Part I

Linear codes

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Many practically important linear codes have also an efficient decoding.

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Comment. To design linear codes we will use Galois fields $GF(q)$ with q being a prime. One can also use Galois fields $GF(q^k)$, $k > 1$, but their structure and operations are defined in a more complex way, see the Appendix.

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Encoding (code) is called systematic if for any $m \in M \subset \Sigma^*$

$$e(m) = mc_m \text{ for some } c_m \in \Sigma^*$$

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A code is called systematic if its encoder transmit a message (an input dataword) w into a codeword of the form wc_w, or (w, c_w). That is if the codeword for the message w consists of two parts: the message w itself (called also information part) and a redundancy part c_w

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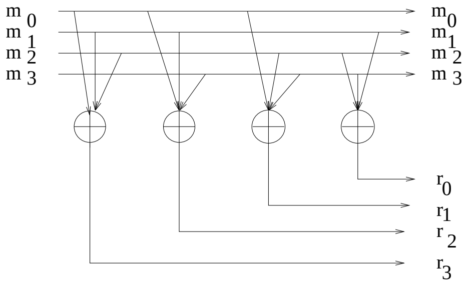
Nowadays most of the stream codes that are used in practice are systematic.

An example of a systematic encoder, that produces so called extended Hamming $(8, 4, 1)$ code is in the following figure.



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In general, does it has a sense to look for such codes that some important sum of any two codewords is again a codeword?

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Linear codes are special sets of words of a fixed length n over an alphabet $\Sigma_q = \{0, .., q-1\}$, where q is a (power of) prime.

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3. There are simple encoding/decoding procedures for linear codes.

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Disadvantages of linear codes are small:

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2. The restriction to linear codes might be a restriction to weaker codes than sometimes desired.

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Proof Operations (a) - (c) just replace one basis by another. Last two operations convert a generator matrix to one of an equivalent code.

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Theorem Let G be a generator matrix of an $[n, k]$-code. Rows of G are then linearly independent .By operations (a) - (e) the matrix G can be transformed into the form: $[I_k|A]$ where I_k is the $k \times k$ identity matrix, and A is a $k \times (n-k)$ matrix.

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Example

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \rightarrow$$

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Proof If $u_1 G - u_2 G = 0$, then

$$0 = \sum_{i=1}^{k} u_{1,i} w_i - \sum_{i=1}^{k} u_{2,i} w_i = \sum_{i=1}^{k} (u_{1,i} - u_{2,i}) w_i$$

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And, therefore, since w_i are linearly independent, $u_1 = u_2$.

LINEAR CODES as SYSTEMATIC CODES

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Since to each linear $[n, k]$-code C there is a generator matrix of the form $G = [I_k|A]$ an encoding of a dataword w with G has the form

$$wG = w \cdot wA$$

Each linear code is therefore equivalent to a systematic code.

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Definition Suppose C is an $[n, k]$-code over F_q^n and $u \in F_q^n$. Then the set

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In practice, this decoding method is too slow and requires too much memory.

How good are particular linear codes?

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SYNDROMES APPROACH to DECODING

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If $u \cdot v = 0$ then words (vectors) u and v are called **orthogonal words**.

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Answer: All words of S have at the end the same symbol as at the beginning.

EXAMPLE

For the $[n, 1]$-repetition (binary) code C, with the generator matrix

$$G = (1, 1, \dots, 1)$$

the dual code $C^\perp$ is $[n, n-1]$-code with the generator matrix $G^\perp$, described by

$$G^\perp = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 \\ & \dots & & & & \\ 1 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

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The rows of a parity check matrix are parity checks on codewords. They actually say that certain linear combinations of elements of every codeword are zeros modulo 2.

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0	1	0	0	1	1	1	1	0	0	0	1	1	0	1	0	0	1
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When preparing a “syndrome decoding” it is sufcient to store only two columns: one for **coset leaders** and one for **syndromes**.

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- **Step 1** Given y compute $S(y)$.
- **Step 2** Locate $z = S(y)$ in the syndrome column.
- **Step 3** Decode y as $y - l(z)$.

Example If $y = 1111$, then $S(y) = 01$ and the above decoding procedure produces

$$1111\text{–}0100 = 1011.$$

Syndrom decoding is much faster than searching for a nearest codeword to a received word. However, for large codes it is still too inefficient to be practical.

In general, the problem of finding the nearest neighbour in a linear code is NP-complete.

KEY OBSERVATION for SYNDROM COMPUTATION

When preparing a "syndrome decoding" it is sufcient to store only two columns: one for **coset leaders** and one for **syndromes**.

Example

coset leaders	syndromes
l(z)	z
0000	00
1000	11
0100	01
0010	10

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In general, the problem of finding the nearest neighbour in a linear code is NP-complete. Fortunately, there are important linear codes with really efficient decoding.

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Properties of binary Hamming codes Coset leaders are precisely words of weight ≤ 1. The syndrome of the word $0 \ldots 010 \ldots 0$ with 1 in j-th position and 0 otherwise is the transpose of the j-th column of H.

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Hamming codes were originally used to deal with errors in long-distance telephon calls.

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In case $q = 0.9$ the probability of correct transmission is 0.6561 in the case no error correction is used and 0.8503 in the case Hamming code is used - an essential improvement.

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G_{24} is $(24, 12, 8)$-code and the weights of all codewords are multiples of 4. G_{23} is obtained from G_{24} by deleting last symbols of each codeword of G_{24}. G_{23} is $(23, 12, 7)$-code.

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GOLAY CODES - CONSTRUCTION

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Rows of the last 11 columns are cyclic permutations of the first row which has 1 at those positions that are squares modulo 11, that is

$$0, 1, 3, 4, 5, 9.$$

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$$M_{2^n} = \begin{bmatrix} M_{2^{n-1}} & M_{2^{n-1}} \\ M_{2^{n-1}} & \bar{M}_{2^{n-1}} n \end{bmatrix}$$

where $\bar{M}_n$ is the complementary matrix to M_n (with 0 and 1 interchanged).

EXAMPLE

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- Matrix Q_r is obtained by considering all combinations of r rows of $G_{1,m}$ and by obtaining products of these rows/vectors, component by component. The result of each of such a multiplication constitues a row of Q_r.

EXAMPLE

$$G_{1,4} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

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Plotkin bound implies that q-nary error-correcting codes with $d \geq n(1 - 1/q)$ have only polynomially many codewords and hence are not very interesting.

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When $d = 1$, then E is an $[n-1, k, 1]$ code, if C has no codeword of weight 1 whose nonzero entry is in last coordinate; otherwise, if $k > 1$, then E is an $[n-1, k-1, d^*]$ code with $d^* > 1$

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They are codes a generator matrix of which has rows labelled by polynomials X^i, $0 \leq i \leq k-1$, columns labeled by elements $0, 1, \ldots, q-1$ and the element in the row labelled by a polynomial p and in the column labelled by an element u is $p(u)$.

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Reed-Solomon codes are used in digital television, satellite communication, wireless communication, barcodes, compact discs, DVD,... They are very good to correct burst errors - such as ones caused by solar energy.

SOCCER GAMES BETTING SYSTEM

Ternary Golay code with parameters (11, 729, 5) can be used to bet for results of 11 soccer games with potential outcomes 1 (if home team wins), 2 (if guest team wins) and 3 (in case of a draw).

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In case one has to bet for 13 games, then one can usually have two games with pretty sure outcomes and for the rest one can use the above ternary Golay code.

APPENDIX

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Parity-check matrices for LDPC codes are often (pseudo)-randomly generated, subject to sparsity constrains. Such LDPC codes are proven to be good with a high probability.

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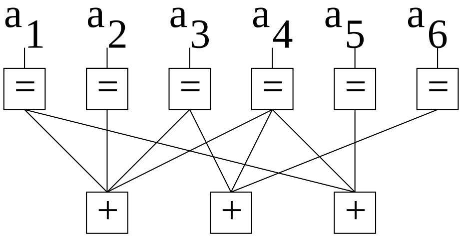
BI-PARTITE (TANNER) GRAPHS REPRESENTATION of LDPC CODES

An $[n, k]$ LDPC code can be represented by a bipartite graph between a set of n top "variable-nodes (v-nodes)" and a set of bottom $(n-k)$ "parity check nodes (pc-nodes)".
Variable nodes:



Parity check nodes:

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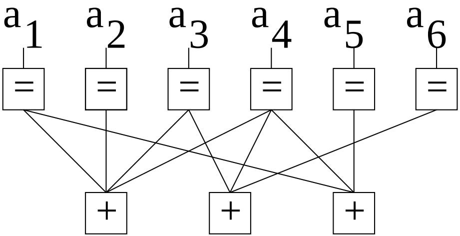

Parity check nodes:

The corresponding parity check matrix has $n-k$ rows and n columns and i-th column has 1 in the j-th row exactly in case if i-th v-node is connected to j-th c-node.

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

TANNER GRAPHS - CONTINUATION

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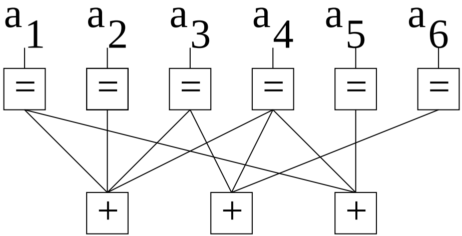
The LDPC-code with the Tanner bipartite graph for $(6,3)$ LDPC-code.



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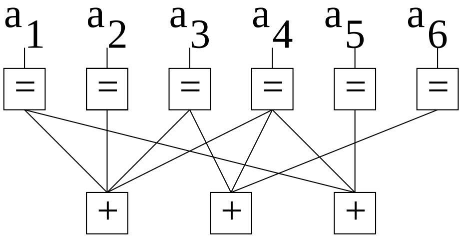

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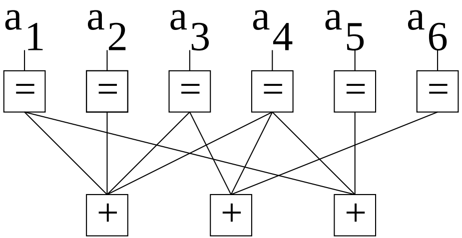

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and therefore the following constrains have to be satisfied:

$$a_1 + a_2 + a_3 + a_4 = 0$$

$$a_3 + a_4 + a_6 = 0$$

$$a_1 + a_4 + a_5 = 0$$

DECODING

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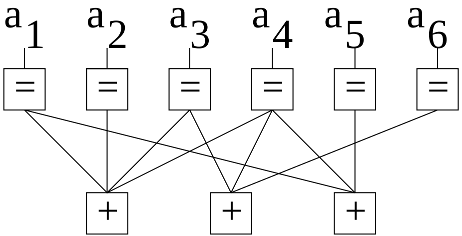

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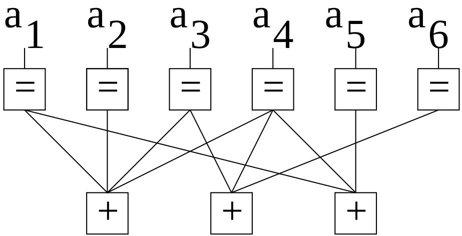

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Let the word ?01?11 be received.

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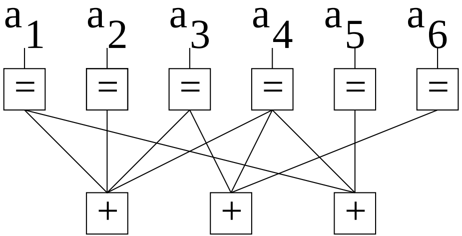

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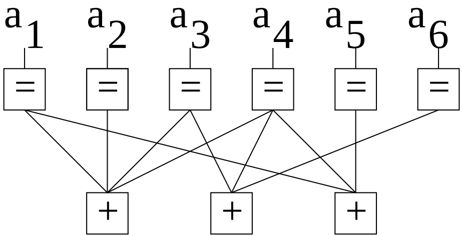

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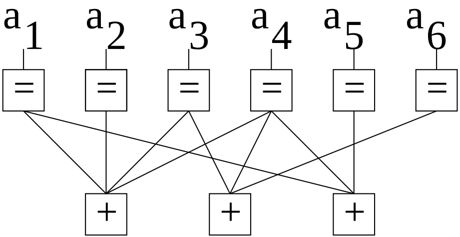

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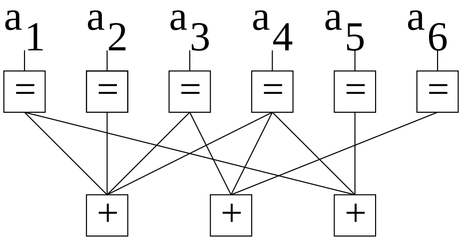

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Using so called **iterative belief propagation techniques**, LDPC codes can be decoded in time linear to their block length.

DESIGN of LDPC codes

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- Some good LDPC codes were designed through randomly chosen parity check matrices.
- Some LDPC codes are based on Reed-Solomon codes, such as the RS-LDPC code used in the 10-gigabit Ethernet standard.