Part I

# Public-key cryptosystems basics: I. Key exchange, knapsack, RSA

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**Symmetric key cyptosystms** are also called **private key cryptosystems** because the communicating parties use unique private key - no one else should know that key.

**The realization that a cryptosystem does not need to be symmetric/private can be seen as the single most important breakthrough in the modern cryptography and as one of the key discoveries leading to the internet and to information society.**

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Supercomputer Salomon in Ostrava, with performance 1.407 petaflops was on 40th place in June 2015; best in India on 79th place.

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Ostrava's Solomon is currently on 282 position. They got a new one, called Barbora, with 8 times larger performance.

# EXASCALE COMPUTERS

# Who is building them?

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**Why they are needed?** Exascale computers would allow to make extremely precise simulations of biological systems what is expected to allow to deal with such problems as climate change and growing food that could withstand drought.

# CHAPTER 5: PUBLIC-KEY CRYPTOGRAPHY I. RSA

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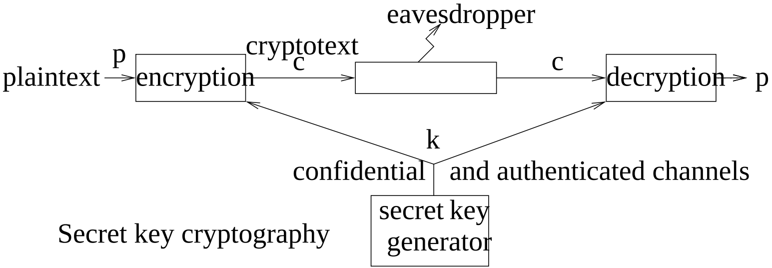
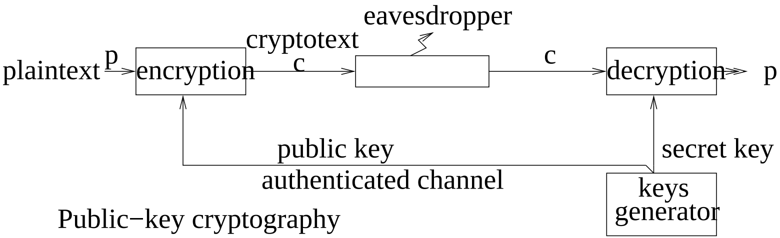
Therefore, **private key cryptography is not a sufficiently good tool for massive communication capable to protect secrecy, privacy and anonymity.**

# SYMMETRIC versus ASYMMETRIC CRYPTOSYSTEMS



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Encryption and decryption keys of public key cryptography could (and should) be different - we can therefore say also that public-key cryptography is **asymmetric cryptography**. Secret key cryptography, that has the same key for encryption and for decryption is then called also as **symmetric cryptography**.

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- For example around 1970 only US government institutions needed to distribute daily tons of keys (on discs, tapes,...) to users they planned to communicate with.

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- Informatization of society was questioned because if governments had problems with key distribution how smaller companies could handle the key distribution problem without bankrupting?
- At the same time, the **key distribution problem** used to be considered, practically by all, as an **unsolvable problem**.

# FIRST INGENIOUS IDEA - KEY PLAYERS

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**Great idea was born. The problem then was to find a computational realization of this great idea.** The first idea - to model locking of padlocks by doing an encryption.

## MERKLE JOINING DIFFIE-HELLMAN

After Diffie and Hellman announced their solution to the key generation problem, Ralph Merkle claimed, and could prove, that he had a similar idea some years ago.

That is the way why some people talk about Merkle-Diffie-Hellman key exchange.

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Let Alice use the encryption substitution.

| | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | b | c | d | e | f | g | h | i | j | k | l | m | n | o | p | q | r | s | t | u | v | w | x | y | z |
| H | F | S | U | G | T | A | K | V | D | E | O | Y | J | B | P | N | X | W | C | Q | R | I | M | Z | L |

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| H | F | S | U | G | T | A | K | V | D | E | O | Y | J | B | P | N | X | W | C | Q | R | I | M | Z | L |

Let Bob use the encryption substitution.

| | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | b | c | d | e | f | g | h | i | j | k | l | m | n | o | p | q | r | s | t | u | v | w | x | y | z |
| C | P | M | G | A | T | N | O | J | E | F | W | I | Q | B | U | R | Y | H | X | S | D | Z | K | L | V |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
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```

Let Bob use the encryption substitution.

```
a b c d e f g h i j k l m n o p q r s t u v w x y z
C P M G A T N O J E F W I Q B U R Y H X S D Z K L V
```

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Message | m | e | e | t | m | e | a | t | n | o | o | n |
| Alice's encrypt. | Y | G | G | C | Y | G | H | C | J | B | B | J |

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a b c d e f g h i j k l m n o p q r s t u v w x y z
C P M G A T N O J E F W I Q B U R Y H X S D Z K L V
```

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Message | m | e | e | t | m | e | a | t | n | o | o | n |
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| H | F | S | U | G | T | A | K | V | D | E | O | Y | J | B | P | N | X | W | C | Q | R | I | M | Z | L |

Let Bob use the encryption substitution.

| | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | b | c | d | e | f | g | h | i | j | k | l | m | n | o | p | q | r | s | t | u | v | w | x | y | z |
| C | P | M | G | A | T | N | O | J | E | F | W | I | Q | B | U | R | Y | H | X | S | D | Z | K | L | V |

| Message | m | e | e | t | m | e | a | t | n | o | o | n |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Alice's encrypt. | Y | G | G | C | Y | G | H | C | J | B | B | J |
| Bob's encrypt. | L | N | N | M | L | N | O | M | E | P | P | E |
| Alice's decrypt. | Z | Q | Q | X | Z | Q | L | X | K | P | P | K |

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| H | F | S | U | G | T | A | K | V | D | E | O | Y | J | B | P | N | X | W | C | Q | R | I | M | Z | L |

Let Bob use the encryption substitution.

| | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | b | c | d | e | f | g | h | i | j | k | l | m | n | o | p | q | r | s | t | u | v | w | x | y | z |
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| Message | m | e | e | t | m | e | a | t | n | o | o | n |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Alice's encrypt. | Y | G | G | C | Y | G | H | C | J | B | B | J |
| Bob's encrypt. | L | N | N | M | L | N | O | M | E | P | P | E |
| Alice's decrypt. | Z | Q | Q | X | Z | Q | L | X | K | P | P | K |
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| H | F | S | U | G | T | A | K | V | D | E | O | Y | J | B | P | N | X | W | C | Q | R | I | M | Z | L |

Let Bob use the encryption substitution.

| | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | b | c | d | e | f | g | h | i | j | k | l | m | n | o | p | q | r | s | t | u | v | w | x | y | z |
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| Message | m | e | e | t | m | e | a | t | n | o | o | n |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Alice's encrypt. | Y | G | G | C | Y | G | H | C | J | B | B | J |
| Bob's encrypt. | L | N | N | M | L | N | O | M | E | P | P | E |
| Alice's decrypt. | Z | Q | Q | X | Z | Q | L | X | K | P | P | K |
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**Observation** The first idea does not work.

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| a | b | c | d | e | f | g | h | i | j | k | l | m | n | o | p | q | r | s | t | u | v | w | x | y | z |
| H | F | S | U | G | T | A | K | V | D | E | O | Y | J | B | P | N | X | W | C | Q | R | I | M | Z | L |

Let Bob use the encryption substitution.

| | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | b | c | d | e | f | g | h | i | j | k | l | m | n | o | p | q | r | s | t | u | v | w | x | y | z |
| C | P | M | G | A | T | N | O | J | E | F | W | I | Q | B | U | R | Y | H | X | S | D | Z | K | L | V |

| Message | m | e | e | t | m | e | a | t | n | o | o | n |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Alice's encrypt. | Y | G | G | C | Y | G | H | C | J | B | B | J |
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| Alice's decrypt. | Z | Q | Q | X | Z | Q | L | X | K | P | P | K |
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| a | b | c | d | e | f | g | h | i | j | k | l | m | n | o | p | q | r | s | t | u | v | w | x | y | z |
| H | F | S | U | G | T | A | K | V | D | E | O | Y | J | B | P | N | X | W | C | Q | R | I | M | Z | L |

Let Bob use the encryption substitution.

| | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | b | c | d | e | f | g | h | i | j | k | l | m | n | o | p | q | r | s | t | u | v | w | x | y | z |
| C | P | M | G | A | T | N | O | J | E | F | W | I | Q | B | U | R | Y | H | X | S | D | Z | K | L | V |

| Message | m | e | e | t | m | e | a | t | n | o | o | n |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Alice's encrypt. | Y | G | G | C | Y | G | H | C | J | B | B | J |
| Bob's encrypt. | L | N | N | M | L | N | O | M | E | P | P | E |
| Alice's decrypt. | Z | Q | Q | X | Z | Q | L | X | K | P | P | K |
| Bob's decrypt. | w | n | n | t | w | n | y | t | x | b | b | x |

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# SECURE COMMUNICATION without ecreret keys

The idea contained in the above mention padlock protocol has been materialized by Shamir as follows:

(**Shamir's "no-key algorithm"**)

**Basic assumption:** Each user $X$ has its own

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2. Bob sends $e_B(e_A(w))$ to Alice

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3. Alice sends $d_A(e_B(e_A(w))) = e_B(w)$ to Bob

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**Disadvantage:** 3 communications are needed (in such a context 3 is a too large number).
**Advantage:** It is a perfect protocol for secret distribution of messages.

# PUBLIC ESTABLISHMENT of SECRET KEYS

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**Diffie-Hellman Protocol:** If two parties, Alice and Bob, want to create a common secret key, then they first agree, somehow, on large primes $p$ and a $q < p$ of large order in $Z_p^*$ and then they perform, using a public channel, the following activities.

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An eavesdropper seems to need, in order to determine $x$ from **X, q, p** and $y$ from **Y, q, p**, a capability to compute discrete logarithms, or to compute $q^{xy}$ from $q^x$ and $q^y$, what is believed to be infeasible.

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That is the way why some people talk now about Merkle-Diffie-Hellman key exchange.

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7. Meanwhile, Eve enjoys reading Alice's message.

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The protocol required still too much communication and a cooperation of both parties for quite a time.

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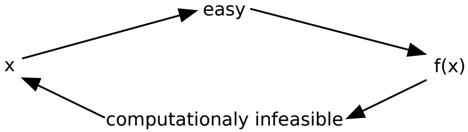
**A search (hunt) for such a function started.**

# ONE-WAY FUNCTIONS

Informally, a function $F : N \to N$ is said to be a one-way function if it is easily computable - in polynomial time - but any computation of its inverse is infeasible.

A one-way permutation is a 1-1 one-way function.



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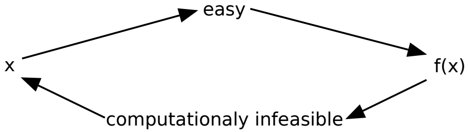


EA more formal approach

Definition A function $f : \{0,1\}^* \rightarrow \{0,1\}^*$ is called a strongly one-way function if the following conditions are satisfied:

1. $f$ can be computed in polynomial time;
2. there are $c, \varepsilon > 0$ such that $|x|^{\varepsilon} \leq |f(x)| \leq |x|^{c}$;
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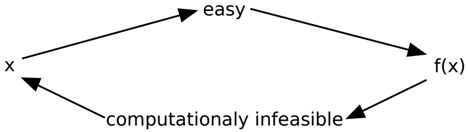


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Candidates: Modular exponentiation: $f(x) = a^x \bmod n$
Modular squaring $f(x) = x^2 \bmod n$, $n$ – $a$ Blum integer
Prime number multiplication $f(p, q) = pq$.

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**New basic idea:** To make a clever use of outcomes of computational complexity theory.

# CRYPTOGRAPHY and COMPUTATIONAL COMPLEXITY

Modern cryptography uses such encryption methods that no "enemy" can have enough computational power and time to do decryption (even those capable to use thousands of supercomputers during tens of years for encryption).

Modern cryptography is based on negative and positive results of complexity theory – on the fact that for some algorithm problems no efficient algorithm seem to exists, surprisingly, and for some "small" modifications of these problems, even more surprisingly, simple, fast and good (randomized) algorithms do exist. Examples:

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There is a list of "most wanted to factor integers". Top recent successes, using thousands of computers for months.

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Primes recognition: Is a given $n$ a prime? – fast randomized algorithms exist (1977). The existence of polynomial deterministic algorithms for primes recognition has been shown only in 2002

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**Knapsack problem:** Given a ( knapsack - integer) vector $X = (x_1, \ldots, x_n)$ and an (integer capacity) $c$, find a binary vector $(b_1, \ldots, b_n)$ such that

$$\sum_{i=1}^{n} b_i x_i = c.$$

Problem is $NP$-hard in general, but easy if $x_i > \sum_{j=1}^{i-1} x_j$, for all $1 < i \leq n$.

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A way to design a trapdoor one-way function is to transform an easy case of a hard (one-way) function to a hard-looking case of such a function, that can be, however, solved easily by those knowing how the above transformation was performed.

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# PROBELMS ITH a MASSIVE USE OF PKC

Once PKC is to be used broadly usual a huge machinery has to be established in a country for generating, storing and validation (of validity,....) of public keys.

# PUBLIC KEY CRYPTOGRAPHY based on KNAPSCAK PROBLEM

Interesting and important public key cryptosystems were developed on the base of the KNAPSACK PROBLEM and its modifications

# GENERAL, UNFEASIBLE, KNAPSACK PROBLEM

**KNAPSACK PROBLEM:** Given an integer-vector $X = (x_1, \ldots, x_n)$ and an integer $c$. Determine a binary vector $B = (b_1, \ldots, b_n)$ (if possible) such that $XB^T = c$.

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**Algorithm** – to solve knapsack problems with superincreasing vectors:

**for** $i = n \leftarrow$ **downto 2 do**
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$X = (1,3,5,10,20,41,94,199)$, $c = 242$

# KNAPSACK and MCELIECE CRYPTOSYSTEMS

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Example
If $A = (74, 82, 94, 83, 39, 99, 56, 49, 73, 99)$ and $B = (1100110101)$ then

$$AB^T =$$

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**Lemma**

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**Proof** Let $X'w^T = c$. Then

$$c' \equiv u^{-1}c \equiv u^{-1}X'w^T \equiv u^{-1}uXw^T \equiv Xw^T (\text{mod } m).$$

Since $X$ is superincreasing and $m > 2x_n$ we have

$$(Xw^T) \bmod m = Xw^T$$

and therefore

$$c' = Xw^T.$$

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In order to encrypt an English plaintext, we first encode its letters by 5-bit numbers _ - 00000, A - 00001, B - 00010,... and then divide the resulting binary strings into blocks of length 10.

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$$w_1 = (0000100110) \qquad w_2 = (1001001001) \qquad w_3 = (0001100001)$$

Encryption:

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**Decryption** of cryptotexts: (2163, 2116, 1870, 3599)

By multiplying with $u^{-1} = 61 \pmod{1250}$ we get new cryptotexts (several new $c'$)

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By multiplying with $u^{-1} = 61 \pmod{1250}$ we get new cryptotexts (several new $c'$)

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# DESIGN of KNAPSACK CRYPTOSYSTEMS – EXAMPLE

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$\mathbf{m} = 1250$, $u = 41$
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In order to encrypt an English plaintext, we first encode its letters by 5-bit numbers _ - 00000, A - 00001, B - 00010,... and then divide the resulting binary strings into blocks of length 10.

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**Remark.** Density of super-increasing vectors of length $n$ is $\leq \frac{n}{n-1}$

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For our version of the **knapsack problem** the term **Merkle-Hellman (Knapsack) Cryptosystem** is often used.

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(McEliece suggested to use $m = 10, t = 50$.)

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6. Observe that the number of potential matrices $S$ and $P$ is so large that probability of guessing these matrices is smaller than probability of guessing correct plaintext!!!
7. It can be shown that it is not safe to encrypt twice the same plaintext with the same public key (and different error vectors).

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- For a standard selection of parameters the public key is more than 521 000 bits long.
- That is why cryptosystem is rarely used in practise in spite of the fact that it has some advantages comparing with RSA cryptosystem discussed next - it has more easy encoding and decoding.

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In doing that we will illustrate modern distributed techniques to factorize very large integers.

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All that means that in modern cryptography we need, for security reasons, to work with numbers that have no correspondence in the physical reality.

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- SSL/TLS use a combination of PKC and SKC. SSL uses mainly **RSA**, TLS uses mainly **ECC** (Elliptic Curves Cryptography).

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Indeed, if $c = w^e$, then $c^2 = (w^e)^2 = w^{2e} = (w^2)^e$.

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**Method 2** It follows from the Euler Totient Theorem, see the Appendix, that

$$m^{-1} \equiv m^{\phi(n)-1} \bmod \phi(n)$$

if $m < n$ and $GCD(m, n) = 1$

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**Theorem** (Fermat's Little Theorem)

$$w^{p-1} \equiv 1 \ (\mathrm{mod}\ p)$$

for any $w$ and any prime $p$.

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**Method 2** It follows from Euler's Totient Theorem that

$$m^{-1} \equiv m^{\phi(n)-1} \bmod \phi(n)$$

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# EXPONENTIATION by squaring

Exponentiation (modular) plays the key role in many cryptosystems. If

$$n = \sum_{i=0}^{k-1} b_i 2^i, \quad b_i \in \{0,1\}$$

then

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**Algorithm for exponentiation**

**begin** $e \leftarrow 1$; $p \leftarrow a$;
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$3^{10000} \mod 13 = 3$
$3^{340} \mod 11 = 1$
$3^{100} \mod 79 = 51$

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$$100, 017, 111, 817, 200, 704$$

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**Plaintext:** KARLSRUHE **First encoding (letters–int.):** 100017111817200704

Since $10^3 < n < 10^4$, the numerical plaintext is divided into blocks of 3 digits $\Rightarrow$ therefore 6 integer plaintexts are obtained

$$100, 017, 111, 817, 200, 704$$

**Encryptions:**

$$100^{23} \bmod 2501, \quad 17^{23} \bmod 2501, \quad 111^{23} \bmod 2501$$
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provide cryptotexts:

$$2306, 1893, 621, 1380, 490, 313$$

# EXAMPLE of ENCRYPTION and DECRYPTION in RSA

**Example** of the design and of the use of RSA cryptosystems.

- By choosing $p = 41, q = 61$ we get $n = 2501, \phi(n) = 2400$
- **By choosing $d = 2087$ we get $e = 23$**
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**encrypted using the RSA cryptosystem with 129 digit number, called also RSA129**

n: 114 381 625 757 888 867 669 235 779 976 146 612 010 218 296 721 242 362 562 561 842 935 706 935 245 733 897 830 597 123 513 958 705 058 989 075 147 599 290 026 879 543 541.

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The problem was solved in 1994 by first factorizing n into one 64-bit prime and one 65-bit prime, and then computing the plaintext

THE MAGIC WORDS ARE SQUEMISH OSSIFRAGE

In 2002 RSA inventors received Turing award.

# Abstract of the US RSA patent 4,405,829

The system includes a communication channel coupled to at least one terminal having an encoding device and to at least one terminal having a decoding device.

**A message-to-be-transferred is enciphered to ciphertext at the encoding terminal by encoding a message as a number, $M$, in a predetermined set.**

That number is then raised to a first predetermined power (associated with the intended receiver) and finally computed. The remainder of residue, $C$, is ... computed when the exponentiated number is divided by the product of two predetermined prime numbers (associated with the predetermined receiver).

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- RSA problem: Given a public key $(n, e)$ and a cryptotext $c$ find an $m$ such that $c = m^e (\bmod n)$.

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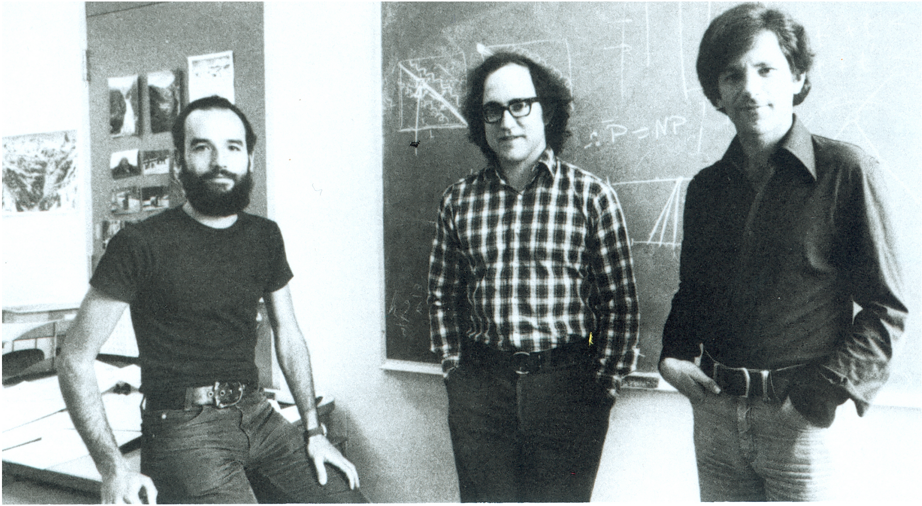
- At night Rivest could not sleep, mediated and all of sudden got an idea. In the morning the paper about RSA was practically written down.

# Ron Rivest, Adi Shamir and Leonard Adleman




Copied from the brochure on LCS

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- This discovery was, however, too early and GCHQ kept it secret and they disclosed their discovery only in 1997, after RSA has been shown very successful.

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- However, if an integer is not a prime then it is very hard to find its factors.

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   - 3.1 Neither $d$ nor $e$ should be small.
   - 3.2 $d$ should not be smaller than $n^{\frac{1}{4}}$. (For $d < n^{\frac{1}{4}}$ a polynomial time algorithm is known to determine $d$).

# WHAT "SMALL" MEANS

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**For example, if $p - q < 2n^{0.25}$**

**(which for even small 1024-bit values of $n$ is about $3 \cdot 10^{77}$)**

**then factoring of $n$ is quite easy.**

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Fast randomized algorithms for prime recognition has been known since 1977. One of the simplest one is due to Rabin and will be presented later.

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Several simple and some sophisticated factorization algorithms will be presented and illustrated in the following.

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In 2009 RSA-768, a 768-bits number, was factorized by a team from several institutions. Time needed would be 2000 years on a single 2.2 GHz AND Opterons. Cash price obtained - 30 000 $.

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Moreover, in such a case s can be obtained by testing.
**Question** Is there enough primes (to choose again and again new ones)?
No problem, the number of primes of length 512 bit or less exceeds $10^{150}$.

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**(which for even small 1024-bit values of $n$ is about $3 \cdot 10^{77}$)**

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2. There is an efficient algorithm, on the next slide, to determine the plaintexts $w$ from the cryptotexts $c$ obtained from $w$ by an RSA-encryption provided the efficiently computable function **half** can be used as the oracle:

# RSA in PRACTICE

- 660-bits integers were already (factorized) broken in practice.
- 1024-bits integers are currently used as moduli.
- 512-bit integers can be factorized with a device costing 5.000 $ in about 10 minutes.
- 1024-bit integers could be factorized in 6 weeks by a device costing 10 millions of dollars.

# ATTACKS on RSA

RSA can be seen as well secure. However, this does not mean that under special circumstances some special attacks can not be successful. Two of such attacks are:

- The first attack succeeds in case the decryption exponent is not large enough. **Theorem** (Wiener, 1990) Let $n = pq$, where $p$ and $q$ are primes such that $q < p < 2q$ and let $(n, e)$ be such that $de \equiv 1 (\mathrm{mod}\, \phi(n))$. If $d < \frac{1}{3}n^{1/4}$. then there is an efficient procedure for computing $d$.
- **Timing attack** P. Kocher (1995) showed that it is possible to discover the decryption exponent by carefully counting the computation times for a series of decryptions. Basic idea: Suppose that Eve is able to observes times Bob needs to decrypt several cryptotext s. Knowing cryptotext and times needed for their decryption, it is possible to determine decryption exponent.

# CASES WHEN RSA IS EASY TO BREAK

- If an user $U$ wants to broadcast a value $x$ to $n$ other users, using for a communication with a user $P_i$ a public key $(e, N_i)$, where $e$ is small, by sending $y_i = x^e \bmod N_i$.
- If $e = 3$ and $2/3$ of the bits of the plaintext are known, then one can decrypt efficiently;
- If 25% of the least significant bits of the decryption exponent $d$ are known, then $d$ can be computed efficiently.
- If two plaintexts differ only in a (known) window of length $1/9$ of the full length and $e = 3$, one can decrypt the two corresponding cryptotext.
- Wiener showed how to get secret key efficiently if $n = pq$, $q < p < 2q$ and $d < \frac{1}{3} n^{0.25}$.

- Imad Khaled Selah, Abdullah Darwish, Saleh Ogeili: Mathematical attacks on RSA Cryptosystem, Journal of Computer Science 2 (8) 665-671, 2006
- Dan Boneh: Twenty years of attacks on RSA Cryptosystems, crypto.stanford.edu/ Dabo/pubs/papers/RSA-surwey.pdf