Computing the Expected Accumulated Reward and Gain for a Subclass of Infinite Markov Chains

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Abstract. We consider the problem of computing the expected accumulated reward and the average gain per transition in a subclass of Markov chains with countable state spaces where all states are assigned a non-negative reward. We state several abstract conditions that guarantee computability of the above properties up to an arbitrarily small (but non-zero) given error. Finally, we show that our results can be applied to probabilistic lossy channel systems, a well-known model of processes communicating through faulty channels.

1 Introduction

Methods for qualitative and quantitative analysis of stochastic systems have been rapidly gaining importance in recent years. Stochastic systems are used for modeling systems that exhibit some kind of uncertainty caused by, e.g., unpredictable errors, randomness, or underspecification. The semantics of stochastic systems is usually defined in terms of Markov chains or Markov decision processes [19, 22]. So far, problems related to formal verification of stochastic systems have been studied mainly for finite-state systems [24, 11, 6, 12, 19, 10]. Only recently, some of these results were extended to certain classes of infinite-state systems, in particular to probabilistic pushdown automata [13, 9, 14, 7], recursive Markov chains [16, 15], and probabilistic lossy channel systems [2, 3, 5, 4, 18, 23].

A more abstract approach has been adopted in [1], where the problems of qualitative and quantitative reachability are studied for a subclass of Markov chains with a finite attractor. In [1], it is shown that the problems of qualitative reachability and qualitative repeated reachability are decidable in the considered subclass, and that the quantitative variants of these problems can be solved up to an arbitrarily small given error. These abstract results are then applied to probabilistic lossy channel systems. Moreover, in the same paper it is shown that the exact probability of (repeated) reachability is not expressible in first order

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theory of the reals for probabilistic lossy channel systems (unlike for probabilistic pushdown automata or recursive Markov chains [13, 9, 16, 15]).

Our contribution: In this paper we adopt an abstract approach similar to the one of [1]. We identify an abstract class of infinite Markov chains where the expected accumulated reward between two given states and the average reward per transition can be effectively approximated up to a given precision. Our results are applicable to a similar class of systems as the results of [1], in particular to various versions of probabilistic lossy channel systems. These problems have previously been considered and solved for probabilistic pushdown automata by showing that these parameters are effectively expressible in first order theory of the reals [14]. However, this approach cannot be used for any class of Markov chains that subsumes probabilistic lossy channel systems; by adapting the results of [1], one can easily show that these values are (provably) not expressible in first order theory of the reals.

The problem of computing the expected accumulated reward can be roughly formulated as follows: assume that each state of a given Markov chain is assigned a rational *reward*, which is collected when the state is visited. We are interested in the expected reward accumulated when going from a given state s to another given state t. In particular, if the reward function returns 1 for every state, then the expected accumulated reward corresponds to the expected number of transitions between s and t, and can also be interpreted as the expected termination time. Another important parameter which is well-known from general theory of Markov chains is the *gain*, i.e., the average reward per transition along a given infinite run. The gain (computed w.r.t. various reward functions) plays an important role in performance analysis and can be used to evaluate various long-run system properties (such as the expected throughput, expected service time, etc.)

Since the expected accumulated reward and the average gain can take irrational values, the best we can hope for is to compute rational lower and upper approximations that are arbitrarily close. Our approach is similar to the one of [21] used for approximating the probability of reaching a given state t from another given state s. Roughly speaking, the algorithm successively computes the probability p_n^- of reaching t from s in at most n steps. This yields a sequence of lower approximations $p_1^-, p_2^-, \ldots$ of p. It holds (without any additional assumptions) that $\lim_{n\to\infty} p_n^-$ equals the probability p of reaching t from s, and that $p_1^- \leq p_2^- \leq \ldots \leq p$. However, it is not clear which p_n^- is "close enough" to p in the sense that $p - p_n \leq \varepsilon$ for a given precision $\varepsilon > 0$. Therefore, one also computes the probabilities d_n of reaching a "dead" state in at most n steps (a state s' is dead if t is not reachable from s'). Putting $p_i^+ = 1 - d_i$ for every $i \in \mathbb{N}$, we obtain a sequence of upper approximations $p_1^+ \geq p_2^+ \geq \cdots$ of p. If the Markov chain contains a finite attractor, then $\lim_{n\to\infty} p_n^- = p = \lim_{n\to\infty} p_n^+$, and it suffices to compute a sufficiently large n such that $p_n^+ - p_n^- \leq \varepsilon$.

We use a similar approach for computing the expected accumulated reward and the average gain by showing that there are effectively computable sequences $E_1^+, E_2^+, \ldots$ and $E_1^-, E_2^-, \ldots$ of upper and lower approximations which converge to the value of the considered parameter. For the expected accumulated reward, the sequence of lower approximations is easy to find, using a similar technique as in the case of reachability. In Section 3 we show how to construct the sequence of upper approximations for a subclass of Markov chains that satisfy certain abstractly formulated conditions. We show that an infinite Markov chain M of this class can be effectively approximated with a sequence of finite-state Markov chains so that the expected accumulated rewards computed in these approximations converge to the expected accumulated reward in M. In order to prove this convergence, we use results of perturbed Markov chains theory [17]. The problem of computing the expected gain is solved along similar lines, but the problem (and hence also the techniques involved) become more complicated. In particular, there is no simple method for constructing a sequence of lower approximations as in the case of the expected accumulated reward, and we have to compute both lower and upper approximating sequences using the sequence of finite-state Markov chains mentioned above.

Due to space constraints, all proofs are omitted. These can be found in a full version of this paper [8].

2 Preliminaries

In the paper we use $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{R}^+$ to denote the sets of rational numbers, real numbers, and non-negative real numbers, respectively. We also use $\mathbb{Q}_\infty$ and $\mathbb{R}^+_\infty$ to denote the set $\mathbb{Q} \cup \{\infty\}$ and $\mathbb{R}^+ \cup \{\infty\}$, respectively. The symbol ∞ is treated according to the standard conventions.

Definition 1. *A (discrete)* Markov chain *is a triple $M = (S, \rightarrow, Prob)$ where S is a finite or countably infinite set of* states*, $\rightarrow \subseteq S \times S$ is a* transition relation*, and $Prob$ is a function which to each transition $s \rightarrow t$ of M assigns its probability $Prob(s \rightarrow t) \in (0, 1]$ so that for every $s \in S$ we have $\sum_{s \rightarrow t} Prob(s \rightarrow t) = 1$.*

In the rest of this paper we write $s \xrightarrow{x} t$ instead of $Prob(s \rightarrow t) = x$. A *path* in M is a finite or infinite sequence $w = s_0, s_1, \ldots$ of states such that $s_i \rightarrow s_{i+1}$ for every i. We say that a state t is *reachable* from a state s if there is a path from s to t. We say that a Markov chain M is *irreducible*, if for all states s, t of M there is a path from s to t in M. The *length* of a given path w is the number of transitions in w. In particular, the length of an infinite path is ∞, and the length of a path s, where $s \in S$, is zero. We also use $w(i)$ to denote the state s_i of w (by writing $w(i) = s$ we implicitly impose the condition that the length of w is at least i). The prefix $s_0, \ldots, s_i$ of w is denoted by w^i. A *run* is an infinite path. The sets of all finite paths and all runs of M are denoted $FPath$ and Run, respectively. Similarly, the sets of all finite paths and runs that start with a given $w \in FPath$ are denoted $FPath(w)$ and $Run(w)$, respectively. In particular, $Run(s)$, where $s \in S$, is the set of all runs initiated in s.

We are interested in probabilities of certain events that are associated with runs. To every $s \in S$ we associate the probabilistic space $(Run(s), \mathcal{F}, \mathcal{P})$ where $\mathcal{F}$ is the σ-field generated by all *basic cylinders* $Run(w)$ where $w \in FPath(s)$, and $\mathcal{P} : \mathcal{F} \to [0,1]$ is the unique probability function such that $\mathcal{P}(Run(w)) = \Pi_{i=0}^{m-1} x_i$ where $w = s_0, \cdots, s_m$ and $s_i \xrightarrow{x_i} s_{i+1}$ for every $0 \le i < m$ (if $m = 0$, we put $\mathcal{P}(Run(w)) = 1$).

For every $s \in S$ and every $A \subseteq S$, we use $\mathcal{P}[s, A]$ to denote the probability of reaching A from s. Formally, $\mathcal{P}[s, A] = \mathcal{P}(\{w \in Run(s) \mid \exists i \ge 0 : w(i) \in A\})$. We write $\mathcal{P}[s, t]$ instead of $\mathcal{P}[s, \{t\}]$.

Definition 2. *A set $A \subseteq S$ is* recurrent *if for all $s \in A$ we have that $s \to t$ implies $\mathcal{P}[t, A] = 1$.*

Note that whenever a run leaves a recurrent set A, then it almost surely (i.e., with probability one) returns back to A in the future.

A *reward function* is a function $f : S \to \mathbb{R}^+$. We extend f to finite paths by putting $f(s_0, \ldots, s_n) = \sum_{i=0}^{n} f(s_i)$. Thus, f assigns to each path its *accumulated reward*. The special reward function which assigns 1 to every $s \in S$ is denoted $\mathbf{1}$ (i.e., $\mathbf{1}(s) = 1$ for each $s \in S$).

3 Computing the expected accumulated reward and gain

In this section we show how to compute certain quantitative properties in certain classes of Markov chains up to an arbitrarily small $\varepsilon > 0$. More precisely, we show that these properties are *effectively approximable* in the following sense:

Definition 3. *Let $\mathcal{O}$ be a class of* objects, *and let $P : \mathcal{O} \to \mathbb{R}^+_\infty$. We say that P is* effectively approximable *if there is an algorithm which, for a given $o \in \mathcal{O}$, enumerates two sequences $E_1^+, E_2^+, \ldots$ and $E_1^-, E_2^-, \ldots$ where $E_i^+, E_i^- \in \mathbb{Q}_\infty$ such that for all $i \ge 1$ we have $E_i^- \le P(o) \le E_i^+$ and $\lim_{i\to\infty} E_i^+ = \lim_{i\to\infty} E_i^-$.*

The sequences $E_1^+, E_2^+, \ldots$ and $E_1^-, E_2^-, \ldots$ are called the upper/lower approximating sequences *of $P(o)$, respectively.*

If P is effectively approximable, then the value of $P(o)$ can effectively be approximated up to an arbitrarily small $\varepsilon > 0$ by enumerating the upper and lower sequences simultaneously until they become sufficiently close.

3.1 The expected accumulated reward

For the rest of this subsection, let us fix a Markov chain $M = (S, \to, Prob)$, two states $s_{in}, s_{fin} \in S$, and a reward function $f : S \to \mathbb{R}^+$. Moreover, we assume that given a state $s \in S$, the set $\{(s, x, t) \mid s \xrightarrow{x} t\}$ of all transitions from s is effectively denumerable.

We define a random variable $R : Run(s_{in}) \to \mathbb{R}^+_\infty$ that counts the reward accumulated between s_{in} and s_{fin}. Formally, given a run $w \in Run(s_{in})$, we define

$$R(w) = \begin{cases} f(w(0), \ldots, w(n-1)) & \exists n : w(n) = s_{fin}, w(i) \neq s_{fin} \text{ for } 1 \le i \le n-1; \\ \infty & \text{otherwise.} \end{cases}$$

The expected value of R is denoted $\mathcal{E}(M, f)$ (the reason why we write $\mathcal{E}(M, f)$ and not just $E(R)$ is that in our proofs we consider various modifications of the chain M and various reward functions, keeping s_{in}, s_{fin} fixed).

Our aim is to show that the function which to a given tuple (M, f, s_{in}, s_{fin}) assigns the value $\mathcal{E}(M, f)$ is effectively approximable (cf. Definition 3) if M and f satisfy certain abstractly formulated conditions. To simplify our notation, we formulate these conditions directly for the previously fixed M and f, and show how to compute the sequences $E_1^+, E_2^+, \ldots$ and $E_1^-, E_2^-, \ldots$ if these conditions are satisfied.

First, let us realize that the lower approximating sequence $E_1^-, E_2^-, \ldots$ can be computed without any additional assumptions about M and f, because

- one can effectively compute a sequence $P_1, P_2, \ldots$ of *finite* sets of finite paths such that $P_i \subseteq P_{i+1}$ for each $i \geq 1$, and $\bigcup_{i=1}^{\infty} P_i$ is exactly the set of all finite paths w where $w(0){=}s_{in}$, $w(k){=}s_{fin}$ for some k, and $w(j){\neq}s_{fin}$ for all $0{\leq}j{<}k$;
- E_i^- can be defined as $\sum_{w \in P_i} \mathcal{P}(Run(w)) \cdot f(w)$

However, the upper approximating sequence $E_1^+, E_2^+, \ldots$ cannot be effectively constructed for general M and f. In order to formulate the promised sufficient conditions, we need to state one auxiliary definition.

Definition 4. *Let $h : S \rightarrow \mathbb{R}^+$ be a reward function, $A \subseteq S$, and $s \in A$. We define a random variable $O_s^{h,A} : Run(s) \rightarrow \mathbb{R}^+$ that counts the reward accumulated "before hitting the set A" as follows:*

$$O_s^{h,A}(w) = \begin{cases} h(w(1), \ldots, w(n{-}1)) & \exists n : w(n) \in A, w(1), \ldots, w(n{-}1) \notin A; \\ \perp & \textit{otherwise.} \end{cases}$$

The symbol $\mathcal{E}O_s^{h,A}$ denotes either the conditional expectation $E(O_s^{h,A} \mid O_s^{h,A}{\neq}\perp)$ or 0, depending on whether $\mathcal{P}(O_s^{h,A}{\neq}\perp)$ is positive or zero, respectively.

The sufficient conditions which (as we shall see) enable an effective construction of $E_1^+, E_2^+, \ldots$ are the following:

1. there is an effectively computable sequence $A_0 \subseteq A_1 \subseteq \cdots$ of finite recurrent sets such that $\bigcup_{i=0}^{\infty} A_i = S$;
2. there is an effectively computable number $\Xi \in \mathbb{R}^+$ such that for all $i \geq 0$ and all $s \in A_i$ we have $\mathcal{E}O_s^{f,A_i} \leq \Xi$ and $\mathcal{E}O_s^{\mathbf{1},A_i} \leq \Xi$ (remember that $\mathbf{1}$ is the reward function which assigns 1 to each state);
3. given a finite set $A \subseteq S$ and $s, t \in A$, it is decidable whether there is a finite path of the form $s{=}s_0, \ldots, s_n{=}t$ where $s_i \notin A$ for all $0 < i < n$ (i.e., whether s can reach t without visiting any state of A in the middle).

As we shall see, these conditions are satisfied by, e.g., Markov chains generated by various variants of probabilistic lossy channel systems. The intuitive meaning of these conditions is explained at appropriate places below. Note that we can safely assume that $s_{in}, s_{fin} \in A_0$.

For the rest of this subsection, let us assume that the conditions 1–3 are satisfied and $s_{in}, s_{fin} \in A_0$. First, let us deal with the case when $\mathcal{P}[s_{in}, s_{fin}] < 1$.

Then clearly $\mathcal{E}(M,f) = \infty$. Moreover, one can easily prove that $\mathcal{P}[s_{in}, s_{fin}] < 1$ if and only if there is a state $s \in A_0$ such that s is reachable from s_{in}, and s_{fin} is *not* reachable from s (we use the fact that A_0 is recurrent and finite). Hence, using condition 3 we can effectively check whether $\mathcal{P}[s_{in}, s_{fin}] < 1$. If this is the case, then $\mathcal{E}(M,f) = \infty$.

Now let us assume that $\mathcal{P}[s_{in}, s_{fin}] = 1$. We show that conditions 1–3 suffice for computing the upper approximating sequence $E_1^+, E_2^+, \dots$ Loosely speaking, the algorithm computes a sequence of finite-state Markov chains that "approximate" the Markov chain M, and the expected reward accumulated between s_{in} and s_{fin} in these chains "approximates" $\mathcal{E}(M,f)$.

We start with some auxiliary definitions. Given a set $A \subseteq S$ and two states $s,t \in A$, we define the set $\mathit{Out}(A,s,t) \subseteq \mathit{Run}(s)$ of runs that reach t without visiting A in the middle:

$$\mathit{Out}(A,s,t) = \{w \in \mathit{Run}(s) \mid \exists n : w(n) = t, w(1), \dots, w(n{-}1) \notin A\}$$

We put $\mathit{Out}(A,s) = \{w \in \mathit{Run}(s) \mid w(1) \notin A\}$, and for all $i \geq 0$ define a Markov chain $M_i = (S_i, \rightarrow_i, \mathit{Prob}_i)$, where $S_i = A_i \cup \{\bar{s} \mid s \in A_i, \mathcal{P}(\mathit{Out}(A_i,s)) > 0\}$ and the transitions are determined as follows:

- if $s,t \in A_i$, then $s \xrightarrow{x}_i t$ iff $s \xrightarrow{x} t$;
- if $s \in A_i$ and $\bar{s} \in S_i$, then $s \xrightarrow{x}_i \bar{s}$ iff $x = \mathcal{P}(\mathit{Out}(A_i,s))$;
- if $s,t \in A_i$ and $\bar{s} \in S_i$, then $\bar{s} \xrightarrow{x}_i t$ iff $x = \mathcal{P}(\mathit{Out}(A_i,s,t) \mid \mathit{Out}(A_i,s)) > 0$.

Note that M_i has finitely many states. Now we define a reward function $f_i : S_i \to \mathbb{R}^+$ where $f_i(s) = f(s)$ and $f_i(\bar{s}) = \mathcal{E}O_s^{f,A_i}$ for every $s \in A_i$. The following (crucial) lemma states that each (M_i, f_i) is a faithful abstraction of (M,f) with respect to the expected reward accumulated between s_{in} and s_{fin}.

Lemma 1. *For all $i \geq 0$ we have that $\mathcal{E}(M,f) = \mathcal{E}(M_i,f_i)$.*

Note that if we were able to compute (M_i, f_i) for some i, we would be done, because the expected accumulated reward can easily be computed for finite-state Markov chains using standard methods. Unfortunately, we cannot compute the transition probabilities of M_i precisely (the transitions of the form $\bar{s} \xrightarrow{x}_i t$ cause the problem), and the definition of f_i is not effective either. However, we can use condition 2 to design a reward function f_i^+ that approximates f_i — for every $i \geq 0$ and every $s \in A_i$ we define $f_i^+(s) = f(s)$ and $f_i^+(\bar{s}) = \Xi$. Condition 2 implies that $f_i \leq f_i^+$ for all $i \geq 0$, hence $\mathcal{E}(M_i,f_i) \leq \mathcal{E}(M_i,f_i^+)$. The following lemma states that the difference between $\mathcal{E}(M_i,f_i)$ and $\mathcal{E}(M_i,f_i^+)$ approaches 0 as i grows.

Lemma 2. *For each $\varepsilon > 0$ there is $i \geq 0$ s.t. $0 \leq \mathcal{E}(M_j,f_j^+) - \mathcal{E}(M_j,f_j) \leq \varepsilon$ for every $j \geq i$.*

The only problem left is that we are not able to compute transition probabilities in the chains M_i. This is overcome by showing that, for a given $\delta > 0$, one can effectively approximate the transition probabilities of M_i and compute a finite-state Markov chain $M_i^\delta = (S_i, \rightarrow_i, \mathit{Prob}_i^\delta)$ with the transition matrix P_i^δ

so that $\|P_i - P_i^\delta\|_\infty \leq \delta$ (the norm $\|\cdot\|_\infty$ of a matrix $P = \{p_{ij}\}$ is defined as $\|P\|_\infty = \max_i \sum_j |p_{ij}|$). Then, we show that for every M_i^δ there is an effectively computable number $c_\delta \in \mathbb{R}^+$ such that $|\mathcal{E}(M_i^\delta, f_i^+) - \mathcal{E}(M_i, f_i^+)| \leq c_\delta \cdot \delta$. Moreover, the number c_δ approaches a *bounded* value as δ goes to zero. Here we employ results of perturbed Markov chains theory and develop some new special tools that suit our purposes. In this way, we obtain the following lemma:

Lemma 3. *For every $i \geq 0$ and every $\varepsilon > 0$ there is an effectively computable $\delta > 0$ such that $|\mathcal{E}(M_i^\delta, f_i^+) - \mathcal{E}(M_i, f_i^+)| \leq \varepsilon$.*

Note that since the definition of M_i^δ is effective, we can compute $\mathcal{E}(M_i^\delta, f_i^+)$ by standard methods for finite-state Markov chains.

Now we can define the upper approximating sequence $E_1^+, E_2^+, \ldots$ of $\mathcal{E}(M, f)$ as follows: For each $i \geq 1$ we put $E_i^+ = \mathcal{E}(M_i^{\delta_i}, f_i^+) + \frac{1}{2^{i+1}}$, where $\delta_i > 0$ is the δ of Lemma 3 computed for the considered i and $\varepsilon = \frac{1}{2^{i+1}}$. Now it is easy to see that $0 \leq E_i^+ - \mathcal{E}(M_i, f_i^+) \leq \frac{1}{2^i}$. By combining this observation together with Lemma 1 and Lemma 2, we obtain that $\lim_{i\to\infty} E_i^+ = \mathcal{E}(M, f)$ and $E_i^+ \geq \mathcal{E}(M, f)$ for all $i \geq 1$. Moreover, the approximations $E_1^+, E_2^+, \ldots$ are effectively computable. Thus, we obtain our first theorem:

Theorem 1. *For every $\varepsilon > 0$ there is an effectively computable number x such that $|\mathcal{E}(M, f) - x| \leq \varepsilon$.*

3.2 The average gain

Similarly as in Section 3.1, we fix a Markov chain $M = (S, \to, \mathit{Prob})$, a state $s_{in} \in S$, and a reward function $f : S \to \mathbb{R}^+$, such that for each $s \in S$ the set $\{(s, x, t) \mid s \xrightarrow{x} t\}$ is effectively denumerable.

We define a function $\mathcal{G}(M, f) : \mathit{Run}(s_{in}) \to \mathbb{R}^+_\infty$ as follows

$$\mathcal{G}(M, f)(w) = \begin{cases} \lim_{n\to\infty} \frac{f(w^n)}{n} & \text{if the limit exists;} \\ \perp & \text{otherwise.} \end{cases}$$

Hence, $\mathcal{G}(M, f)(w)$ corresponds to the *gain* (i.e., "average reward per transition"), which is a standard notion in stochastic process theory (see, e.g., [20]). As we shall see in Section 3.3, the gain can be used to compute other interesting characteristics which reflect long-run properties of a given system.

Note that $\mathcal{G}(M, f)(w)$ can be undefined for some $w \in \mathit{Run}(s_{in})$. As we shall see, for Markov chains that satisfy conditions 1–3 of Section 3.1, the total probability of all such runs is zero. Since $\mathcal{G}(M, f)(w)$ can take infinitely many values, a standard problem of stochastic process theory is to compute $E(\mathcal{G}(M, f))$, the expected value of $\mathcal{G}(M, f)$. However, one should realize that the information provided by $E(\mathcal{G}(M, f))$ is relevant only in situations when a system is repeatedly restarted and runs "sufficiently long" so that the average reward per transition approaches its limit. In our setup, we can provide a bit more detailed information about the runs of $\mathit{Run}(s_{in})$, which is not reflected in the "ensemble average"

$E(\mathcal{G}(M,f))$. We show that in the subclass of Markov chains that satisfy conditions 1–3, the variable $\mathcal{G}(M,f)$ can take *only finitely many values with a positive probability*, and we give an algorithm which approximates these values as well as the associated probabilities up to an arbitrarily small $\varepsilon > 0$. Thus, we obtain a "complete picture" about possible limit behaviours of runs initiated in s_{in}. Note that $E(\mathcal{G}(M,f))$ can be effectively approximated simply by taking the weighted sum of the finitely many admissible values of $\mathcal{G}(M,f)$. It is worth noting that similar results have recently been achieved for an incomparable class of Markov chains generated by probabilistic pushdown automata [7] by using completely different methods.

The class of Markov chains considered in this subsection is the same as in Section 3.1, i.e., we assume that the previously fixed chain M satisfies conditions 1–3 (cf. Section 3.1). We also assume (without restrictions) that $s_{in} \in A_0$. Since the constructions and techniques employed in this section are hard to explain at an intuitive level, we only state our main theorem and refer to [8] for missing details.

Theorem 2. *There are finitely many pairwise disjoint sets* $\mathcal{R}_1, \ldots, \mathcal{R}_n \subseteq \mathit{Run}(s_{in})$ *and numbers* $x_1, \ldots, x_n \in \mathbb{R}^+$ *such that*

- $\mathit{Prob}(\bigcup_{i=1}^{n} \mathcal{R}_i) = 1$*, and* $\mathit{Prob}(\mathcal{R}_i) > 0$ *for every* $1 \le i \le n$*;*
- *for every* $1 \le i \le n$ *and every* $w \in \mathcal{R}_i$ *we have* $\mathcal{G}(M,f)(w) = x_i$*;*
- *for every* $\varepsilon > 0$ *and every* $1 \le i \le n$*, there is an effectively computable number* y_i *such that* $|x_i - y_i| \le \varepsilon$*;*
- *for every* $1 \le i \le n$*, it is decidable whether* $\mathit{Prob}(\mathcal{R}_i) = 1$*; moreover, for every* $\varepsilon > 0$ *and every* $1 \le i \le n$*, there is an effectively computable number* r_i *such that* $|\mathit{Prob}(\mathcal{R}_i) - r_i| \le \varepsilon$*.*

3.3 The average ratio

The gain can be used to define some interesting characteristics of Markov chains, like, e.g., the frequency of visits to a distinguished family of states along an infinite run. In performance analysis, one is also interested in features that cannot be directly specified as gains, but as limits of *fractions* of two reward functions.

Let us start with a simple motivating example. Let $M = (S, \rightarrow, \mathit{Prob})$ be a Markov chain, $s_{in} \in S$ an initial state, $f : S \rightarrow \mathbb{R}^+$ a reward function, and $T \subseteq S$ a set of *triggers*. Intuitively, a trigger is a state initiating a finite "service" of a certain request. Hence, each run with infinitely many triggers can be seen as an infinite sequence of finite services, where each service corresponds to a finite path between two consecutive occurrences of a trigger. What we are interested in is the average accumulated reward per service. Formally, the average accumulated reward per service can be defined as follows: we fix another reward function g where $g(s)$ returns 1 if $s \in T \cup \{s_{in}\}$, and 0 otherwise. Now we define a random variable $R : \mathit{Run}(s_{in}) \rightarrow \mathbb{R}^+_\infty$ as follows:

$$R(w) = \begin{cases} \lim_{n\to\infty} \frac{f(w^n)}{g(w^n)} & \text{if the limit exists;} \\ \perp & \text{otherwise.} \end{cases}$$

It is easy to see that $R(w)$ indeed corresponds to the average accumulated reward per service in the run w. For the reasons which have been discussed at the beginning of Section 3.2, we are interested not only in $E(R)$ (the expected average accumulated reward per service), but in a complete classification of admissible values of R and their associated probabilities. Of course, this is possible only under some additional assumptions about the chain M; as we shall see, conditions 1–3 are sufficient.

Now we move from the above example to a general setup. For the rest of this section, we fix a Markov chain $M = (S, \rightarrow, \mathit{Prob})$, a state $s_{in} \in S$, and two reward functions $f, g : S \rightarrow \mathbb{R}^+$. In order to simplify our presentation, we assume that $g(s_{in}) > 0$. We define the *average ratio* $\mathcal{R}_g^f : \mathit{Run}(s_{in}) \rightarrow \mathbb{R}_\infty^+$ as follows.

$$\mathcal{R}_g^f(w) = \begin{cases} \lim_{n\rightarrow\infty} \frac{f(w^n)}{g(w^n)} & \text{if the limit exists;} \\ \perp & \text{otherwise.} \end{cases}$$

First we observe that the average ratio can be expressed in terms of gains.

Lemma 4. *Let $w \in \mathit{Run}(s_{in})$ be a run. If both $\mathcal{G}(M,f)(w)$ and $\mathcal{G}(M,g)(w)$ are defined and finite, and if $\mathcal{G}(M,f)(w) + \mathcal{G}(M,g)(w) > 0$, then*

$$\mathcal{R}_g^f(w) = \frac{\mathcal{G}(M,f)(w)}{\mathcal{G}(M,g)(w)}$$

Here we use the convention that $c/0 = \infty$ for $c > 0$.

For the rest of this section, we assume that the chain M satisfies conditions 1–3 of Section 3.1. First, let us consider the special case when the chain M_0 is irreducible. It follows from Theorem 2 that $\mathcal{G}(M,f)$ and $\mathcal{G}(M,g)$ are constant almost everywhere and finite. Moreover, the values $\mathcal{G}(M,f)$ and $\mathcal{G}(M,g)$ can be effectively approximated up to a given $\varepsilon > 0$. The case when $\mathcal{G}(M,g) = 0$ requires some attention.

Lemma 5. $\mathcal{G}(M,g) = 0$ *iff for all $s \in S$ reachable from s_{in} we have that $g(s)$=0.*

Now we consider the general case when M_0 is not necessarily irreducible. We obtain that the values of $\mathcal{G}(M,g)$ are determined by the bottom strongly connected components of the underlying transition system $\mathcal{T}_{M_0}$ of M_0. The value associated with a given component C is 0 iff all states $s \in S$ that are reachable from a state of $C \cap A_0$ satisfy $g(s) = 0$ iff for all runs $w \in \mathit{Run}(s_{in})$ that enter a state of $C \cap A_0$ there is $k \geq 0$ such that for all $j \geq k$ we have $g(w(j)) = 0$. Thus, we obtain the following generalization of Theorem 2.

Theorem 3. *Let us assume that for each $s \in A_0$ it is decidable whether there is $t \in S$ reachable from s such that $g(t) > 0$, and the same for the reward function f. Then there are finitely many pairwise disjoint sets $\mathcal{Z}_f, \mathcal{Z}_g, \mathcal{Z}_{f,g}, \mathcal{R}_1, \ldots, \mathcal{R}_n \subseteq \mathit{Run}(s_{in})$ and numbers $x_1, \ldots, x_n \in \mathbb{R}^+$ such that*

- *$\mathit{Prob}(\bigcup_{i=1}^n \mathcal{R}_i \cup \mathcal{Z}_f \cup \mathcal{Z}_g \cup \mathcal{Z}_{f,g}) = 1$, and*
  - *$\mathcal{R}_g^f(w) = x_i > 0$ for all $w \in \mathcal{R}_i$ and all $1 \leq i \leq n$;*

  - $\mathcal{R}_g^f(w) = 0$ *for all* $w \in \mathcal{Z}_f$*;*
  - $\mathcal{R}_g^f(w) = \infty$ *for all* $w \in \mathcal{Z}_g$.
- *for all* $w \in \mathcal{Z}_{f,g}$ *there is* $k \geq 0$ *such that* $j \geq k$ *implies* $f(w(j)) = g(w(j)) = 0$.
- *for every* $\varepsilon > 0$ *there are effectively computable* $y_1, \ldots, y_n$ *such that* $|x_i - y_i| \leq \varepsilon$ *for* $1 \leq i \leq n$*;*
- *the probabilities* $Prob(\mathcal{R}_i)$ *for* $1 \leq i \leq n$*,* $Prob(\mathcal{Z}_f)$*,* $Prob(\mathcal{Z}_g)$*, and* $Prob(\mathcal{Z}_{f,g})$ *can be effectively approximated up to a given* $\varepsilon > 0$*; moreover, for each of these probabilities, it is decidable whether the probability is equal to* 1 *or not.*

4 Probabilistic lossy channel systems

Lossy channel systems (LCS) [2] have been proposed as a model for processes communicating via faulty communication channels. A lossy channel system consists of a finite-state control unit and a finite set of FIFO channels. A *configuration* of LCS consists of the current control state and the current contents of the channels. A computational step from a given configuration consists of adding/removing one message to/from a channel, and possibly changing the control state. Moreover, during each transition, one or more messages can be lost from the channels.

A probabilistic lossy channel system (PLCS) is a probabilistic variant of LCS. In PLCS, transitions and message losses are chosen randomly according to a given probability distribution. There are several models of PLCS that differ mainly in the treatment of message losses. The model considered in [18] assumes that each step of a system is either a message loss or a "perfect" step that is performed consistently with transition function. There is a fixed probability $\lambda > 0$ that the next step will be a message loss. This model is called a global-fault model in [23]. Another variant of PLCS was considered in [5], where it is assumed that each message can be lost independently of the other messages with some given probability $\lambda > 0$. Then each step of a system consists of a perfect step followed by a loss of (zero or more) messages, where each message is lost with the probability λ, independently of the other messages. This model is also called a local-fault model. See [23] for a deeper explanation of the above models of PLCS.

We show that the abstract results of Section 3 are applicable both to the global-fault and the local-fault variant of PLCS. In our discussion, we use the following result for one-dimensional random walks: For each $0 < \lambda < 1$ we define a Markov chain $M_\lambda = (\mathbb{N}_0, \rightarrow, Prob)$ where the transitions are defined as follows. For all $n \geq 0$ we put $n \xrightarrow{1-\lambda} n+1$, for all $n \geq 1$ we put $n \xrightarrow{\lambda} n-1$, and we also put $0 \xrightarrow{\lambda} 0$. It is easy to prove that if $\lambda > \frac{1}{2}$, then the expected number of transitions needed to reach 0 from 1 equals $\frac{1}{2\lambda-1}$.

Let $\mathcal{L}$ be a PLCS, and let us assume that f is a reward function that assigns a rational reward to configurations of $\mathcal{L}$. Moreover, let us assume that f is effectively bounded, i.e., there is an effectively computable constant ξ such that for every configuration s we have $f(s) \leq \xi$.

Let us first consider the global-fault model. We argue that if $\lambda > \frac{1}{2}$, then the conditions 1–3 of Section 3.1 are satisfied, and hence Theorems 1, 2, and 3 apply. For all $i \geq 0$ we define the set A_i consisting of all configurations where the total number of messages stored in the channels is bounded by i. Since at most one message can be added into channels during a perfect step and each step is lossy with probability λ, we obtain that the expected time to reach A_i after leaving A_i is bounded from above by the expected number of transitions needed to reach 0 from 1 in M_λ. Hence, condition 2 is satisfied because f is effectively bounded. Condition 3 can be proved using similar arguments as in Theorem 8 in [1].

In the local-fault model, the probability of a message loss converges to 1 as the number of stored messages increases. In particular, there is $n \in \mathbb{N}$ such that for each configuration where the total number of stored messages exceeds n we have that the probability of losing at least two messages in the next step is greater than $\frac{1}{2}$ (since at most one message can be added to channels in a single step, the number of messages stored in the next configuration decreases with probability greater than $\frac{1}{2}$). It is easy to see that the number n is computable from λ. Hence, if we define A_i to be the set of all configurations where the number of stored messages is less than or equal to $n + i$, we obtain that conditions 2 and 3 are satisfied, using the same argument as for the global-fault model above. Hence, the general results of Theorems 1, 2, and 3 apply.

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