphi(c, \bar{b}[\bar{v}]),$$

for every formula φ *over* U *and all indices* $\bar{u}, \bar{v} \subseteq J$ *with* $\mathrm{ord}(s\bar{u}) = \mathrm{ord}(s\bar{v})$.

Proof. Replacing $\bar{a}^v$ by $\bar{a}^v|_p$ we may assume that p is the only U-class. Let J be a complete dense order extending I and let $(\bar{b}^v)_{v \in J}$ be an indiscernible sequence extending $(\bar{a}^v)_{v \in I}$.

If $(\bar{b}^v)_v$ is indiscernible over $U \cup \{c\}$ then there is nothing to do. Otherwise, there are a formula φ and tuples $\bar{u}, \bar{v} \subseteq J$ with $\mathrm{ord}(\bar{u}) = \mathrm{ord}(\bar{v})$ such that

$$\mathfrak{M} \vDash \varphi(c, \bar{b}[\bar{u}]) \wedge \neg\varphi(c, \bar{b}[\bar{v}]).$$

We can choose $\bar{u}$ and $\bar{v}$ such that there is exactly one index i with $u_i \neq v_i$. Hence, we may assume w.l.o.g. that $\bar{u} = u\bar{r}^0\bar{r}^1$ and $\bar{v} = v\bar{r}^0\bar{r}^1$ where

$$r^0_0 < \cdots < r^0_{m-1} < u < v < r^1_0 < \cdots < r^1_{l-1}.$$

Fix the interval $J_0 := (r^0_{m-1}, r^1_0) \subseteq J$. The sequence $(\bar{b}^v)_{v \in J_0}$ is indiscernible over $U \cup \bar{b}[\bar{r}^0\bar{r}^1]$. We can apply Corollary 4.7 to the element c and the sequence $(\bar{b}^v)_{v \in J_0}$ to find an index $s \in J_0$ and a relation $\sigma \in \{=, \neq, <, \leq\}$ such that

$$\mathfrak{M} \vDash \varphi(c, \bar{b}^x, \bar{b}[\bar{r}^0\bar{r}^1]) \quad \text{iff} \quad x \,\sigma\, s, \quad \text{for all } x \in J_0.$$

We claim that s is the desired index.

Suppose otherwise. Then there is some formula ψ and indices $\bar{u}, \bar{v} \subseteq J$ with $\mathrm{ord}(s\bar{u}) = \mathrm{ord}(s\bar{v})$ such that

$$\mathfrak{M} \vDash \psi(c, \bar{b}[\bar{u}]) \wedge \neg\psi(c, \bar{b}[\bar{v}]).$$

Again we may assume that $\bar{u} = u\bar{r}^2\bar{r}^3$ and $\bar{v} = v\bar{r}^2\bar{r}^3$ with $r^2_0 < \cdots < r^2_{m'-1} < u < v < r^3_0 < \cdots < r^3_{l'-1}$. Let $J_1 := (r^2_{m'-1}, r^3_0) \subseteq J$. As above there is some index $t \in J_1$ and some $\rho \in \{=, \neq, <, \leq\}$ such that

$$\mathfrak{M} \vDash \psi(c, \bar{b}^x, \bar{b}[\bar{r}^2\bar{r}^3]) \quad \text{iff} \quad x \,\rho\, t.$$

$\mathrm{ord}(su) = \mathrm{ord}(sv)$ and $u \leq t \leq v$ implies that $t \neq s$. Hence, there exist infinite convex subsets $I_0 \subseteq J_0$ and $I_1 \subseteq J_1$ with $s \in I_0$ and $t \in I_1$ such that $I_0 \cap I_1 = \varnothing$,

$\bar r^0\bar r^1 \cap I_1 = \varnothing$, and $\bar r^2\bar r^3 \cap I_0 = \varnothing$. Furthermore, there are formulae $\varphi'(x,\bar y)$ and $\psi'(x,\bar y)$ with monadic parameters such that

$$\begin{aligned}\mathfrak{M} &\vDash \varphi'(c,\bar b^x) \quad\text{iff}\quad x = s\,, &&\text{for all } x \in I_0\,,\\ \mathfrak{M} &\vDash \psi'(c,\bar b^x) \quad\text{iff}\quad x = t\,, &&\text{for all } x \in I_1\,.\end{aligned}$$

For $u \in I_0$ and $v \in I_1$, fix order isomorphisms $\alpha_u : I_0 \to I_0$ and $\beta_v : I_1 \to I_1$ with $\alpha_u(s) = u$ and $\beta_v(t) = v$. Let π_{uv} be a U-automorphism such that

$$\begin{aligned}\pi_{uv}(\bar b^x) &= \bar b^{\alpha_u(x)} &&\text{for } x \in I_0\,,\\ \pi_{uv}(\bar b^x) &= \bar b^{\beta_v(x)} &&\text{for } x \in I_1\,,\\ \pi_{uv}(\bar b^x) &= \bar b^x &&\text{for } x \in J \smallsetminus (I_0 \cup I_1)\,,\end{aligned}$$

and set $c^{uv} := \pi_{uv}(c)$. For $u, s \in I_0$ and $v, t \in I_1$, it follows that

$$\mathfrak{M} \vDash \varphi'(c^{uv},\bar b^s) \wedge \psi'(c^{uv},\bar b^t) \quad\text{iff}\quad u = s \text{ and } v = t\,.$$

Contradiction. □

Lemma 4.9. *Suppose that $\mathfrak{M}$ does not admit coding. Let $(\bar a^v)_{v\in I}$ be an indiscernible sequence over U, $c \in \mathbb{M}$ an element, $\varphi(z,\bar x^0,\dots,\bar x^{m-1})$ a formula over U, and p a U-class. Set*

$$\varphi[c,\bar v] := \varphi(c,\bar a^{v_0}|_p,\dots,\bar a^{v_{m-1}}|_p)\,.$$

If there are indices $\bar u,\bar v \in I^m$ such that

$$\mathfrak{M} \vDash \varphi[c,\bar u] \wedge \neg\varphi[c,\bar v]$$

then there either exists a formula $\vartheta(\bar x,\bar y)$ (with monadic parameters) such that

$$\mathfrak{M} \vDash \vartheta(\bar a^x|_p,\bar a^y|_p) \quad\text{iff}\quad x \leq y\,,$$

or there exist an index $s \in I$ such that $\operatorname{equ}(s\bar x) = \operatorname{equ}(s\bar y)$ *implies*

$$\mathfrak{M} \vDash \varphi[c,\bar x] \leftrightarrow \varphi[c,\bar y]\,.$$

Proof. By Lemma 4.8, there is an index s such that the truth value of $\varphi[c,\bar x]$ only depends on $\operatorname{ord}(s\bar x)$. Suppose that there are indices $\bar u,\bar v \in I^m$ with $\operatorname{equ}(s\bar u) = \operatorname{equ}(s\bar v)$ and

$$\mathfrak{M} \vDash \varphi[c,\bar u] \wedge \neg\varphi[c,\bar v]\,.$$

We construct a formula ϑ that defines the ordering of I. By adding unused variables to φ we may assume that $s \in \bar u$. Furthermore, by changing φ we may assume that $u_i \neq u_k$ and $v_i \neq v_k$, for $i \neq k$. Let k be the minimal index such that

$$\mathfrak{M} \vDash \neg\varphi[c, v_0 \dots v_k u_{k+1} \dots u_{m-1}]\,.$$

Since

$$\mathfrak{M} \vDash \varphi[c, v_0 \dots v_{k-1} u_k u_{k+1} \dots u_{m-1}] \wedge \neg\varphi[c, v_0 \dots v_{k-1} v_k u_{k+1} \dots u_{m-1}]$$

we may assume that there is some index k such that $u_i = v_i$, for $i \neq k$. W.l.o.g. assume that $k = 0$ and that $u_0 < v_0$. Since $\operatorname{ord}(\bar u) \neq \operatorname{ord}(\bar v)$ there must be at least one index $k > 0$ with $u_0 < u_k < v_0$. By a similar argument as above we may assume that there is exactly one such index. Hence, we may assume that

$$\bar u = ut\bar r_0\bar r_1 \quad\text{and}\quad \bar v = vt\bar r_0\bar r_1 \quad\text{where } \bar r_0 < u < t < v < \bar r_1\,.$$

We consider two cases.

(a) Suppose that $t \neq s$. Then $\operatorname{equ}(s\bar u) = \operatorname{equ}(s\bar v)$ implies that $s \in \bar r_0\bar r_1$. Since $\operatorname{ord}(vu\bar r_0\bar r_1) = \operatorname{ord}(vt\bar r_0\bar r_1)$ it follows that

$$\mathfrak{M} \vDash \varphi[c, uv\bar r_0\bar r_1] \wedge \neg\varphi[c, vu\bar r_0\bar r_1]\,.$$

Fix a linear order $J \supseteq I$ and a strictly increasing function $\alpha : I \to J$ such that $\alpha(\bar r_0) < I < \alpha(\bar r_1)$. Let $(\bar b^v)_{v\in J}$ be an indiscernible sequence extending $(\bar a^v)_{v\in I}$ and fix a U-automorphism π such that $\pi(\bar a^x) = \bar b^{\alpha(x)}$. We set $d := \pi(c)$. For $x, y \in I$ with $x \neq y$ it follows that

$$\mathfrak{M} \vDash \varphi[d, xy\alpha(\bar r_0)\alpha(\bar r_1)] \quad\text{iff}\quad x < y\,.$$

Hence, we can define

$$\vartheta(\bar x,\bar y) := \bar x = \bar y \vee \varphi(d,\bar x,\bar y,\bar b[\alpha(\bar r_0\bar r_1)])\,.$$

(b) It remains to consider the case that $t = s$. Then we have

$$\mathfrak{M} \vDash \varphi[c, us\bar r_0\bar r_1] \wedge \neg\varphi[c, vs\bar r_0\bar r_1]\,.$$

Fix a linear order $J \supseteq I$, tuples $\bar w_0, \bar w_1 \subseteq J$ with $\bar w_0 < I < \bar w_1$, and an indiscernible sequence $(\bar b^v)_{v\in J}$ extending $(\bar a^v)_{v\in J}$. For each $t \in I$, let $\alpha_t : I \to J$ be an order embedding such that $\alpha_t(s\bar{r}_0\bar{r}_1) = t\bar{w}_0\bar{w}_1$ and choose a U-automorphism π_t with $\pi(\bar{a}^x) = \bar{b}^{\alpha(x)}$. Setting $c^t := \pi_t(c)$ it follows, for $x \neq t$, that

$$\mathfrak{M} \models \varphi[c^t, xt\bar{w}_0\bar{w}_1] \quad \text{iff} \quad x < t\,.$$

By Theorem 2.18, there is a formula χ (with monadic parameters) such that

$$\mathfrak{M} \models \chi(c, \bar{a}) \quad \text{iff} \quad \bar{a} = \bar{a}^x \text{ and } c = c^x\,, \text{ for some } x \in I\,.$$

If we define

$$\vartheta(\bar{x}, \bar{y}) := \bar{x} = \bar{y} \vee \exists z\big(\chi(z, \bar{y}) \wedge \varphi(z, \bar{x}, \bar{y}, \bar{b}[\bar{w}_0\bar{w}_1])\big)$$

it follows that

$$\mathfrak{M} \models \vartheta(\bar{a}^x, \bar{a}^y) \quad \text{iff} \quad x \leq y\,. \qquad \square$$

Next we consider the case that there are several U-classes. The following lemma roughly states that, when adding an element c to U, the partition into U-classes does not change.

Lemma 4.10. *Let $(\bar{a}^v)_{v\in I}$ be an infinite indiscernible sequence over U, $c \in \mathbb{M}$ be an element, $\varphi(z, \bar{x}^0, \dots, \bar{x}^{m-1})$ a formula over U, and let $p_0, \dots, p_{k-1}$ be the U-classes corresponding to the variables in $\bar{x}^0, \dots, \bar{x}^{m-1}$. For indices $\bar{v}_0, \dots, \bar{v}_{k-1} \in I^m$, we set*

$$\varphi[c, \bar{v}_0, \dots, \bar{v}_{k-1}] := \varphi\big(c, \bar{a}^{v_0^0}|_{p_0} \dots \bar{a}^{v_{k-1}^0}|_{p_{k-1}}, \dots, \bar{a}^{v_0^{m-1}}|_{p_0} \dots \bar{a}^{v_{k-1}^{m-1}}|_{p_{k-1}}\big)\,.$$

If there are indices $\bar{u}_0, \bar{u}_1, \bar{v}_0, \dots, \bar{v}_{k-1} \in I^m$ such that $\mathrm{ord}(\bar{u}_i) = \mathrm{ord}(\bar{v}_i)$, for $i < 2$, and

$$\begin{aligned}
&\mathfrak{M} \models \varphi[c, \bar{v}_0, \bar{v}_1, \bar{v}_2, \dots, \bar{v}_{k-1}]\\
&\mathfrak{M} \models \neg\varphi[c, \bar{u}_0, \bar{v}_1, \bar{v}_2, \dots, \bar{v}_{k-1}]\\
&\mathfrak{M} \models \neg\varphi[c, \bar{v}_0, \bar{u}_1, \bar{v}_2, \dots, \bar{v}_{k-1}]
\end{aligned}$$

then $\mathfrak{M}$ admits coding.

Proof. For a contradiction, suppose that $\mathfrak{M}$ does not admit coding. Since the sequence $(\bar{a}^v|_{p_0\cup p_1})_v$ is indiscernible over $U \cup \bar{a}|_{p_2\cup\dots\cup p_{k-1}}[I]$, we may w.l.o.g. assume that $k = 2$. Further, note that the sequence $(\bar{a}^v|_{p_0})_v$ is indiscernible over $U \cup \bar{a}|_{p_1}[I]$.

For fixed $\bar{y} \in I^m$, there are two cases. The truth value of $\varphi[c, \bar{x}, \bar{y}]$ might only depend on $\mathrm{ord}(\bar{x})$. Otherwise, we may assume, by Lemma 4.8, that there exists a unique index $s(\bar{y})$ such that the truth value of $\varphi[c, \bar{x}, \bar{y}]$ only depends on $\mathrm{ord}(\bar{x}s(\bar{y}))$. Similarly, if, for $\bar{x} \in I^m$, $\varphi[c, \bar{x}, \bar{y}]$ depends on more that just $\mathrm{ord}(\bar{y})$ then there exists a unique index $t(\bar{x})$ such that the truth value of $\varphi[c, \bar{x}, \bar{y}]$ only depends on $\mathrm{ord}(\bar{y}t(\bar{x}))$.

By compactness, we may assume that $I = \mathbb{R}$. For every pair of order automorphisms $\alpha, \beta : I \to I$, fix a U-automorphism $\pi_{\alpha\beta}$ such that

$$\begin{aligned}
\pi_{\alpha\beta}(\bar{a}^v|_{p_0}) &= \bar{a}^{\alpha(v)}|_{p_0}\,,\\
\pi_{\alpha\beta}(\bar{a}^v|_{p_1}) &= \bar{a}^{\beta(v)}|_{p_1}\,.
\end{aligned}$$

First, we prove that we have $s(\bar{y}) = s(\bar{y}')$, for all $\bar{y}, \bar{y}' \in I^m$ such that $s(\bar{y})$ and $s(\bar{y}')$ are defined. For a contradiction, suppose that $s(\bar{y}) < s(\bar{y}')$. For $u < v$ in I let $\alpha_{uv} : I \to I$ be an order isomorphism such that $\alpha_{uv}(s(\bar{y})) = u$ and $\alpha_{uv}(s(\bar{y}')) = v$, and set $c^{uv} := \pi_{\alpha_{uv},\mathrm{id}}(c)$. We construct formulae $\psi(z, \bar{x})$ and $\psi'(z, \bar{x})$ (with monadic parameters) such that

$$\begin{aligned}
&\mathfrak{M} \models \psi(c, \bar{a}^v|_{p_0}) \quad \text{iff} \quad v = s(\bar{y})\,,\\
\text{and} \quad &\mathfrak{M} \models \psi'(c, \bar{a}^v|_{p_0}) \quad \text{iff} \quad v = s(\bar{y}')\,.
\end{aligned}$$

Let χ_{p_0} be the formula from Theorem 2.18 defining the relation $\{\, \bar{a}^v|_{p_0} \mid v \in I \,\}$. If the linear ordering on the sequence $(\bar{a}^v|_{p_0})_{v\in I}$ is definable by a formula over $U \cup \{c\} \cup \bar{a}|_{p_1\cup\dots\cup p_{m-1}}[I]$ then we can define $\psi(z, \bar{x})$ by

$$\begin{aligned}
\chi_{p_0}(\bar{x}) \wedge \forall\bar{u}^0\dots\forall\bar{u}^{m-1}\forall\bar{v}^0\dots\forall\bar{v}^{m-1}&\\
\Big[\bigwedge_{i<m}(\chi_{p_0}(\bar{u}^i) \wedge \chi_{p_0}(\bar{v}^i))&\\
\wedge\, \mathrm{ord}(\bar{x}\bar{u}^0 \dots \bar{u}^{m-1}) = \mathrm{ord}(\bar{x}\bar{v}^0 \dots \bar{v}^{m-1})&\\
\to (\varphi'(z, \bar{u}^0, \dots, \bar{u}^{m-1}) \leftrightarrow \varphi'(z, \bar{v}^0, \dots, \bar{v}^{m-1}))\Big]&
\end{aligned}$$

where $\varphi'(z, \bar{x}^0, \dots, \bar{x}^{m-1})$ is an abbreviation for

$$\varphi\big(z,\ \bar{x}^0,\ \bar{a}^{y_0}|_{p_1}, \dots,\ \bar{x}^{m-1},\ \bar{a}^{y_{m-1}}|_{p_1}\big)\,.$$

If the ordering is not definable then it follows by Lemma 4.9 that the truth value of $\varphi[c, \bar{u}, \bar{y}]$ only depends on $\mathrm{equ}(s\bar{u})$. In this case we can replace the condition

$\operatorname{ord}(\bar{x}\bar{u}^0 \dots) = \operatorname{ord}(\bar{x}\bar{v}^0 \dots)$ in the above formula by the formula

$$\operatorname{equ}(\bar{x}\bar{u}^0 \dots \bar{u}^{m-1}) = \operatorname{equ}(\bar{x}\bar{v} \dots \bar{v}^{m-1}) .$$

The formula $\psi'(z, \bar{x})$ is defined analogously. It follows that

$$\mathfrak{M} \vDash \psi(c^{uv}, \bar{a}^x|_{p_0}) \wedge \psi'(c^{uv}, \bar{a}^y|_{p_0}) \quad \text{iff} \quad x = u \text{ and } y = v .$$

Fixing disjoint intervals $I_0, I_1 \subseteq I$ with $I_0 < I_1$ we obtain a definable bijection $\bar{a}|_{p_0}[I_0] \times \bar{a}|_{p_0}[I_1] \to \{ c^{uv} \mid u \in I_0, v \in I_1 \}$. Contradiction.

In the same way it follows that $t(\bar{x}) = t(\bar{x}')$ if these values are defined. By assumption, there are indices $\bar{x} := \bar{v}_0$ and $\bar{y} := \bar{v}_1$ such that $s(\bar{y})$ and $t(\bar{x})$ are defined. Let us denote these values by s and t. As above we can construct formulae $\vartheta_0(z, \bar{x})$ and $\vartheta_1(z, \bar{y})$ such that

$$\mathfrak{M} \vDash \vartheta_0(c, \bar{a}^x|_{p_0}) \quad \text{iff} \quad x = s ,$$
$$\text{and} \quad \mathfrak{M} \vDash \vartheta_1(c, \bar{a}^y|_{p_1}) \quad \text{iff} \quad y = t .$$

For $u, v \in I$, Let $\alpha_u, \beta_v : I \to I$ be order isomorphisms such that $\alpha_u(s) = u$ and $\beta_v(t) = v$, and set $c^{uv} := \pi_{\alpha_u, \beta_v}$. It follows that

$$\mathfrak{M} \vDash \vartheta_0(c^{uv}, \bar{a}^x|_{p_0}) \wedge \vartheta_1(c^{uv}, \bar{a}^y|_{p_1}) \quad \text{iff} \quad x = u \text{ and } y = v .$$

Consequently, $\mathfrak{M}$ admits coding. □

Lemma 4.11. *Suppose that $\mathfrak{M}$ does not admit coding. Let $(\bar{a}^v)_{v \in I}$ be an indiscernible sequence over U. For every element c such that $(\bar{a}^v)_v$ is not indiscernible over $U \cup \{c\}$, there exist a linear order $J \supseteq I$, an indiscernible sequence $(\bar{b}^v)_{v \in J}$ with $\bar{b}^v = \bar{a}^v$, for $v \in I$, and a unique index $s \in J$ such that*

$$\mathfrak{M} \vDash \varphi(c, \bar{b}[\bar{u}]) \leftrightarrow \varphi(c, \bar{b}[\bar{v}]) ,$$

for all formulae φ over U and all tuples $\bar{u}, \bar{v} \subseteq J$ with $\operatorname{ord}(s\bar{u}) = \operatorname{ord}(s\bar{v})$.

Proof. Let $\alpha := |\bar{a}^v|$. By Lemma 4.10, there is a U-class p such that the sequence $(\bar{a}^v|_{\alpha \setminus p})_v$ is indiscernible over $U \cup \bar{a}|_p[I] \cup \{c\}$. Furthermore, by Lemma 4.8 there exists a linear order $J \supseteq I$, an indiscernible sequence $(\bar{b}^v)_{v \in J}$ with $\bar{b}^v = \bar{a}^v$, for $v \in I$, and an index $s \in J$ such that

$$\mathfrak{M} \vDash \varphi(c, \bar{b}|_p[\bar{u}]) \leftrightarrow \varphi(c, \bar{b}|_p[\bar{v}]) ,$$

for all formulae φ over $U \cup \bar{b}|_{\alpha \setminus p}[J]$ and all indices $\bar{u}, \bar{v} \subseteq J$ with $\operatorname{ord}(s\bar{u}) = \operatorname{ord}(s\bar{v})$. It follows that

$$\mathfrak{M} \vDash \varphi(c, \bar{b}[\bar{u}]) \leftrightarrow \varphi(c, \bar{b}[\bar{v}]) ,$$

for all formulae φ over U and all indices $\bar{u}, \bar{v} \subseteq J$ with $\operatorname{ord}(s\bar{u}) = \operatorname{ord}(s\bar{v})$. □

It follows that we can generalise Corollary 4.7 to sequences with several U-classes.

Corollary 4.12. *Suppose that $\mathfrak{M}$ does not admit coding and let $(\bar{a}^v)_{v \in I}$ be an indiscernible sequence over U where the order I has no minimal and no maximal element.*

For each element c and all formulae $\varphi(x, \bar{y})$ over U, one of the following cases holds:

- $\left|[\![\varphi(c, \bar{a}^v)]\!]_v\right| \leq 1$
- $\left|[\![\neg\varphi(c, \bar{a}^v)]\!]_v\right| \leq 1$
- $[\![\varphi(c, \bar{a}^v)]\!]_v$ *is an initial segment of I.*
- $[\![\varphi(c, \bar{a}^v)]\!]_v$ *is a final segment of I.*

Combining the preceding lemmas we finally obtain the main result of this section. The next theorem states that we can extend each indiscernible sequence to cover every given element.

Theorem 4.13. *Suppose that $\mathfrak{M}$ does not admit coding. Let $(\bar{a}^v)_{v \in I}$ be an indiscernible sequence over U. For every element c, there exist a linear order $J \supseteq I$ and an indiscernible sequence $(\bar{b}^v c^v)_{v \in J}$ over U such that $\bar{b}^v = \bar{a}^v$, for $v \in I$, and $c = c^v$, for some $v \in J$.*

Proof. W.l.o.g. assume that I is infinite and complete. If $(\bar{a}^v)_v$ is indiscernible over c then we can set $c^v := c$, for all v. Otherwise, it follows by Lemma 4.11 that there exist a linear order $J \supseteq I$, an indiscernible sequence $(\bar{b}^v)_{v \in J}$ with $\bar{b}^v = \bar{a}^v$, for $v \in I$, and a unique index $s \in J$ such that

$$\mathfrak{M} \vDash \varphi(c, \bar{b}[\bar{u}]) \leftrightarrow \varphi(c, \bar{b}[\bar{v}]) ,$$

for all formulae φ over U and all tuples $\bar{u}, \bar{v} \subseteq J$ such that $\operatorname{ord}(s\bar{u}) = \operatorname{ord}(s\bar{v})$.

For $u \in J$, let $\alpha_u : J \to J$ be an order isomorphism with $\alpha_u(s) = u$. Choose U-automorphisms π_u with $\pi_u(\bar{b}^v) = \bar{b}^{\alpha_u(v)}$ and set $c^u := \pi_u(c)$.

Let Φ be the set of all formulae $\varphi(\bar{x}, \bar{y})$ such that, for some infinite subset $J_0 \subseteq J$, we have

$$\mathfrak{M} \vDash \varphi(\bar{b}[\bar{u}], c[\bar{u}]), \quad \text{for all increasing sequences } \bar{u} \subseteq J_0 .$$

For every formula φ we have $\varphi \in \Phi$ or $\neg\varphi \in \Phi$, by Ramsey's theorem. Furthermore, Φ is closed under entailment. Let $\Psi \subseteq \Phi$ be a maximal consistent subset of Φ. If there were a formula φ with $\varphi \notin \Psi$ and $\neg\varphi \notin \Psi$ then $\Psi \cup \{\varphi\}$ and $\Psi \cup \{\neg\varphi\}$ were inconsistent. Hence, we would have $\Psi \vDash \neg\varphi$ and $\Psi \vDash \varphi$. This implies that $\Psi \vDash \varphi \wedge \neg\varphi$ and Ψ is inconsistent. Contradiction.

It follows that Ψ is a complete type. Let $(\hat{b}^\nu \hat{c}^\nu)_{\nu \in J}$ be a sequence realising Ψ. Since $\operatorname{tp}(\hat{c}^s / U \cup (\hat{b}^\nu)_\nu) = \operatorname{tp}(c / U \cup (\bar{b}^\nu)_\nu)$ there exists an U-isomorphism π with $\pi(\hat{c}^s) = c$ and $\pi(\hat{b}^\nu) = \bar{b}^\nu$, for all $\nu \in J$. It follows that the sequence $(\bar{b}^\nu \pi(\hat{c}^\nu))_\nu$ is the desired indiscernible sequence. □

By induction it follows that we can extend each indiscernible sequence to cover every given set of elements.

Corollary 4.14. *Suppose that $\mathfrak{M}$ does not admit coding. Let $(\bar{a}^\nu)_{\nu \in I}$ be an indiscernible sequence of α-tuples over U. For every set $C \subseteq \mathbb{M}$, there exist a linear order $J \supseteq I$ and an indiscernible sequence $(\bar{b}^\nu)_{\nu \in J}$ of β-tuples over U with $\beta \geq \alpha$ such that $C \subseteq \bar{b}[J]$ and $\bar{a}^\nu = \bar{b}^\nu|_\alpha$, for $\nu \in I$.*

We conclude this section by an investigation of the U-partition of a sequence of the form $(\bar{a}^\nu|_N)_\nu$, for an arbitrary set $N \subseteq \alpha$. We start with a generalisation of Lemma 4.10.

Lemma 4.15. *Let $(\bar{a}^\nu c^\nu)_{\nu \in I}$ be an infinite indiscernible sequence over U and let P be the minimal U-partition for the sequence $(\bar{a}^\nu)_{\nu \in I}$. Let $\varphi(\bar{z}, \bar{x}^0, \dots, \bar{x}^{m-1})$ be a formula over U and let $p_0, \dots, p_{k-1} \in P$ be the U-classes corresponding to the variables in $\bar{x}^0, \dots, \bar{x}^{m-1}$. For $\bar{t}, \bar{v}_0, \dots, \bar{v}_{k-1} \in I^m$, we set*

$$\varphi[\bar{t}, \bar{v}_0, \dots, \bar{v}_{k-1}] :=$$
$$\varphi\big(c[\bar{t}], \bar{a}^{v_0^0}|_{p_0} \dots \bar{a}^{v_{k-1}^0}|_{p_{k-1}}, \dots, \bar{a}^{v_0^{m-1}}|_{p_0} \dots \bar{a}^{v_{k-1}^{m-1}}|_{p_{k-1}}\big) .$$

If there are indices $\bar{u}_0, \bar{u}_1, \bar{v}_0, \dots, \bar{v}_{k-1}, \bar{t} \in I^m$ such that $\operatorname{ord}(\bar{u}_i) = \operatorname{ord}(\bar{v}_i)$, for $i < 2$, and

$$\mathfrak{M} \vDash \varphi[\bar{t}, \bar{v}_0, \bar{v}_1, \bar{v}_2, \dots, \bar{v}_{k-1}]$$
$$\mathfrak{M} \vDash \neg\varphi[\bar{t}, \bar{u}_0, \bar{v}_1, \bar{v}_2, \dots, \bar{v}_{k-1}]$$
$$\mathfrak{M} \vDash \neg\varphi[\bar{t}, \bar{v}_0, \bar{u}_1, \bar{v}_2, \dots, \bar{v}_{k-1}]$$

then $\mathfrak{M}$ admits coding.

Proof. Replacing U by $U \cup \bar{a}|_{p_2 \cup \dots \cup p_{k-1}}[I]$ we may assume that $k = 2$. By assumption there are tuples $\bar{u}_0, \bar{u}'_0, \bar{u}_1, \bar{u}'_1 \in I^m$ such that

$$\operatorname{ord}(\bar{u}_i) = \operatorname{ord}(\bar{u}'_i) = \operatorname{ord}(\bar{v}_i)$$

and

$$\mathfrak{M} \vDash \varphi[\bar{t}, \bar{u}_0, \bar{v}_1] \wedge \neg\varphi[\bar{t}, \bar{u}'_0, \bar{v}_1] ,$$
$$\mathfrak{M} \vDash \varphi[\bar{t}, \bar{v}_0, \bar{u}_1] \wedge \neg\varphi[\bar{t}, \bar{v}_0, \bar{u}'_1] .$$

As usual we may assume that $\bar{u}_i$ and $\bar{u}'_i$ differ only in one component. Thus, suppose that $\bar{u}_i = u_i \bar{r}_i$ and $\bar{u}'_i = u'_i \bar{r}_i$. Furthermore, we may assume that

$$|[u_i, u'_i] \cap \bar{t}| \leq 1$$

since, if $u_i \leq t_k < t_l \leq u'_i$ then we can replace either u_i or u'_i by some index between t_k and t_l. Hence, suppose that there are indices k and l such that

$$[u_0, u'_0] \cap \bar{t} \subseteq \{t_k\} \quad \text{and} \quad [u_1, u'_1] \cap \bar{t} \subseteq \{t_l\} .$$

Let α be an order isomorphism with $\alpha(t_l) = t_k$. W.l.o.g. suppose that $k = 0$ and let $\bar{t} = t_0 \bar{t}'$. It follows that

$$\mathfrak{M} \vDash \varphi[t_0 \alpha(\bar{t}'), \alpha(\bar{v}_0), \alpha(\bar{u}_0)] \wedge \neg\varphi[t_0 \alpha(\bar{t}'), \alpha(\bar{v}_0), \alpha(\bar{u}'_0)] .$$

Fix indices s_-, s_+ such that

$$s_- < u_0 u'_0 \alpha(u_1) \alpha(u'_1) < s_+ \quad \text{and} \quad (s_-, s_+) \cap \bar{t} = \{t_0\} ,$$

and set $J := (s_-, s_+)$. The subsequence $(\bar{a}^\nu)_{\nu \in J}$ is indiscernible over the set $V := U \cup \bar{a}[I \setminus J]$. Defining

$$\psi(z, \bar{x}_0 \bar{y}_0, \bar{x}_1 \bar{y}_1) := \varphi(z, c[\bar{t}'], \bar{x}_0, \bar{x}_1) \wedge \varphi(z, c[\alpha(\bar{t}')], \bar{y}_0, \bar{y}_1)$$

we obtain a formula over V such that

$$\mathfrak{M} \vDash \psi[c^{t_0}, \bar{u}_0 \alpha(\bar{v}_0), \bar{v}_1 \alpha(\bar{u}_1)] ,$$
$$\mathfrak{M} \vDash \neg\psi[c^{t_0}, \bar{u}'_0 \alpha(\bar{v}_0), \bar{v}_1 \alpha(\bar{u}_1)] ,$$
$$\mathfrak{M} \vDash \neg\psi[c^{t_0}, \bar{u}_0 \alpha(\bar{v}_0), \bar{v}_1 \alpha(\bar{u}'_1)] .$$

By Lemma 4.10 it follows that $\mathfrak{M}$ admits coding. □

It follows that the Δ-dependence of two indices i and k is a 'local' property since it only depends on the sequence $(a_i^\nu a_k^\nu)_\nu$, not on all of $(\bar{a}^\nu)_\nu$.

Theorem 4.16. *Suppose that $\mathfrak{M}$ does not admit coding. Let $(\bar{a}^\nu)_{\nu\in I}$ be an indiscernible sequence over U with $|\bar{a}^\nu| = \alpha$, and let $N \subseteq \alpha$. If P is the U-partition of $(\bar{a}^\nu)_\nu$ then the U-partition of $(\bar{a}^\nu|_N)_\nu$ is $\{\, p \cap N \mid p \in P \,\}$.*

Proof. It is sufficient to consider the case that $N = \alpha \setminus \{n\}$. Then the general case will follow by induction. Let P be the U-partition of $(\bar{a}^\nu|_N)_\nu$. Consider a formula $\varphi(z^0\bar{x}^0, \dots, z^{m-1}\bar{x}^{m-1})$ over U where the variables z^i correspond to a_n^ν while $\bar{x}^i$ correspond to $\bar{a}^\nu|_N$. Let $p_0, \dots, p_{k-1} \subseteq N$ be the U-classes appearing in the variables $\bar{x}^i$. By Lemma 4.15, it follows that, for every $\bar{t} \in I^m$, there exists some class p_l such that the truth value of φ only depends on the class p_l, i.e.,

$$\begin{aligned}\mathfrak{M} \vDash \varphi\big(&a_n^{t_0}\bar{a}^{u_0^0}|_{p_0}\dots\bar{a}^{u_{k-1}^0}|_{p_{k-1}}, \dots, a_n^{t_{m-1}}\bar{a}^{u_0^{m-1}}|_{p_0}\dots\bar{a}^{u_{k-1}^{m-1}}|_{p_{k-1}}\big)\\ \leftrightarrow \varphi\big(&a_n^{t_0}\bar{a}^{v_0^0}|_{p_0}\dots\bar{a}^{v_{k-1}^0}|_{p_{k-1}}, \dots, a_n^{t_{m-1}}\bar{a}^{v_0^{m-1}}|_{p_0}\dots\bar{a}^{v_{k-1}^{m-1}}|_{p_{k-1}}\big),\end{aligned}$$

for all indices $\bar{u}^i, \bar{v}^i \in I^m$ with $\operatorname{ord}(\bar{u}^i) = \operatorname{ord}(\bar{v}^i)$, for $i < k$, and $\bar{u}^l = \bar{v}^l$. By indiscernibility, this index is the same for all $\bar{t}$. It follows that the U-class of n is either $\{n\}$ or $p_l \cup \{n\}$, while the other U-classes are p_j, $j \neq l$. $\square$

5 Conclusion

We have developed a structure theory for indiscernible sequences in structures that do not admit coding. In particular, we have introduced the notion of a U-class and we have shown that, for every U-class p of an indiscernible sequence $(\bar{a}^\nu)_{\nu\in I}$ and all indices $i, k \in p$, we can define the map $a_i^\nu \mapsto a_k^\nu$, $\nu \in I$. Further, we have shown that these U-classes behave well under extensions of the indiscernible sequence. Finally, our main theorem states that we can extend every indiscernible sequence (in width or in length) to cover any given subset of the universe.

These results show that theories that do not admit coding are nicer behaved than dp-minimal theories (where our main theorem fails). It might be hoped that theories without coding can be used as a test bed for the investigation of dp-minimal theories: they provide a simple context in which hypotheses can be tested and proved more easily, before they are generalised to cover all dp-minimal theories.

The original motivation of our work comes from the model theory of monadic second-order logic. In particular, we aimed at solving the conjecture of Seese, which is equivalent to the statement that every structure with no MSO-definable pairing function has a finite partition width. In a forthcoming paper [2] (see also [5]), we will give a partial answer to this conjecture by showing that the partition width of every structure not admitting coding is bounded by $2^{2^{\aleph_0}}$.

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