Part I

Elliptic curves cryptography and factorization

EMPIRICAL NOTION of SECRECY of CRYPTOSYSTEMS

A cryptographic system is consider as sufficiently secure until someone finds an attack against it.

ELLIPTIC CURVES - PRELIMINARIES

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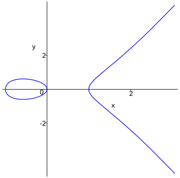

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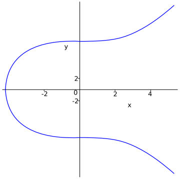

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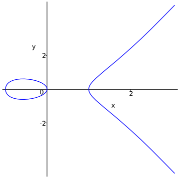

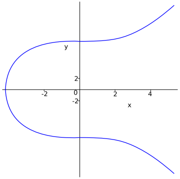

$y^2 = x(x+1)(x-1)$ $\qquad$ $y^2 = x^3 + 73$

Elliptic curves cryptography is based on a special operation of addition of any two points on an elliptic curve such that it is easy to make addition of any two points, but it is in general unfeasible to find the first point given the sum of two points and

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- In August 2015 NSA announced plans to replace the ECC cryptography by, not yet determined, a post-quantum cryptography .

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- Both of these uses of elliptic curves, ECC cryptography and ECC based integer factorization are dealt with in this chapter.

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- Abel has been considered, by his contemporaries, as mathematical genius that left enough for mathematicians to study for next 500 years.

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It is amazing how practical is the elliptic curve cryptography that is based on very strangely looking and very theoretical concepts.

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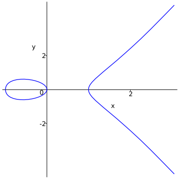
In case coefficients and x, y can be any rational numbers, a graph of an elliptic curve has one of the forms shown in the following figure. The graph depends on whether the polynomial $x^3 + ax + b$ has three or only one real root.



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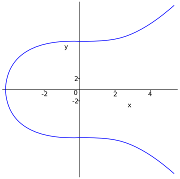

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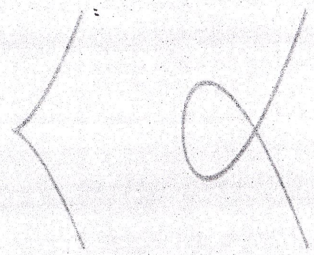
The graph of a non-singular curve has two components if its discriminant is positive, and one component if it is negative.

EXAMPLES OF SINGULAR "ELLIPTIC CURVES"



Types of singularities: on the left, a curve with a cusp ($y^2 = x^3$). On the right, a curve with a self-intersection ($y^2 = x^3 - 3x + 2$). None of them is a valid elliptic curve.

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The reason is that if we are working with rational coefficients or mod p, where $p > 3$ is a prime, then such a general equation can be transformed to our special case of equation - see the Appendix. In other cases, it may be indeed necessary to consider the most general form of equation.

ELLIPTIC CURVES - GENERALITY

A general elliptic curve over Z_{p^m} where p is a prime is the set of points (x, y) satisfying so-called Weierstrass equation

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for some constants u, v, a, b, c together with a single element $\mathbf{0}$, called the point of infinity.

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If $p \neq 2$ Weierstrass equation can be simplified by transformation

$$y \to \frac{y - (ux + v)}{2}$$

to get the equation

$$y^2 = x^3 + dx^2 + ex + f$$

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- **Elliptic curves are currently behind practically most preferred methods of cryptographic security.**
- Elliptic curves are also a basis of very important factorization method.

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ADDITION of POINTS on ELLIPTIC CURVES - GEOMETRY

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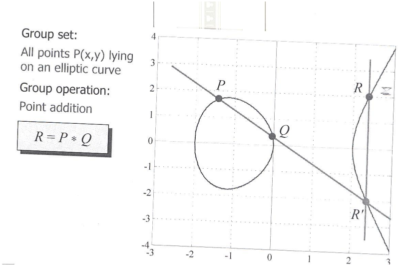
It is now easy to verify that the above addition of points forms Abelian group with ∞ as the identity (null) element.

ADDITION of POINTS - EXAMPLES 1 and 2

The following pictures show some cases of points additions



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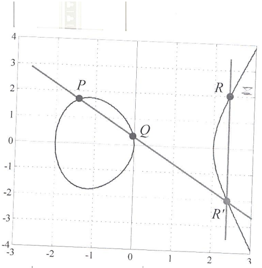
Group set:

All points P(x,y) lying on an elliptic curve

Group operation:

Point addition

$R = P * Q$



Inverse element:

P'(x,-y) = P(x,y)

is mirrored on x-axis

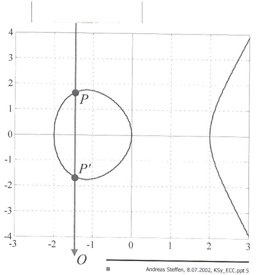
Point addition with inverse element:

P ∗ P' = O

results in a neutral element O(x,∞) at infinity

Neutral element:

P ∗ O = P



ADDITION of POINTS - EXAMPLES 3 and 4

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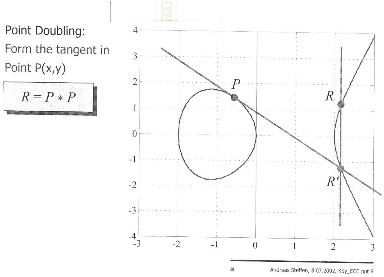

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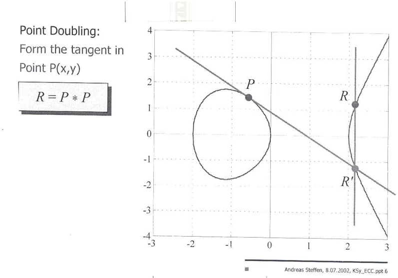

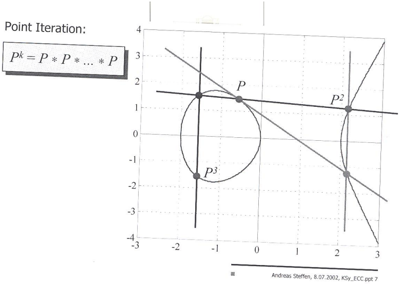

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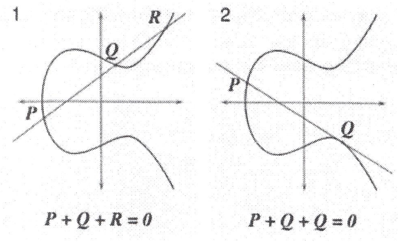

ADDITION of POINTS - EXAMPLES 5 and 6

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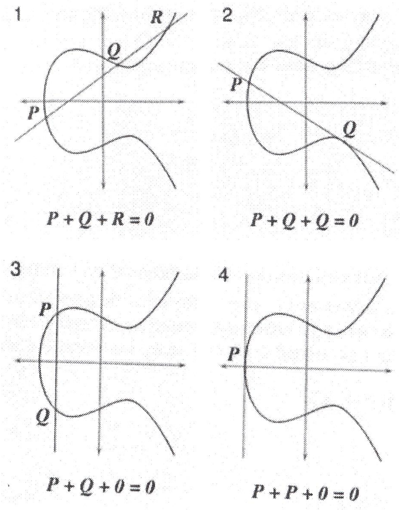

Formulas

Addition of points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ of an elliptic curve $E : y^2 = x^3 + ax + b$ can be easily computed using the following formulas:

$$P_1 + P_2 = P_3 = (x_3, y_3)$$

where

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Since its two roots have coordinates x_1 and x_2 for the third, x_3, it has to hold

$$x_3 = \lambda^2 - (x_1 + x_2) = \lambda^2 - x_1 - x_2,$$

because $-\lambda^2$ is the coefficient at x^2 and therefore $x_1 + x_2 + x_3 = -(-\lambda^2) = \lambda^2$.

ELLIPTIC CURVES mod n

The points on an elliptic curve

$$E : y^2 = x^3 + ax + b \ (\text{mod } n),$$

where a and b are integers, notation $E_n(a, b)$ are such pairs of integers (x,y), $|x| \leq n$, $|y| \leq n$, that satisfy the above equation, along with the point ∞ at infinity.

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EXAMPLE OF AN ELLIPTIC CURVE OVER A PRIME

Points of the elliptic curve $y^2 = x^3 + x + 6$ over Z_{11}

x	$x^3 + x + 6 \pmod{11}$	in QR_{11}	y
0	6	no	
1	8	no	
2	5	yes	4,7
3	3	yes	5,6
4	8	no	
5	4	yes	2,9
6	8	no	
7	4	yes	2,9
8	9	yes	3,8
9	7	no	
10	4	yes	2,9

The number of points of an elliptic curve over Z_p is in the interval

$$(p + 1 - 2\sqrt{p}, p + 1 + 2\sqrt{p})$$

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Observe that if this gcd-value is between 1 and n we have a factor of n.

POINTS on CURVE $y^2 = x^3 + x + 6 \mod 11$

x	y^2	$y_{1,2}$	P(x,y)	P'(x,y)
0	6	-		
1	8	-		
2	5	4,7	(2,4)	(2,7)
3	3	5,6	(3,5)	(3,6)
4	8	-		
5	4	2,9	(5,2)	(5,9)
6	8	-		
7	4	2,9	(7,2)	(7,9)
8	9	3,8	(8,3)	(8,8)
9	7	-		
10	4	2,9	(10,2)	(10,9)

There are 12 points lying on the elliptic curve.

Together with the point O at infinity, the points on the elliptic curve form a group with n=13 elements.

n is called the order of the elliptic curve group and depends on the choice of the curve parameters a and b.

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$$y_3 = \lambda(x_1 - x_3) - y_1 \equiv 8(2 - 5) - 7 \equiv -31 \equiv -9 \equiv 2 \mod 11$$

PROPERTIES of ELLIPTIC CURVES MODULO p

- Elliptic curves modulo an integer p have finitely many points and are finitely generated - all points can be obtained from few given points using the operation of addition.

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- Hasse's theorem If an elliptic curve E_p has $|E_p|$ points then $||E_p| - p - 1| < 2\sqrt{p}$

  In other words, the number of points of a curve grows roughly as the number of elements in the field. The exact number of such points is, however, rather difficult to calculate.

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- However, no proof of security of ECC has been published so far.

USE OF ELLIPTIC CURVES IN CRYPTOGRAPHY

Let E be an elliptic curve and A, B be its points such that $B = kA = (A + A + \ldots A + A)$ – k times – for some k. The task to find (given A and B) such a k is called the discrete logarithm problem for elliptic curves.

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No efficient algorithm to compute discrete logarithm problem for elliptic curves is known and also no good general attacks. Elliptic curves based cryptography is based on these facts.

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- Change, in the cryptographic protocol P, each exponentiation to a multiplication of points of the elliptic curve E by integers.
- To the point of the elliptic curve E that results from such a protocol assign a message (cryptotext).

POWERS of POINTS

The following table shows powers of various points of the curve

$$y^2 = x^3 + x + 6 \mod 11$$

k	P^k	s	y_0
1	(2,4)	3	9
2	(5,9)	9	8
3	(8,8)	8	10
4	(10,9)	2	0
5	(3,5)	1	2
6	(7,2)	4	7
7	(7,9)	1	2
8	(3,6)	2	0
9	(10,2)	8	10
10	(8,3)	9	8
11	(5,2)	3	9
12	(2,7)	∞	-

Given an elliptic curve

$y^2 = x^3 + ax + b \mod p$

and a basis point P, we can compute

$Q = P^k$

through k-1 iterative point additions. Fast algorithms for this task exist. Unfortunately most of them are patented by Certicom and others.

Question: Is it possible to compute k when the point Q is known?

Answer: This is a hard problem known as the Elliptic Curve Discrete Logarithm.

where instead of λ an s is written.

MAPPING MESSAGES into POINTS of ELLIPTIC CURVES I.

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Basic idea: Given an elliptic curve $E(\bmod p)$, the problem is that not to every x there is an y such that (x, y) is a point of E.

Given a message (number) m we adjoin to m few bits at the end of m and adjust them until we get a number x such that $x^3 + ax + b$ is a square mod p.

EFFICIENCY of various CRYPTO GRAPHIC SYSTEMS

The following pictures show how many bits need keys of different crypto graphic systems to achieve the same security.

Equivalent Cryptographic Strength

Symmetric	56	80	112	128	192	256
RSA n	512	1024	2048	3072	7680	15360
ECC p	112	161	224	256	384	512
Key size ratio	5:1	6:1	9:1	12:1	20:1	30:1

ELLIPTIC CURVES KEY EXCHANGE

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- Bob chooses an integer n_B, computes $n_B P$ and sends it to Alice.
- Alice computes $n_A(n_B P)$ and Bob computes $n_B(n_A P)$. This way they have the same key.

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To send a message m ALice expresses m as a point X on E_p, chooses a random number r, computes

$$A = rP \; ; \; B = X + rQ$$

and sends the pair (A, B) to Bob who decrypts by calculating $X = B - xA$.

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Warning Observe that actually $rr^{-1} = 1 + tn$ for some t. For the above verification procedure to work we then have to use the fact that $nP = \infty$ and therefore $P + t \cdot \infty = P$

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Federal (USA) elliptic curve digital signature standard (ECDSA) was introduced in 2005.

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Elliptic curve method was used to factor Fermat numbers F_{10} (308 digits) and F_{11} (610 digits).

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To use ECC, all parties involved have to agree on all basic elements concerning the elliptic curve E being used:

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- SPH method factors the order of a curve into small primes and solves the discrete logarithm problem as a combination of discrete logarithms for small numbers.
- Computation time of the square root method is proportional to $O(\sqrt{e^n})$ where n is the order of the based element of the curve.

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- For example, for 128-bit security one needs a curve over $\underline{F}_q$, where $q \approx 2^{256}$.
- This can be contrasted with RSA cryptography that requires 3072-bit public and private keys to keep the same level of security.

BREAKING ECC

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So called Fundamental theorem of arithmetic, known since Euclid, claims that factorization of an integer n into a power of primes

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is unique when primes p_i are ordered. However, theorem provides no clue how to find such a factorization and till now **no classical polynomial factorization algorithm is know.**

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Factorization could be done in polynomial time using Shor's algorithm and a powerful quantum computer, as discussed later.

RABIN-MILLER's PRIME RECOGNITION III.

Rabin-Miller's Monte Carlo prime recognition algorithm is based on the following result from the number theory.

Lemma Let $n \in N$. Denote, for $1 \leq x \leq n$, by $C(x)$ the condition:

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Claim: If $C(x_i)$ holds for some i, then n is not a prime for sure. Otherwise n is declared to be prime. Probability that this is not the case is 2^{-m}.

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1732 L. Euler found that $F_5 = 4294967297 = 641 \cdot 6700417$

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- The fastest known factorization algorithm has time

$$e^{(1.9 \ln n)^{1/3}(\ln \ln n)^{2/3})}$$

and with it we can factor 140 digit numbers in reasonable time.

BASIC FACTORIZATION METHODS

These methods are actually heuristics, and for each of them a variety of modifications is known.

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Notation $L(\varepsilon, c)$ is used to denote complexity

$$O(e^{(c+o(1))(\ln n)^{\varepsilon}(\ln \ln n)^{1-\varepsilon}})$$

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Denote then

$$k = gcd(a - c, d - b) \quad h = gcd(a + c, d + b)$$

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In such a case either both k and h are even or both m and l are even. In the first case

$$n = ((\frac{k}{2})^2 + (\frac{h}{2})^2)(l^2 + m^2)$$

Unfortunately, disadvantage of Euler's factorization method is that it cannot be applied to factor an integer having a prime factor of the form $4k + 3$.

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Therefore, in order to find a factor of n, we need only to investigate the values

$$x = a^2 - n$$

$$\text{for } a = \left\lceil \sqrt{n} \right\rceil + 1, \left\lceil \sqrt{n} \right\rceil + 2, \ldots, \frac{(n-1)}{2}$$

until a perfect square for x is found.

SIMPLE but POWER IDEAS

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To factorize an integer n:

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2. Two versions:

Version 1: Compute $gcd(x_i - x_j, n)$ for $i = 1, 2, \ldots$ and $j = 1, 2, \ldots, i-1$ until a factor of n is found.

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The second method was used to factor 8-th Fermat number F_8 with 78 digits.

$$f(x) = x^2 + x + 1$$

$$n = 18923; \quad x = y = x_0 = 2347$$

$$x \leftarrow f(x) \bmod n; y \leftarrow f(f(y)) \bmod n$$

$$gcd(x - y, n) =?$$

x	$=$	4164	y	$=$	9593	$gcd(x-y,n)$	$=$	1
x	$=$	9593	y	$=$	2063	gcd	$=$	1
x	$=$	12694	y	$=$	14985	gcd	$=$	1
x	$=$	2063	y	$=$	14862	gcd	$=$	1
x	$=$	358	y	$=$	3231	gcd	$=$	1
x	$=$	14985	y	$=$	3772	gcd	$=$	1
x	$=$	5970	y	$=$	16748	gcd	$=$	1
x	$=$	14862	y	$=$	3586	gcd	$=$	1
x	$=$	5728	y	$=$	16158	gcd	$=$	149

Pollard $p-1$ ALGORITHM - FIRST VERSION

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Algorithm was invented J. Pollard in 1987 and has time complexity $O(B(\log n)^p)$. It works well if both $p|n$ and $p-1$ have only small prime factors.

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By Fermat's Little Theorem, this implies that $p|(a^m - 1)$ for any integer a and therefore by computing

$$gcd(a^m - 1, n)$$

(for some a) some factor p of n can be obtained.

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Basis idea: To factorize an integer n choose an elliptic curve E_n, a point P on E and compute, modulo n, either iP for $i = 2, 3, 4, \ldots$ or $2^j P$ for $j = 1, 2, \ldots$.

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Example: If curve $E : y^2 = x^3 + 4x + 4 \pmod{2773}$ and its point $P = (1, 3)$ are used, then $2P = (1771, 705)$ and in order to compute 3P one has to compute $gcd(1770, 2773) = 59$ – factorization is done.

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5. Try to compute mP.

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Example: For the elliptic curve

$$E : y^2 = x^3 + x - 1 \pmod{35}$$

and its point $P = (1, 1)$ we have

$$2P = (2, 32); 4P = (25, 12); 8P = (6, 9)$$

and at the attempt to compute 9P one needs to compute $gcd(15, 35) = 5$ and factorization is done.

It remains to be explored how efficient this method is and when it is more efficient than other methods.

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- The number with only small factors is called smooth and if all prime factors are smaller than an b, then it is called b-smooth.

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- The number with only small factors is called smooth and if all prime factors are smaller than an b, then it is called b-smooth.

It can be shown that the density of smooth integers is so large that if we choose a random elliptic curve E_n then it is a reasonable chance that n is smooth.

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Idea: If one searches for m-digits factors, one chooses k in such a way that k is a multiple of as many as possible of those m-digit numbers which do not have too large prime factors. In such a case one has a good chance that k is a multiple of the number of elements of the group of points of the elliptic curve modulo n.

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Method 1: One chooses an integer B and takes as k the product of all maximal powers of primes smaller than B.

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Method 1: One chooses an integer B and takes as k the product of all maximal powers of primes smaller than B.

Example: In order to find a 6-digit factor one chooses B=147 and $k = 2^7 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot \ldots \cdot 139$. The following table shows B and the number of elliptic curves one has to test:

Digits of to-be-factors	6	9	12	18	24
B	147	682	2462	23462	162730
Number of curves	10	24	55	231	833

Computation time by the elliptic curves method depends on the size of factors.

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Quantum computers works with superpositions of basic quantum states on which very special (unitary) operations are applied and very special quantum features (non-locality) are used.

Quantum computers work not with bits, that can take on any of two values 0 and 1, but with qubits (quantum bits) that can take on any of infinitely many states $\alpha|0\rangle + \beta|1\rangle$, where α and β are complex numbers such that $|\alpha|^2 + |\beta|^2 = 1$.

REDUCTIONS

Shor's polynomial time quantum factorization algorithm is based on an understanding that factorization problem can be reduced

1. first on the problem of solving a simple modular quadratic equation;
2. second on the problem of finding periods of functions $f(x) = a^x \bmod n$.

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Lemma If there is a polynomial time deterministic (randomized) [quantum] algorithm to find a nontrivial solution of the modular quadratic equations

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Proof. Let $a \neq \pm 1$ be such that $a^2 \equiv 1 \pmod{n}$. Since

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if n is not prime, then a prime factor of n has to be a prime factor of either $a+1$ or $a-1$.

By using Euclid's algorithm to compute

$$gcd(a+1, n) \text{ and } gcd(a-1, n)$$

we can find, in $O(\lg n)$ steps, a prime factor of n.

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1. *Choose randomly $1 < a < n$.*
2. *Compute $gcd(a, n)$. If $gcd(a, n) \neq 1$ we have a factor.*
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If this algorithm stops, then $a^{r/2}$ is a non-trivial solution of the equation

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EXAMPLE

Let $n = 15$. Select $a < 15$ such that $gcd(a, 15) = 1$.
$\{$The set of such a is $\{2, 4, 7, 8, 11, 13, 14\}\}$

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Hence $a^{r/2} = 11 \pmod{15}$. Therefore

$$gcd(15, 12) = 3, \qquad gcd(15, 10) = 5$$

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Lemma If $1 < a < n$ satisfying $gcd(n, a) = 1$ is selected in the above algorithm randomly and n is not a power of prime, then

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Corollary If there is a polynomial time randomized [quantum] algorithm to compute the period of the function

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then there is a polynomial time randomized [quantum] algorithm to find non-trivial solution of the equation $a^2 \equiv 1 \pmod{n}$ (and therefore also to factorize integers).

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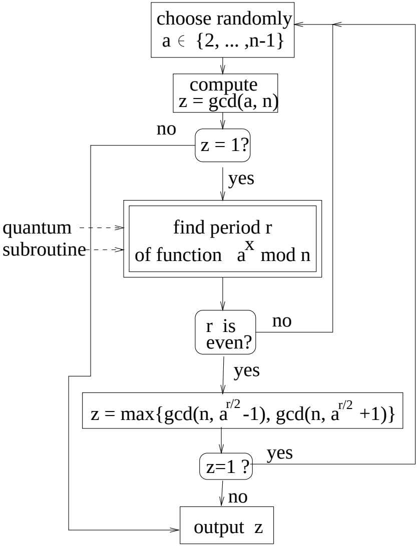
The following flow diagram shows the general scheme of Shor's quantum factorization algorithm



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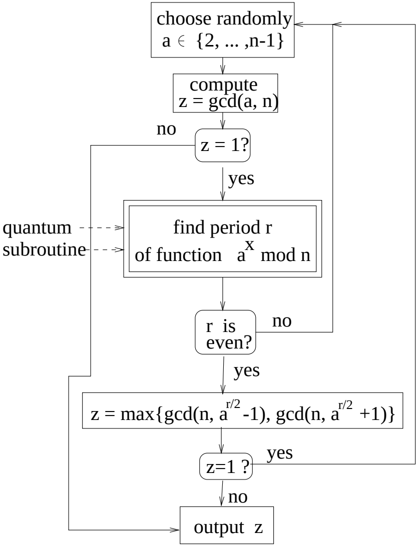

QUADRATIC SIEVE METHOD of FACTORIZATION - BASIC IDEAS

Step 1 To factorize an n one finds many integers x such that $x^2 - n$ has only small factors and decomposition of $x^2 - n$ into small factors.

Example - $n = 7429$

$$\left.\begin{array}{l} 83^2 - 7429 = -540 = (-1) \cdot 2^2 \cdot 3^3 \cdot 5 \\ 87^2 - 7429 = 140 = 2^2 \cdot 5 \cdot 7 \\ 88^2 - 7429 = 315 = 3^2 \cdot 5 \cdot 7 \end{array}\right\} \text{relations}$$

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If this is to form a square the following equations have to hold

$$\begin{array}{ll} \lambda_1 & \equiv 0 \mod 2 \\ \lambda_1 + \lambda_2 + \lambda_3 & \equiv 0 \mod 2 \\ \lambda_2 + \lambda_3 & \equiv 0 \mod 2 \end{array}$$

Therefore:

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Step 1 To factorize an n one finds many integers x such that $x^2 - n$ has only small factors and decomposition of $x^2 - n$ into small factors.

Example - $n = 7429$

$$\left.\begin{array}{l} 83^2 - 7429 = -540 = (-1) \cdot 2^2 \cdot 3^3 \cdot 5 \\ 87^2 - 7429 = 140 = 2^2 \cdot 5 \cdot 7 \\ 88^2 - 7429 = 315 = 3^2 \cdot 5 \cdot 7 \end{array}\right\} \text{relations}$$

Step 2 One multiplies some of the relations such that their product is a square. For example

$$(87^2 - 7429)(88^2 - 7429) = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 = 210^2$$

Now, compute product modulo n:

$$(87^2 - 7429)(88^2 - 7429) \equiv (87 \cdot 88)^2 = 7656^2 \equiv 227^2 \mod 7429$$
$$\text{and therefore } 227^2 \equiv 210^2 \mod 7429$$

Hence 7429 divides $227^2 - 210^2$ and therefore $17 = 227 - 210$ is a factor of 7429.

A method to choose relations to form equations: For the i-th relation one takes a variable λ_i and forms the expression

$$((-1) \cdot 2^2 \cdot 3^3 \cdot 5)^{\lambda_1} \cdot (2^2 \cdot 5 \cdot 7)^{\lambda_2} \cdot (3^2 \cdot 5 \cdot 7)^{\lambda_3} = (-1)^{\lambda_1} \cdot 2^{2\lambda_1 + 2\lambda_2} \cdot 3^{2\lambda_1 + 2\lambda_2} \cdot 5^{\lambda_1 + \lambda_2 + \lambda_3} \cdot 7^{\lambda_2 + \lambda_3}$$

If this is to form a square the following equations have to hold

$$\begin{aligned} \lambda_1 &\equiv 0 \mod 2 \\ \lambda_1 + \lambda_2 + \lambda_3 &\equiv 0 \mod 2 \\ \lambda_2 + \lambda_3 &\equiv 0 \mod 2 \end{aligned}$$

Therefore: $\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = 1$

QUADRATIC SIEVE FACTORIZATION - SKETCH of METHODS

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Step 1 One chooses a set of primes that can be factors – a so-called factor basis.

One chooses an m such that $m^2 - n$ is small and considers numbers $(m+u)^2 - n$ for $-k \leq u \leq k$ for small k.
One then tries to factor all $(m+u)^2 - n$ with primes from the factor basis, from the smallest to the largest - see table for n=7429 and m=86.

u	-3	-2	-1	0	1	2	3
$(m+u)^2 - n$	-540	-373	-204	-33	140	315	492
Sieve with 2	-135		-51		35		123
Sieve with 3	-5		-17	-11		35	41
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In order to factor a 129-digit number from the RSA challenge they used

8 424 486 relations
569 466 equations
544 939 elements in the factor base

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The most efficient factorization method, for factorization of integers with more than 100 digits, is the general number field sieve method (superpolynomial but sub-exponential); The second fastest is the quadratic sieve method.

APPENDIX

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The reason is that if we are working with rational coefficients or mod p, where $p > 3$ is a prime, then such a general equation can be transformed to our special case of equation. In other cases, it may be indeed necessary to consider the most general form of equation.

ELLIPTIC CURVES - GENERALITY

A general elliptic curve over Z_{p^m} where p is a prime is the set of points (x, y) satisfying so-called Weierstrass equation

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- For example, for 128-bit security one needs a curve over $\underline{F}_q$, where $q \approx 2^{256}$.
- This can be contrasted with RSA cryptography that requires 3072 public and private keys.

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- SPH method factors the order of a curve into small primes and solves the discrete logarithm problem as a combination of discrete logarithms for small numbers.
- Computation time of the square root method is proportional to $O(\sqrt{e^n})$ where n is the order of the based element of the curve.

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- The binary field case was broken in April 2004 using 2600 computers for 17 months.

GOOD ELLIPTIC CURVES

- NIST recommended 5 elliptic curves for prime fields, one for prime sizes 192, 224, 256, 384 and 521 bits.
- NIST also recommended five elliptic curves for binary fields $\mathbf{F}_{2^m}$ one for m equal 163, 233, 283, 409 and 571.