Exercises – set 6

TAIL PROBABILITIES and CHERNOFF BOUNDS

Shenggen Zheng, April 4th, 2013, 8:30–9:45 B411

1. Alice has a funny father, every day he tosses 8 coins of 1 dollar. If the outcome of the coin is head, then the coin will be given to Alice as her allowance. Let X be the money that Alice gets from her father and let p be the probability of the event that $X \geq 6$.

   (a) Calculate p.

   (b) Compare the upper bounds on p that you can obtain using Markov's inequality, Chebyshev's inequality and Chernoff bounds.

   (c) Is it true that Chernoff bounds are always better than Chebyshev bounds? Why?

2. (Useful trick) We know that, for $0 < x < 1$,

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i. \tag{1}$$

   Prove the following equalities:

$$\text{(a)} \quad \frac{1}{(1-x)^2} = \sum_{i=0}^{\infty} (i+1)x^i. \tag{2}$$

$$\text{(b)} \quad \frac{2}{(1-x)^3} = \sum_{i=0}^{\infty} (i+1)(i+2)x^i. \tag{3}$$

$$\text{(c)} \quad \sum_{i=1}^{\infty} i^2 x^i = \frac{x^2+x}{(1-x)^3}. \tag{4}$$

$$\tag{5}$$

   Let X be a geometric random variable with parameter p. Use the above equalities to prove that

$$Var[X] = \frac{1-p}{p^2}.$$

3. Let $W = (X, Y, Z)$ be a point chosen randomly in a $2 \times 2 \times 2$ cube centered in $(0, 0, 0)$. Try to give an estimation for π.
   (Hint: consider the sphere of radius 1 centered in the point $(0, 0, 0)$.)