Exercises – set 5

Basic tools - moments and deviations

Shenggen Zheng, March 28, 2013, 8:30–9:45 B411

0 .(Even a died thin camel looks larger than a horse) Suppose that a camel is always bigger than a horse and there are camels and horses standing in a square $M = (a_{ij})_{i,j=1}^n$ with every row (column) has at least one camel and one horse. (a) Prove that

$$\max_i \min_j a_{ij} \leq \min_j \max_i a_{ij}. \tag{1}$$

(b) Prove that the above inequality still holds for matrix M with real numbers as elements.

1. Alice has a funny father, every day he tosses 6 coins of 1 dollar. If the outcome of the coin is head, then the coin will be given to Alice as her allowance. Let X denote the money that Alice gets from her father.
   (a) What is the probability that $Pr(X \geq i)$, where $i = 1, 2, \cdots, 6$?
   (b) Verify that $Pr(X \geq i) \leq \frac{E(X)}{i}$.
   (c) If Alice tosses n coins instead of 6, prove that $Pr(X \geq i) \leq \frac{E(X)}{i}$.
2. This problem shows that Markov's inequality is as tight as it could possibly be. Given a positive integer k, describe a random variable X that assumes only nonnegative values such that

$$Pr(X \geq kE[x]) = \frac{1}{k}. \tag{2}$$

3. suppose that each box of chocolate contains one of 4 different coupons. Once you obtain 4 different coupons, you can send in for a prize. What is the expectation number of boxes of chocolate must you buy in order to get 4 different coupons.
   (a) What is the expectation number of boxes of chocolate you must buy in order to get 1 different coupons?
   (b) What is the expectation number of boxes of chocolate you must buy in order to get 2 different coupons?
   (c) What is the expectation number of boxes of chocolate you must buy in order to get 3 different coupons?
   (d) What is the expectation number of boxes of chocolate you must buy in order to get 4 different coupons?
   (e) If there are n different coupons, what is the expectation number of boxes of chocolate you must buy in order to get n different coupons?

   (Hints: let X be the number of boxes bought until at least one of every type of coupon is obtained, let p_i be the probability that to buy a new coupons in one trail while you had exactly $i-1$ different coupons. If X_i be the number of boxes bought in order to have i-th new coupons while you have exactly $i-1$ different coupons, then clearly $X = \sum_{i=1}^n X_i$.)