Exercises – set 8
Probabilistic Method II
Shenggen Zheng, April 18th, 2013, 8:30–9:45 B411

1. Consider an instance of SAT with m clauses, where every clause has exactly k literals.
   (a) Prove that there is an assignment that satisfies at least $m(1-2^{-k})$ clauses.
   (b) Give a Las Vegas algorithm that finds an assignment satisfying at least $m(1-2^{-k})$ clauses, and analyze its expected running time.

2. (Apple Game) Consider the following two-player game. The game begins with n apples on the table. Each round, one player, called Alice, takes no more than 3 but at least 1 apple out of the table. The second player, called Bob, also takes no more than 3 but at least 1 apple out of the table. The last one to take apples on the table wins the game.
   a) Is there any winning strategy for Alice when $n = 18$?
   b) Is there any winning strategy for Bob for some n?
   c) Prove that if n is randomly chosen. It is better to be the first one who takes apples from the table.

3. Let us consider an $n \times m$ boolean matrix M, for example

$$\begin{pmatrix} 1 & 0 & \cdots & 1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1 \end{pmatrix}_{n\times m}.$$

   The entry of $M_{i,j} = 1$ means that deterministic algorithm A_i accepts input w_j and $M_{i,j} = 0$ means that deterministic algorithm A_i rejects input w_j, respectively.

   a) Let $N = \sum_{i,j} M_{i,j}$. How large must N be in order to imply that there must exist an i such that $\sum_j M_{i,j} = m$ (means that there must exist an A_i accepts every input).
   b) Suppose that there is a randomized algorithm A and the number of inputs for A is m. For every input w,
   $$Pr(A \text{ accepts } w) = \frac{1}{n}\sum_{i=1}^{n} Pr(A_i \text{ accepts } w)$$
   is the probability that A accepts the input w, where A_i is a deterministic algorithm (means that $Pr(A_i \text{ accepts } w) = 1$ or $Pr(A_i \text{ accepts } w) = 0$). There exists an ε such that for every input w,
   $$Pr(A \text{ accepts } w) > 1 - \varepsilon.$$
   Prove that there exists an A_i that accepts every input w if $\varepsilon < \frac{1}{m}$.