Part I

Public-key cryptosystems II. Other cryptosystems and cryptographic primitives

CHAPTER 6: OTHER CRYPTOSYSTEMS and BASIC CRYPTOGRAPHY PRIMITIVES

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Finally, we will discuss, in some details, such very important cryptography primitives as **pseudo-random number generators** and **hash functions** .

STORY of SQUARE ROOTS

and

QUADRATIC RESIDUES

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However, in case n is a prime or a product of two odd primes, such a polynomial squaring algorithm exists.

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for some $y \in Z_n^\star$, otherwise x is called a **quadratic nonresidue**.

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For any prime p the set QR(p) has $\frac{p-1}{2}$ elements.

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So called **Euler criterion** says that c is a quadratic residue modulo prime p iff

$$c^{(p-1)/2} \equiv 1 \ (\text{mod}\ p).$$

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$$1^2 \equiv 1 \pmod{15},\ 2^2 \equiv 4 \pmod{15},\ 4^2 \equiv 1 \pmod{15},$$
$$7^2 \equiv 4 \pmod{15},\ 8^2 \equiv 4 \mod 15,$$
$$11^2 \equiv 1 \pmod{15},\ 13^2 \equiv 4 \pmod{15},\ 14^2 \equiv 1 \pmod{15}$$

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- The inverse function is $f^{-1}(x) = x^{((p-1)(q-1)+4)/8} \bmod n$

EXAMPLE

For $n = 21 = 3 \times 7$

$$Z_{21}^{*} = \{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$$

$$QR(21) = \{1, 4, 16\}$$

and

$$1^2 = 1 \bmod 21 \quad 4^2 = 16 \bmod 21 \quad 16^2 = 4 \bmod 21$$

DISCRETE SQUARE ROOTS CRYPTOSYSTEMS

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That is, likely, why Rabin did not propose this system as a practical cryptosystem.

COMPUTATION of SQUARE ROOTS MODULO PRIMES

In case of Blum primes p and q and Blum integer $n = pq$, in order to solve the equation $x^2 \equiv a (\text{mod } n)$, one needs to compute squares of a modulo p and modulo q and then to use the Chinese remainder theorem to solve the equation $x^2 = a \pmod{pq}$.

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Using the Chinese Remainder Theorem we then get

$$x \equiv \pm 15, \pm 29 \pmod{77}.$$

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Example If $m_1 = 2, m_2 = 3, m_3 = 5$, then $(1, 0, 2)$ represents integer 27.
Advantage: With such a modular representation addition, subtraction and multiplication can be done component-wise and therefore in parallel time.

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- However, this is not the case if w is an arbitrary string.

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- Since $x^2 \equiv (a^2p^2s^2 + b^2q^2r^2) \pmod{n}$ and $ap + bq = 1$ we have $bq \equiv 1 \pmod{p}$ and therefore $x^2 \equiv r^2 \pmod{p}$;

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- Similarly we get $x^2 \equiv s^2 \pmod q$ and the Chinese remainder theorem then implies $x^2 \equiv c \pmod n$;
- Similarly we get $y^2 \equiv c \pmod n$.

GENERALIZED RABIN CRYPTOSYSTEM

Public key: n, B $(0 \leq B < n)$

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$$x, \quad -x, \quad \omega\left(x + \frac{B}{2}\right) - \frac{B}{2}, \quad -\omega\left(x + \frac{B}{2}\right) - \frac{B}{2}$$

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$e\left(\omega\left(x + \frac{B}{2}\right) - \frac{B}{2}\right) = \left(\omega\left(x + \frac{B}{2}\right) - \frac{B}{2}\right)\left(\omega\left(x + \frac{B}{2}\right) + \frac{B}{2}\right) = \omega^2\left(x + \frac{B}{2}\right)^2 - \left(\frac{B}{2}\right)^2 = x^2 + Bx = e(x)$

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Decryption of the generalized Rabin cryptosystem can be reduced to the decryption of the original Rabin cryptosystem.

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Therefore decryption can be done by factoring n and solving congruences

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Therefore computation of $\gcd(x_1 + r, n)$ or $\gcd(x_1 - r, n)$ must yield factors of n.

DISCRETE LOGARITHM CRYPTOSYSTEMS

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Note: Security of the ElGamal cryptosystem is based on infeasibility of the discrete logarithm computation.

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Its main advantage is that encryption and decryption are very similar, and even identical in some cases, and then the same hardware can be used for both encryption and decryption.

Let F a be a so-called round function and $K_0, K_1, \dots, K_n$ be sub-keys for rounds $0, 1, 2, \dots, n$.

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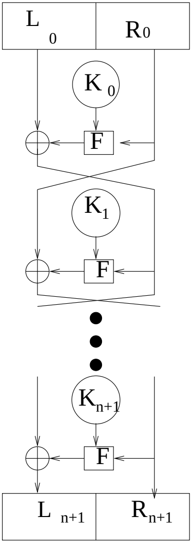

WHEN ARE ENCRYPTIONS PERFECTLY SECURE?

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- Given e_k, it should be unfeasible to determine d_k.

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Definition – computational distinguishability Let $X = \{X_n\}_{n \in N}$ and $Y = \{Y_n\}_{n \in N}$ be **probability ensembles** such that each X_n and Y_n ranges over strings of length n. We say that X and Y are computationally indistinguishable if for every feasible algorithm A the difference

$$d_A(n) = | Pr[A(X_n) = 1] - Pr[A(Y_n) = 1] |$$

is a negligible function in n.

SECURE ENCRYPTION – FIRST DEFINITION

Definition – semantic security of encryption A cryptographic system with an encryption function e is **semantically secure** if for every feasible algorithm A, there exists a feasible algorithm B so that for every two functions

$$f, h : \{0,1\}^* \rightarrow \{0,1\}^n$$

and all probability ensembles $\{X_n\}_{n \in N}$, where X_n ranges over $\{0,1\}^n$

$$Pr[A(e(X_n), h(X_n)) = f(X_n)] < Pr[B(h(X_n)) = f(X_n)] + \mu(n),$$

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RSA cryptosystem is not secure in the above sense. However, randomized versions of RSA are semantically secure.

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Quantum processes can generate perfect randomness and on this basis quantum (almost perfect) generators of randomness are already commercially available.

STORY of RANDOMNESS

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By Epikurus, there exists a true randomness that is independent of our knowledge.

Einstein also accepted the notion of randomness only in the relation to incomplete knowledge.

VIEWS on RANDOMNESS in 19th CENTURY

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There are only two possibilities, either a big chaos conquers the world, or order and law.

Marcus Aurelius

EINSTEIN versus BOHR

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Famous reply by Niels Bohr - one of the fathers of quantum mechanics.

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2. Quantum measurement yields, in principle, random outcomes.

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- There is no proof that perfect randomness exists in the real world.
- More exactly, there is no proof that quantum mechanical phenomena of the microworld can be exploited to provide a perfect source of randomness for the macroworld.

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compute x^2 and take a sequence of the middle digits of x^2 as a new "seed" x.

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Example. **Linear congruential generator**

SIMPLE PSEUDORANDOM GENERATORS

Informally, a **pseudorandom generator** is a deterministic polynomial time algorithm which expands short random sequences (called **seeds**) into longer bit sequences such that the resulting probability distribution is in polynomial time indistinguishable from the uniform probability distribution.

Example. **Linear congruential generator**

One chooses n-bit numbers m, a, b, X_0 and generates an n^2 element sequence

$$X_1 X_2 \dots X_{n^2}$$

of n-bit numbers by the iterative process

$$X_{i+1} = (aX_i + b) \bmod m.$$

CRYPTOGRAPHY and RANDOMNESS

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   Basic question: When is a pseudo-random generator good enough for cryptographical purposes?

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Definition. Let $l(n) : N \rightarrow N$ be such that $l(n) > n$ for all n. A **(cryptographically strong) pseudorandom generator with a stretch function l**, is an efficient deterministic algorithm which on the input of a random n-bit seed outputs a $l(n)$-bit sequence which is computationally indistinguishable from any random $l(n)$-bit sequence.

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A very fundamental concept: A predicate b is a hard core predicate of the function f if b is easy to evaluate, but $b(x)$ is hard to predict from f(x). (That is, it is unfeasible, given f(x) where x is uniformly chosen, to predict $b(x)$ substantially better than with the probability $1/2$.)

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Theorem Let f be a one-way function which is length preserving and efficiently computable, and b be a hard core predicate of f, then

$$G(s) = b(s) \cdot b(f(s)) \cdots b\left(f^{l(|s|)-1}(s)\right)$$

is a (cryptographically strong) pseudorandom generator with stretch function $l(n)$.

THEOREM

Theorem A cryptographically strong (perfect) pseudorandom generator exists if one-way functions exist.

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for one-time pad for encoding and decoding.

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It has been shown that if integer factoring is intractable, then the so-called *BBS* pseudo-random generator, discussed below, is unpredictable to the left.

(We make use of the fact that if factoring is unfeasible, then for almost all quadratic residues x mod n, coin-tossing is the best possible way to estimate the least significant bit of x after seeing x^2 mod n.)

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Let n be a Blum integer. Choose a random quadratic residue x_0 (modulo n).

For $i \geq 0$ let

$$x_{i+1} = {x_i}^2 \bmod n, \quad b_i = \text{the least significant bit of } x_I$$

For each integer i, let

$$BBS_{\ n,i}(x_0) = b_0 \ldots b_{i-1}$$

be the first i bits of the pseudo-random sequence generated from the seed x_0 by the *BBS* pseudo-random generator.

PERFECTLY SECURE CIPHERS - EXAMPLES

RANDOMIZED VERSION of RSA-LIKE CRYPTOSYSTEM

The scheme works for **any trapdoor function** (as in case of RSA),

$$f : D \to D, D \subset \{0,1\}^n,$$

for any **pseudorandom generator**

$$G : \{0,1\}^k \to \{0,1\}^l,\ k << l$$

and any **hash function**

$$h : \{0,1\}^l \to \{0,1\}^k,$$

where $\mathbf{n = l + k}$.

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Encryption of a message $m \in \{0,1\}^l$ is done as follows:

1. A random string $r \in \{0,1\}^k$ is chosen.
2. Set $x = (m \oplus G(r)) \| (r \oplus h(m \oplus G(r)))$. (If $x \notin D$ go to step 1.)
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Decryption of a cryptotext c.

- Compute $f^{-1}(c) = a\|b$, $|a| = l$ and $|b| = k$.
- Set $r = h(a) \oplus b$ and get $m = a \oplus G(r)$.

Comment: Operation "$\|$" stands for a concatenation of strings.

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Private key: Blum primes p and q.

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Encryption of $x \in \{0,1\}^m$.

1. Randomly choose $s_0 \in \{0, 1, \ldots, n\}$.
2. For i = 1, 2, . . . , m + 1 compute

$$s_i \leftarrow s_{i-1}^2 \mod n$$

   and $\sigma_i = lsb(s_i)$. ——{lsb – least significant bit}

The cryptotext is then (s_{m+1}, y), where $y = x \oplus \sigma_1\sigma_2 \ldots \sigma_m$.

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Decryption: of the cryptotext (r, y):
Let $d = 2^{-m} \bmod \phi(n))$.

- Let $s_1 = r^d \mod n$.
- For i = 1, . . . , m, compute $\sigma_i = lsb(s_i)$ and $s_{i+1} \leftarrow s_i^2 \mod n$

The plaintext x can then be computed as $y \oplus \sigma_1\sigma_2 \ldots \sigma_m$.

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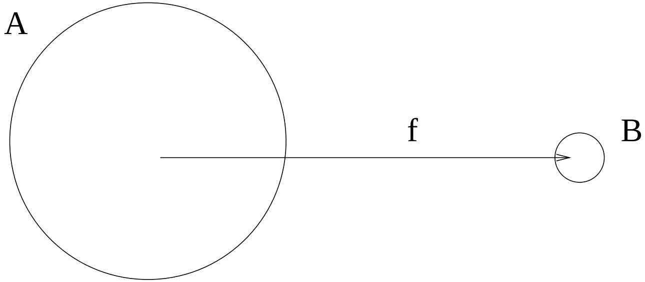

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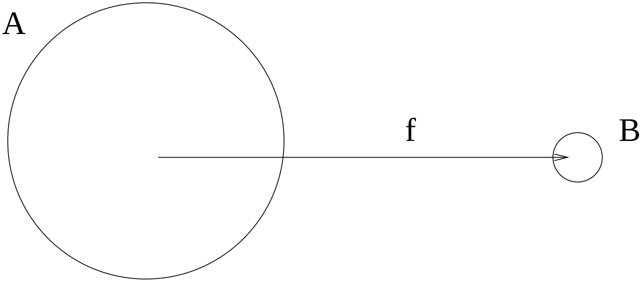

Cryptographic hash functions are hash functions that satisfy well enough basic cryptographic properties.

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- to help to solve a variety of cryptographic problems.

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Hash function have a variety applications, especially in the design of efficient algorithms and in cryptography.

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In cryptographic practice "**difficult**" generally means "**almost certainly beyond the reach of any adversary who must be prevented from breaking the system for as long as the security of the system is considered to be very important**".

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In 2013 a long-term **Password Hashing Competition** was announced to choose a new, standard algorithm for password hashing.

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In addition, to send reliably a message w through an unreliable (and cheap) channel, one sends also its (small) hash h(w) through a very secure (and therefore expensive) channel.

The receiver, familiar also with the hash function h that is being used, can then verify the integrity of the message w' he receives by computing h(w') and comparing

h(w) and h(w') .

EXAMPLES

Example 1 For a vector $a = (a_1, \ldots, a_k)$ of integers let

$$H(a) = \sum_{i=0}^{k} a_i \mod n$$

where n is a product of two large primes.
This hash functions does not meet any of the three properties mentioned above.

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This function is one-way, but it is not weakly collision resistant.

HASH FUNCTIONS h from CRYPTOSYSTEMS

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If

$$x = x_1 \| x_2 \| \ldots \| x_m$$

is the decomposition of x into substrings of length n, g_0 is a random string, and

$$g_i = f(x_i, g_{i-1})$$

for $i = 1, \ldots, m$, where f is a function that "incorporates" encryption functions e_j of the cryptosystem, for suitable keys k_j, then

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For example such good properties have these two functions:

$$f(x_i, g_{i-1}) = e_{g_{i-1}}(x_i) \oplus x_i$$
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PRACTICALLY USED HASH FUNCTIONS

A variety of hash functions has been constructed. Very often used hash functions were MD4, MD5 (created by Rivest in 1990 and 1991 and producing 128 bit message digest).

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Observe that every cryptographic hash function is vulnerable to a collision attack using so called birthday attack. Due to the **birthday problem** a hash of n bits can be broken in $\sqrt{2^n}$ evaluations of the hash function - much faster than the brute force attack.

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- In order to ensure long-term robustness of applications that use hash functions a public competition was announced by NIST to replace SHA-2.
- On October 2012 Keccak was selected as the winner and a version of this algorithm is expected to be a new standard (since 2014) under the name SHA-3.

MD5

Often used in practise has been hash function MD5 designed in 1991 by Rivest. It maps any binary message into 128-bit hash.

The input message is broken into 512-bit blocks, divided into 16 words-states (of 32 bits) and padded if needed to have final length divisible by 512. Padding consists of a bit 1 followed by so many 0's as required to have the length up to 64 bits fewer than a multiple of 512. Final 64 bits represent the length of the original message modulo 2^{64}.

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The main MD5 algorithm operates on 128-bits words that are divided into four 32-bits words A, B, C, D initialized to some fixed constants. The main algorithm then operates on 512 bit message blocks in turn - each block modifying the state.

The processing of a message consists of four rounds. j-th round is composed of 16 similar operations using non-linear functions F_j and left rotations by s_j places where s_j varies for each round - see next figure.

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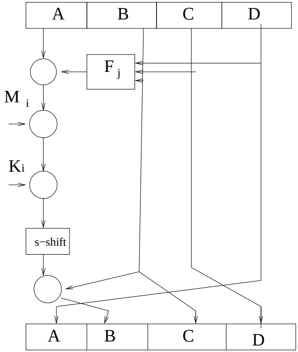
The processing of a message consists of four rounds. j-th round is composed of 16 similar operations using non-linear functions F_j and left rotations by s_j places where s_j varies for each round - see next figure. K_i and M_i are 32-bits keys and messages.



BREAKING MD5

- In 2006 Vladimír Klima published an algorithm to find a collision for MD5 within one minute on a notebook.
- In 2010 T. Xie, O. Feng published single-block MD5 collision.

HOW to FIND COLLISIONS of HASH FUNCTIONS

The most basic method is based on so-called birthday paradox related to so-called the birthday problem.

BIRTHDAY PROBLEM and its VARIATIONS

It is well known that if there are 23 (29) [40] $\{57\}$ $< 100 >$ people in one room, then the probability that two of them have the same birthday is more than 50% (70%)[89%] $\{99\%\}$ $< 99.99997\% >$ — this is called a Birthday paradox.

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More generally, if we have n objects and r people, each choosing one object (so that several people can choose the same object), then if $r \approx 1.177\sqrt{n}(r \approx \sqrt{2n\lambda})$, then probability that two people choose the same object is 50% $((1 - e^{-\lambda})\%)$.

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Another version of the birthday paradox: Let us have n objects and two groups of r people. If $r \approx \sqrt{\lambda n}$, then probability that someone from one group chooses the same object as someone from the other group is $(1 - e^{-\lambda})$.

BASIC DERIVATIONS related to BIRTHDAY PARADOX

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For the probability $\bar{p}(n)$ that all $n < 366$ people in a room have birthday in different days, it holds

$$\bar{p}(n) = \prod_{i=1}^{n-1}\left(\frac{365-i}{365}\right) = \frac{\prod_{i=1}^{n-1}(365-i)}{365^{n-1}} = \frac{365!}{365^{n}(365-n)!}$$

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Probability $p(n)$ that at least two person have the same birthday is therefore

$$p(n) = 1 - \bar{p}(n)$$

This probability is larger than 0.5 first time for $n = 23$.

FINDING COLLISIONS USING BIRTHDAY PARADOX

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To find collisions, that is two x_1 and x_2 such that $h(x_1) = h(x_2)$ is easier, thanks to the birthday paradox and can be done by the following algorithm:

ALGORITHM

Input: A hash function h onto a domain of size n, a real θ and an empty hash table.
Output: A pair (x_1, x_2) such that $x_1 \neq x_2$ and $h(x_1) = h(x_2)$

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1. **for** $\theta\sqrt(n)$ **different** x **do**
2. **compute** $y = h(x)$
3. **if there is a** (y, x') **pair in the hash table then**
4. **yield** (x, x') **and stop**
5. **add** (y, x) **to the hash table**
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Theorem If we pick the numbers x with uniform distribution in $\{1, 2, \ldots, n\}$ $\theta\sqrt{n}$ times, then we get at least one number twice with probability converging (for $n \to \infty$) to

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For $n = 365$ we get triples: $(\theta, \theta\sqrt{n}, \text{probability})$ as follows: (0.79, 15, 25%); (1.31, 25, 57%); (2.09, 40, 89%)

WHY CURRENTLY BROADLY USED HASHES HAVE 160 BITS?

The birthday paradox imposes also a lower bound on the sizes of hashes of the cryptographically good hash functions.

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For example, a 40-bit hashes would be insecure because a collision could be found with probability 0.5 with just over 40^{20} random guesses.

Minimum acceptable size of hashes seems to be 128 and therefore 160 are used in such important systems as DSS – Digital Signature Schemes (a standard).