Part I

Quantum cryptography

QUANTUM CRYPTOGRAPHY

Quantum cryptography is an area of science and technology that explores and utilizes potential of quantum phenomena for getting higher quality (security) for cryptography tasks.

A new and important feature of quantum cryptography is that security of quantum cryptographical protocols is based on the laws of nature – of quantum physics, and not on the unproven assumptions of computational complexity.

Quantum cryptography is the first area of information processing and communication in which quantum physics laws are directly exploited to bring an essential advantage in information processing.

MAIN OUTCOMES – so far

- It has been shown that with quantum computers, we could design absolutely secure quantum generation of shared and secret random classical keys.
- It has been proven that even without quantum computers unconditionally secure quantum generation of classical secret and shared keys is possible (in the sense that any eavesdropping is detectable).
- Unconditionally secure basic quantum cryptography primitives, such as bit commitment and oblivious transfer, are impossible.
- Quantum teleportation and pseudo-telepathy are possible.
- Quantum cryptography and quantum networks are already in the developmental stages. Quantum communication between satellites and ground stations were already demonstrated for 1200 km in 2016 in China. That indicates that quantum internet seems possible.

BASICS of QUANTUM INFORMATION PROCESSING

As an introduction to quantum cryptography

the very basic motivations, experiments, principles, concepts and results of quantum information processing and communication

will be presented in the next few slides.

BASIC MOTIVATION

In quantum information processing we witness an interaction between the two most important areas of science and technology of 20-th century, between

quantum physics and informatics.

This is very likely to have important consequences for 21th century.

QUANTUM PHYSICS

Quantum physics deals with fundamental entities of physics – **particles** (waves?) like

- protons, electrons and neutrons (from which matter is built);
- photons (which carry electromagnetic radiation)
- various "elementary particles" which mediate other interactions in physics.
- We call them particles in spite of the fact that some of their properties are totally unlike the properties of what we call particles in our ordinary classical world.

  For example, a quantum particle "can go through two places at the same time" and can interact with itself.

  Quantum physics is full of counter-intuitive, weird, mysterious and even paradoxical events.

FEYNMAN's VIEW

I am going to tell you what Nature behaves like ...

However, do not keep saying to yourself, if you can possibly avoid it,

BUT HOW CAN IT BE LIKE THAT?

Because you will get "down the drain" into a blind alley from which nobody has yet escaped

NOBODY KNOWS HOW IT CAN BE LIKE THAT

Richard Feynman (1965): The character of physical law.

CLASSICAL versus QUANTUM INFORMATION

Main properties of classical information:

1. It is easy to store, transmit and process classical information in time and space.
2. It is easy to make (unlimited number of) copies of classical information
3. One can measure classical information without disturbing it.

Main properties of quantum information:

1. It is difficult to store, transmit and process quantum information
2. There is no way to copy perfectly unknown quantum information
3. Measurement of quantum information destroys it, in general.

CLASSICAL versus QUANTUM COMPUTING

The essence of the difference between
classical computers and **quantum computers**
is in the way information is stored and processed.

In classical computers, information is represented on macroscopic level by bits, which can take one of the two values

0 or 1

In quantum computers, information is represented on microscopic level using qubits, (quantum bits) which can take on any from the following uncountable many values

$$\alpha|0\rangle + \beta|1\rangle$$

where α, β are arbitrary complex numbers such that

$$|\alpha|^2 + |\beta|^2 = 1.$$

CLASSICAL versus QUANTUM REGISTERS

An n bit classical register can store at any moment exactly one n-bit string.

An n-qubit quantum register can store at any moment a superposition of all 2^n n-bit strings.

Consequently, on a quantum computer one can "compute' in a single step all 2^n values of a function defined on n-bit inputs.

This enormous massive parallelism is one reason why quantum computing can be so powerful.

BASIC EXPERIMENTS

BASIC EXPERIMENTS

CLASSICAL EXPERIMENTS

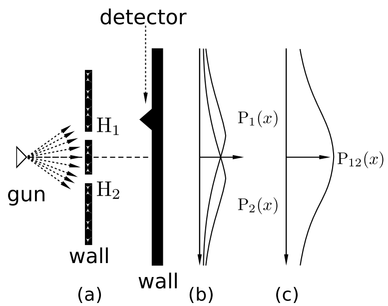


Figure 1: Experiment with bullets

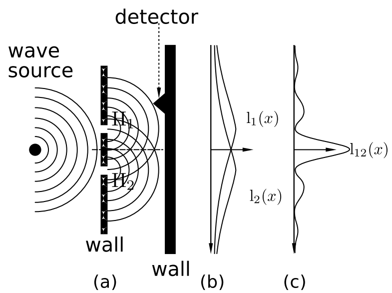
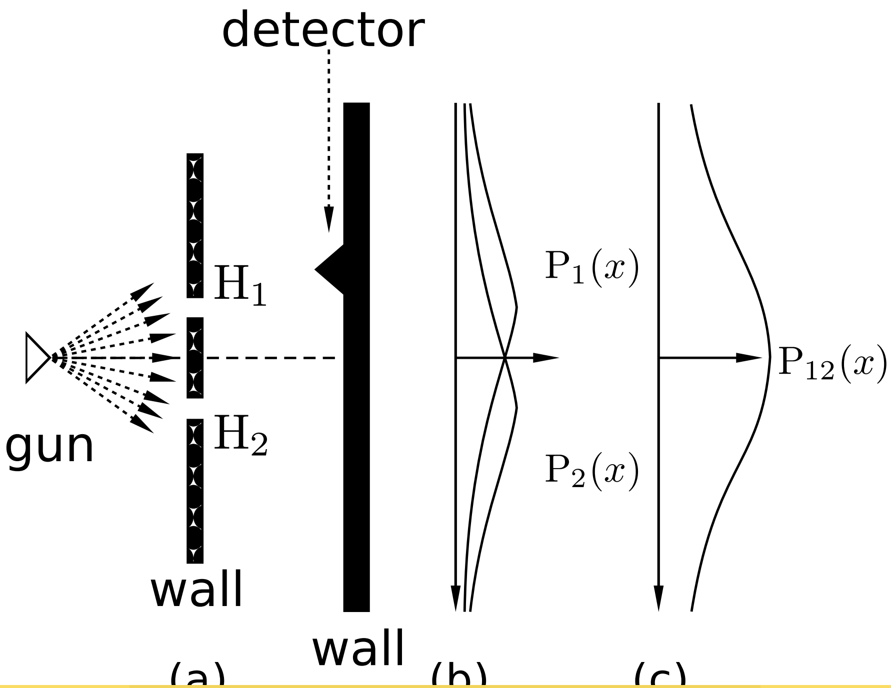
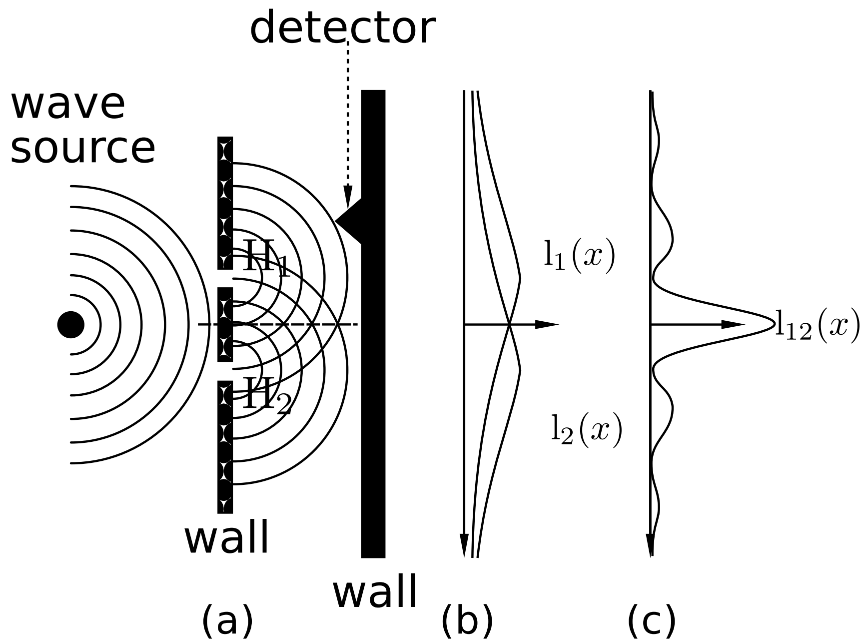


Figure 2: Experiments with waves

CLASSICAL EXPERIMENT with bullets



CLASSICAL EXPERIMENT with waves



QUANTUM EXPERIMENTS

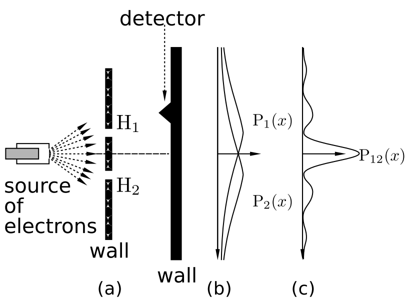


Figure 3: Two-slit experiment

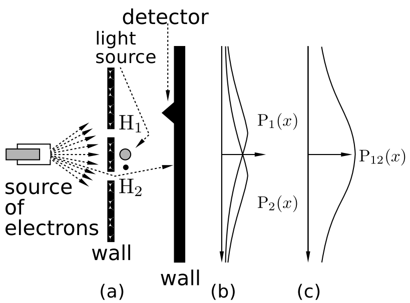
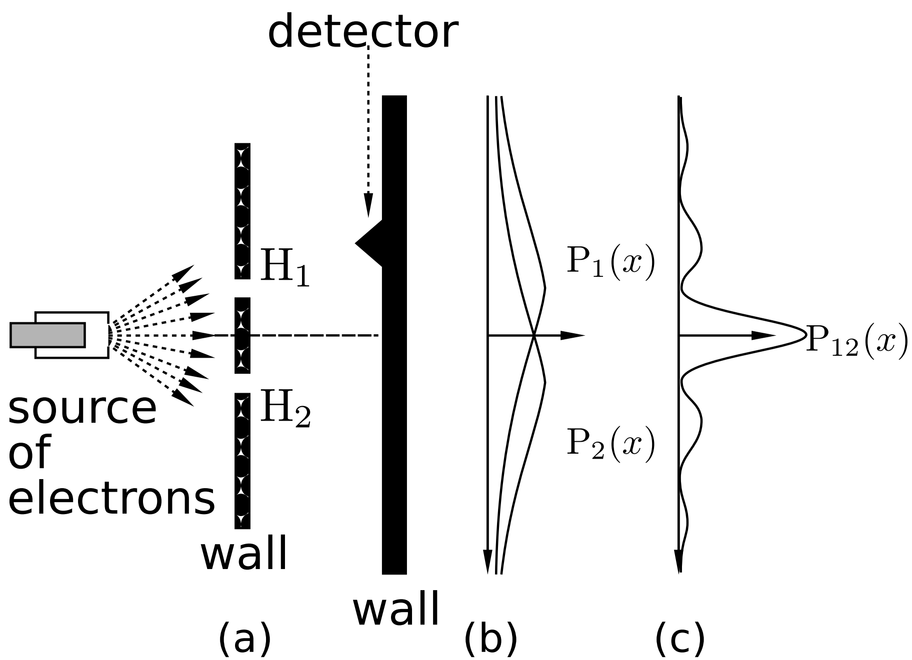
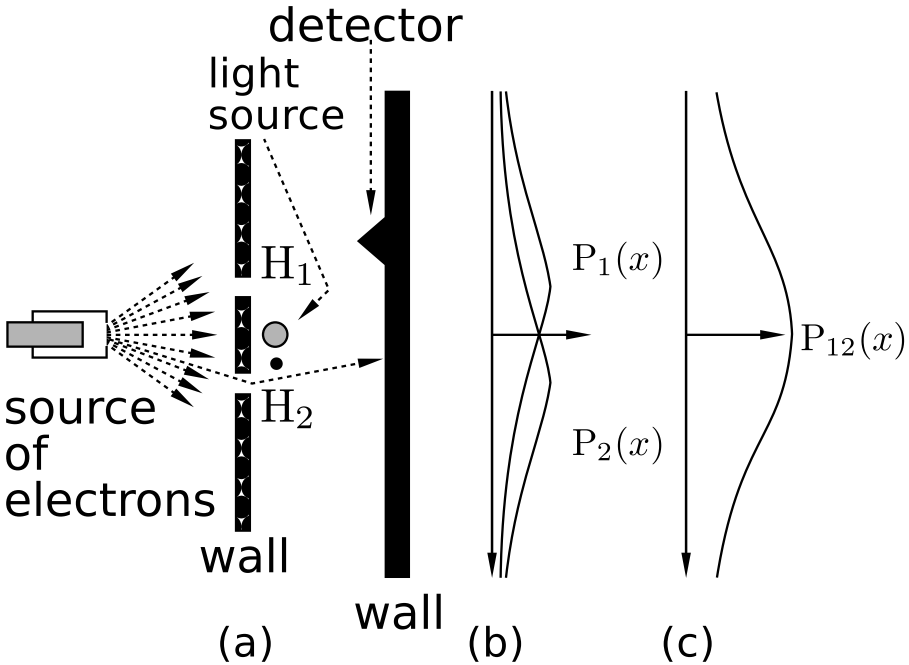


Figure 4: Two-slit experiment with an observation

TWO-SLIT EXPERIMENT



TWO-SLIT EXPERIMENT with OBSERVATION



THREE BASIC PRINCIPLES of QUANTUM WORLD

P1 To each transfer from a quantum state ϕ to a state ψ a complex number

$$\langle\psi|\phi\rangle$$

is associated. This number is called the probability amplitude of the transfer and

$$|\langle\psi|\phi\rangle|^2$$

is then the **probability** of the transfer.

P2 If a transfer from a quantum state ϕ to a quantum state ψ can be decomposed into two subsequent transfers

$$\psi \leftarrow \phi' \leftarrow \phi$$

then the resulting amplitude of the transfer is the product of amplitudes of subtransfers: $\langle\psi|\phi\rangle = \langle\psi|\phi'\rangle\langle\phi'|\phi\rangle$

P3 If a transfer from a state ϕ to a state ψ has two independent alternatives

then the resulting amplitude is the sum of amplitudes of two subtransfers.

QUANTUM SYSTEMS = HILBERT SPACE

Hilbert space H_n is an n-dimensional complex vector space with

scalar product

$$\langle\psi|\phi\rangle = \sum_{i=1}^{n} \phi_i\psi_i^* \text{ of vectors } |\phi\rangle = \begin{vmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_n\end{vmatrix}, |\psi\rangle = \begin{vmatrix}\psi_1\\ \psi_2\\ \vdots\\ \psi_n\end{vmatrix},$$

This allows to define the norm of vectors as

$$\|\phi\| = \sqrt{|\langle\phi|\phi\rangle|}.$$

Two vectors $|\phi\rangle$ and $|\psi\rangle$ are called **orthogonal** if $\langle\phi|\psi\rangle = 0$.

A **basis** B of H_n is any set of n vectors $|b_1\rangle, |b_2\rangle, \ldots, |b_n\rangle$ of the norm 1 which are mutually orthogonal.

Given a basis $B = \{|b_i\rangle\}_{i=1}^n$, any vector $|\psi\rangle$ from H_n can be uniquely expressed in the form:

$$|\psi\rangle = \sum_{i=1}^{n} \alpha_i|b_i\rangle.$$

BRA-KET NOTATION

Dirac introduced a very handy notation, so called bra-ket notation, to deal with amplitudes, quantum states and linear functionals $f : H \rightarrow C$.

If $\psi, \phi \in H$, then

$\langle\psi|\phi\rangle$ – scalar product of ψ and ϕ (an amplitude of going from ϕ to ψ).

$|\phi\rangle$ – ket-vector (a column vector) - an equivalent to ϕ

$\langle\psi|$ – bra-vector (a row vector) a linear functional on H

such that $\langle\psi|(|\phi\rangle) = \langle\psi|\phi\rangle$

EXAMPLES

Example For states $\phi = (\phi_1, \dots, \phi_n)$ and $\psi = (\psi_1, \dots, \psi_n)$ we have

$$|\phi\rangle = \begin{pmatrix} \phi_1 \\ \dots \\ \phi_n \end{pmatrix}, \langle\phi| = (\phi_1^*, \dots, \phi_n^*); \langle\phi|\psi\rangle = \sum_{i=1}^{n} \phi_i^* \psi_i;$$

$$|\phi\rangle\langle\psi| = \begin{pmatrix} \phi_1\psi_1^* & \dots & \phi_1\psi_n^* \\ \vdots & \ddots & \vdots \\ \phi_n\psi_1^* & \dots & \phi_n\psi_n^* \end{pmatrix}$$

QUANTUM EVOLUTION / COMPUTATION

EVOLUTION in QUANTUM SYSTEM	COMPUTATION in HILBERT SPACE

is described by

Schrödinger linear equation

$$i\hbar \frac{\partial |\Phi(t)\rangle}{\partial t} = H(t)|\Phi(t)\rangle$$

where $\hbar$ is Planck constant, H(t) is a Hamiltonian (total energy) of the system that can be represented by a Hermitian matrix, and $\Phi(t)$ is the state of the system in time t.
If the Hamiltonian is time independent then the above Schrödinger equation has solution

$$|\Phi(t)\rangle = U(t)|\Phi(0)\rangle$$

where

$$U(t) = e^{\frac{iHt}{\hbar}}$$

is the evolution operator that can be represented by a **unitary matrix**. **A step of such an evolution is therefore a multiplication of a "unitary matrix" A with a vector $|\psi\rangle$, i.e. A $|\psi\rangle$**

UNITARY MATRICES

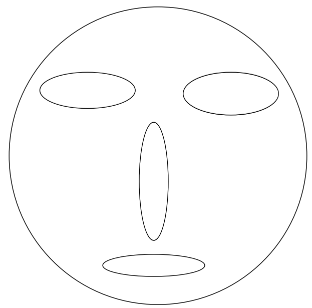
A matrix A is **unitary** if

$$A \cdot A^\dagger = A^\dagger \cdot A = I$$

where the matrix $A^\dagger$ is obtained from the matrix A by revolving A around the main diagonal and changing all elements by their complex conjugates.

QUANTUM (PROJECTION) MEASUREMENTS

A quantum state is always observed (measured) with respect to an **observable** O – a decomposition of a given Hilbert space into orthogonal subspaces (where each vector can be uniquely represented as a sum of vectors of these subspaces).



There are two outcomes of a projection measurement of a state $|\phi\rangle$ with respect to O:

1. Into classical world comes information into which subspace projection of $|\phi\rangle$ was made.
2. In the classical world projection of the measured state (as a new state) $|\phi'\rangle$ stays in one of the above subspaces.

The subspace into which projection is made is chosen **randomly** and the corresponding probability is uniquely determined by the amplitudes at the representation of $|\phi\rangle$ as a sum of states of the subspaces.

QUANTUM STATES and PROJECTION MEASUREMENT

In case an orthonormal basis $\{\beta_i\}_{i=1}^n$ is chosen in a Hilbert space H_n, then any state $|\phi\rangle \in H_n$ can be expressed in the form

$$|\phi\rangle = \sum_{i=1}^{n} a_i|\beta_i\rangle, \qquad \sum_{i=1}^{n} |a_i|^2 = 1$$

where

$a_i = \langle\beta_i|\phi\rangle$ are called probability amplitudes

and

their squares provide probabilities

that if the state $|\phi\rangle$ is measured with respect to the basis $\{\beta_i\}_{i=1}^n$, then the state $|\phi\rangle$ collapses into the state $|\beta_i\rangle$ with probability $|a_i|^2$.

The classical "outcome" of the measurement of the state $|\phi\rangle$ with respect to the basis $\{\beta_i\}_{i=1}^n$ is the index i of that state $|\beta_i\rangle$ into which the state $|\phi\rangle$ collapses.

QUBITS

A **qubit** is a quantum state in H_2

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where $\alpha, \beta \in C$ are such that $|\alpha|^2 + |\beta|^2 = 1$ and

$\{|0\rangle, |1\rangle\}$ is a (**standard**) basis of H_2

EXAMPLE: Representation of qubits by

(a) electron in a Hydrogen atom

(b) a spin-1/2 particle

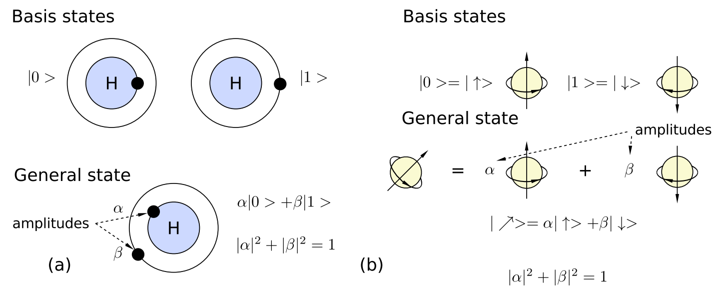


Figure 5: Qubit representations by energy levels of an electron in a hydrogen atom and by a spin-1/2 particle. The condition $|\alpha|^2 + |\beta|^2 = 1$ is a legal one if $|\alpha|^2$ and $|\beta|^2$ are to be the probabilities of being in one of two basis states (of electrons or photons).

HILBERT SPACE H_2

STANDARD BASIS

$|0\rangle, |1\rangle$

$$\begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix}$$

DUAL BASIS

$|0'\rangle, |1'\rangle$

$$\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix} \begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}$$

Hadamard matrix

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$

$$H|0\rangle = |0'\rangle \qquad H|0'\rangle = |0\rangle$$
$$H|1\rangle = |1'\rangle \qquad H|1'\rangle = |1\rangle$$

transforms one of the basis into another one.

General form of a unitary matrix of degree 2

$$U = e^{i\gamma}\begin{pmatrix}e^{i\alpha} & 0\\0 & e^{-i\alpha}\end{pmatrix}\begin{pmatrix}\cos\theta & i\sin\theta\\i\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}e^{i\beta} & 0\\0 & e^{-i\beta}\end{pmatrix}$$

PAULI MATRICES

Very important one-qubit unary operators are the following Pauli operators, expressed in the standard basis as follows;

$$\sigma_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \quad \sigma_y = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}, \quad \sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

Observe that Pauli matrices transform a qubit state $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ as follows

$$\sigma_x(\alpha|0\rangle + \beta|1\rangle) = \beta|0\rangle + \alpha|1\rangle$$
$$\sigma_z(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle - \beta|1\rangle$$
$$\sigma_y(\alpha|0\rangle + \beta|1\rangle) = \beta|0\rangle - \alpha|1\rangle$$

Operators σ_x, σ_z and σ_y represent therefore a bit error, a sign error and a bit-sign error.

QUANTUM MEASUREMENT of QUBITS

of a qubit state

A qubit state can "contain" unboundly large amount of classical information. However, **an unknown quantum state cannot be identified.**

By a **measurement** of the qubit state

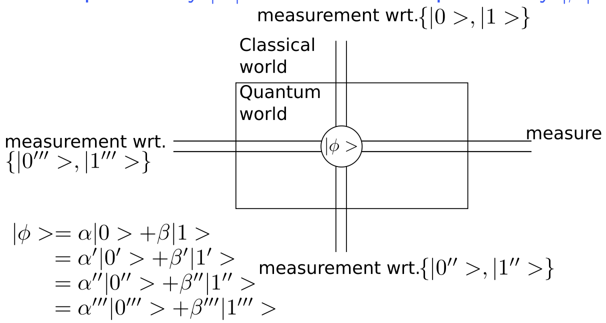
$$\alpha|0\rangle + \beta|1\rangle$$

with respect to the basis

$$\{|0\rangle, |1\rangle\}$$

we can obtain only classical information and only in the following random way:

0 with probability $|\alpha|^2$ 1 with probability $|\beta|^2$



MIXED STATES – DENSITY MATRICES

A probability distribution $\{(p_i, |\phi_i\rangle)\}_{i=1}^k$ on pure states is called a mixed state to which it is assigned a density operator

$$\rho = \sum_{i=1}^{n} p_i|\phi\rangle\langle\phi_i|.$$

One interpretation of a mixed state $\{(p_i, |\phi_i\rangle)\}_{i=1}^k$ is that a source X produces the state $|\phi_i\rangle$ with probability p_i.

Any matrix representing a density operator is called density matrix.

Density matrices are exactly Hermitian, positive matrices with trace 1.

To two different mixed states can correspond the same density matrix.

Two mixes states with the same density matrix are physically undistinguishable.

MAXIMALLY MIXED STATES

To the maximally mixed state,

$$\left(\frac{1}{2}, |0\rangle\right), \left(\frac{1}{2}, |1\rangle\right)$$

representing a random bit, corresponds the density matrix

$$\frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}(1,0) + \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}(0,1) = \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \frac{1}{2}I_2$$

Surprisingly, many other mixed states have density matrix that is the same as that of the maximally mixed state.

QUANTUM ONE-TIME PAD CRYPTOSYSTEM

CLASSICAL ONE-TIME PAD cryptosystem

plaintext	an n-bit string p
shared key	an n-bit string k
cryptotext	an n-bit string c
encoding	$c = p \oplus k$
decoding	$p = c \oplus k$

QUANTUM ONE-TIME PAD cryptosystem

plaintext:	an n-qubit string $|p\rangle = |p_1\rangle \dots |p_n\rangle$
shared key:	two n-bit strings k,k'
cryptotext:	an n-qubit string $|c\rangle = |c_1\rangle \dots |c_n\rangle$
encoding:	$|c_i\rangle = \sigma_x^{k_i}\sigma_z^{k_i'}|p_i\rangle$
decoding:	$|p_i\rangle = \sigma_z^{k_i'}\sigma_x^{k_i}|c_i\rangle$

where $|p_i\rangle = \begin{pmatrix}a_i\\b_i\end{pmatrix}$ and $|c_i\rangle = \begin{pmatrix}d_i\\e_i\end{pmatrix}$ are qubits and $\sigma_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ with $\sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ are Pauli matrices.

UNCONDITIONAL SECURITY of QUANTUM ONE-TIME PAD

In the case of encryption of a qubit

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$$

by QUANTUM ONE-TIME PAD cryptosystem, what is being transmitted is the mixed state

$$\left(\frac{1}{4}, |\phi\rangle\right), \left(\frac{1}{4}, \sigma_x|\phi\rangle\right), \left(\frac{1}{4}, \sigma_z|\phi\rangle\right), \left(\frac{1}{4}, \sigma_x\sigma_z|\phi\rangle\right)$$

whose density matrix is

$$\frac{1}{2}I_2$$

This density matrix is identical to the density matrix corresponding to that of a random bit, that is to the mixed state

$$\left(\frac{1}{2}, |0\rangle\right), \left(\frac{1}{2}, |1\rangle\right)$$

SHANNON's THEOREMS

Shannon classical encryption theorem says that n bits are necessary and sufficient to encrypt securely n bits.

Quantum version of Shannon encryption theorem says that 2n classical bits are necessary and sufficient to encrypt securely n qubits.

COMPOSED QUANTUM SYSTEMS (1)

Tensor product of vectors

$$(x_1,\dots,x_n)\otimes(y_1,\dots,y_m) = (x_1y_1,\dots,x_1y_m,x_2y_1,\dots,x_2y_m,\dots,x_2y_m,\dots,x_ny_1,\dots,x_ny_m)$$

Tensor product of matrices $A\otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{pmatrix}$

where $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$

Example $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{21} & a_{22} \end{pmatrix}$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & a_{12} & 0 \\ 0 & a_{11} & 0 & a_{12} \\ a_{21} & 0 & a_{22} & 0 \\ 0 & a_{21} & 0 & a_{22} \end{pmatrix}$$

COMPOSED QUANTUM SYSTEMS II

Tensor product of Hilbert spaces $H_1 \otimes H_2$ is the complex vector space spanned by tensor products of vectors from H_1 and H_2 . That corresponds to the quantum system composed of the quantum systems corresponding to Hilbert spaces H_1 and H_2.

An important difference between classical and quantum systems

A state of a compound classical (quantum) system can be (cannot be) always composed from the states of the subsystem.

QUANTUM REGISTERS

A general state of a 2-qubit register is:

$$|\phi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

where

$$|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$$

and $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ are vectors of the "standard" basis of H_4, i.e.

$$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \; |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \; |10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \; |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

An important unitary matrix of degree 4, to transform states of 2-qubit registers:

$$CNOT = XOR = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}$$

It holds:

$$\text{CNOT} : |x, y\rangle \Rightarrow |x, x \oplus y\rangle$$

NO-CLONING THEOREM

INFORMAL VERSION: Unknown quantum state cannot be cloned.

FORMAL VERSION: There is no unitary transformation U such that for any qubit state $|\psi\rangle$

$$U(|\psi\rangle|0\rangle) = |\psi\rangle|\psi\rangle$$

PROOF: Assume U exists and for two different states $|\alpha\rangle$ and $|\beta\rangle$

$$U(|\alpha\rangle|0\rangle) = |\alpha\rangle|\alpha\rangle \qquad U(|\beta\rangle|0\rangle) = |\beta\rangle|\beta\rangle$$

Let

$$|\gamma\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle + |\beta\rangle)$$

Then

$$U(|\gamma\rangle|0\rangle) = \frac{1}{\sqrt{2}}(|\alpha\rangle|\alpha\rangle + |\beta\rangle|\beta\rangle) \neq |\gamma\rangle|\gamma\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle|\alpha\rangle + |\beta\rangle|\beta\rangle + |\alpha\rangle|\beta\rangle + |\beta\rangle|\alpha\rangle)$$

However, CNOT can make copies of the basis states $|0\rangle, |1\rangle$: Indeed, for $x \in \{0,1\}$,

$$CNOT(|x\rangle|0\rangle) = |x\rangle|x\rangle$$

BELL STATES

States

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \qquad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \qquad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

form an orthogonal (so called Bell) basis in H_4 and play an important role in quantum computing.

Theoretically, there is an observable for this basis. However, no one has been able to construct a device for Bell measurement using linear elements only.

QUANTUM n-qubit REGISTERS

A general state of an n-qubit register has the form:

$$|\phi\rangle = \sum_{i=0}^{2^n-1} \alpha_i|i\rangle = \sum_{i\in\{0,1\}^n} \alpha_i|i\rangle, \text{ where } \sum_{i=0}^{2^n-1} |\alpha_i|^2 = 1$$

and $|\phi\rangle$ is a vector in H_{2^n}.
Operators on n-qubits registers are unitary matrices of degree 2^n.
Is it difficult to create a state of an n-qubit register?
In general yes, in some important special cases not.For example, if n-qubit Hadamard transformation

$$H_n = \otimes_{i=1}^n H.$$

is used then

$$H_n|0^{(n)}\rangle = \otimes_{i=1}^n H|0\rangle = \otimes_{i=1}^n|0'\rangle = |0'^{(n)}\rangle = \frac{1}{\sqrt{2^n}}\sum_{i=0}^{2^n-1}|i\rangle = \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}|x\rangle$$

and, in general, for $x \in \{0,1\}^n$

$$H_n|x\rangle = \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}(-1)^{x\cdot y}|y\rangle.$$ [1]

[1] The dot product is defined as follows: $x \cdot y = \otimes_{i=1}^n x_i y_i$.

QUANTUM PARALLELISM

If

$$f : \{0, 1, \ldots, 2^n - 1\} \Rightarrow \{0, 1, \ldots, 2^n - 1\}$$

then the mapping

$$f' : (x, 0) \Rightarrow (x, f(x))$$

is one-to-one and therefore there is a unitary transformation U_f such that.

$$U_f(|x\rangle|0\rangle) \Rightarrow |x\rangle|f(x)\rangle$$

Let us now have the state

$$|\Psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle|0\rangle$$

With a single application of the mapping U_f we then get

$$U_f|\Psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} U_f(|i\rangle|0\rangle) = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle|f(i)\rangle$$

OBSERVE THAT IN A SINGLE COMPUTATIONAL STEP 2^n VALUES OF f ARE COMPUTED!

IN WHAT LIES POWER OF QUANTUM COMPUTING?

In quantum superposition or in quantum parallelism?
NOT,
in QUANTUM ENTANGLEMENT!

Let

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

be a state of two very distant particles, **for example** on two planets
Measurement of one of the particles, with respect to the standard basis, makes the above state to collapse to one of the states

$$|00\rangle \text{ or } |11\rangle.$$

This means that subsequent measurement of other particle (on another planet) provides the same result as the measurement of the first particle. This indicate that in quantum world non-local influences, correlations, exist.

POWER of ENTANGLEMENT

Quantum state $|\Psi\rangle$ of a composed bipartite quantum system $A \otimes B$ is called entangled if it cannot be decomposed into tensor product of the states from A and B.

Quantum entanglement is an important quantum resource that allows

- To create phenomena that are impossible in the classical world (for example teleportation)
- To create quantum algorithms that are asymptotically more efficient than any classical algorithm known for the same problem.
- To create communication protocols that are asymptotically more efficient than classical communication protocols for the same task
- To create, for two parties, shared secret binary keys
- To increase capacity of quantum channels

CLASSICAL versus QUANTUM CRYPTOGRAPHY

- Security of classical cryptography is based on unproven assumptions of computational complexity (and it can be jeopardize by progress in algorithms and/or technology).

  Security of quantum cryptography is based on laws of quantum physics that allow to build systems where undetectable eavesdropping is impossible.
- Since classical cryptography is vulnerable to technological improvements it has to be designed in such a way that a secret is secure with respect to **future technology**, during the whole period in which the secrecy is required.

  Quantum key generation, on the other hand, needs to be designed only to be secure against technology available at the moment of key generation.

QUANTUM KEY GENERATION

Quantum protocols for using quantum systems to achieve unconditionally secure generation of secret (classical) keys by two parties are one of the main theoretical achievements of quantum information processing and communication research.

Moreover, experimental systems for implementing such protocols are one of the main achievements of experimental quantum information processing research.

It is believed and hoped that it will be

quantum key generation (QKG)

another term is

quantum key distribution (QKD)

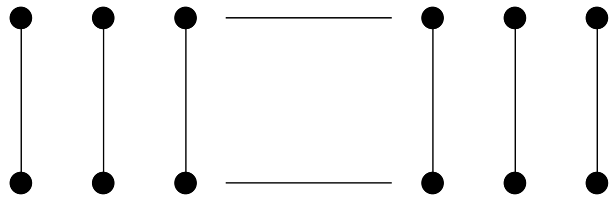
where one can expect the first

transfer from the experimental to the application stage.

QUANTUM KEY GENERATION – EPR METHOD

Let Alice and Bob share n pairs of particles in the entangled EPR-state.

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$



n pairs of particles in EPR state

If both of them measure their particles in the standard basis, then they get, as the classical outcome of their measurements the same random, shared and secret binary key of length n.

POLARIZATION of PHOTONS

Polarized photons are currently mainly used for experimental quantum key generation.

Photon, or light quantum, is a particle composing light and other forms of electromagnetic radiation.

Photons are electromagnetic waves and their electric and magnetic fields are perpendicular to the direction of propagation and also to each other.

An important property of photons is polarization – it refers to the bias of the electric field in the electromagnetic field of the photon.

LINEAR POLARIZATION - visualization

You can think of light as traveling in waves. One way to visualize these waves is to imagine taking a long rope and tying one end in a fixed place and to move the free end in some way.

Moving the free end of the rope up and down sets up a "wave" along the rope which also moves up and down. If you think of he rope as as representing a beam of light, the light would be a "vertically polarized".

If the free end of the rope is moved from side to side a wave that moves from from side to side is set up. If this way moves a light beam, it is called "horizontally polarized".

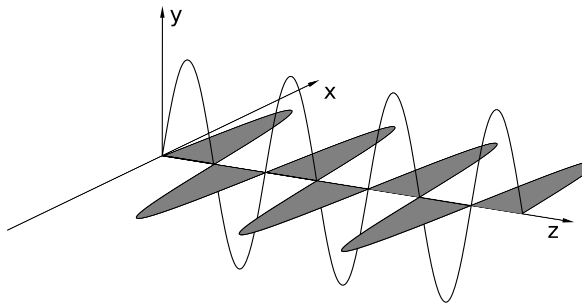


Figure: Linearly polarized photons - visualization

Both vertical and horizontal polarizations are examples of "linear polarizations"

CIRCULAR POLARIZATION

If the free end of the rope is moved around in a circle, then we would get a wave that looks like a corkscrew. This would visualize circular polarization"

POLARIZATION of PHOTONS III

Generation of orthogonally polarized photons.

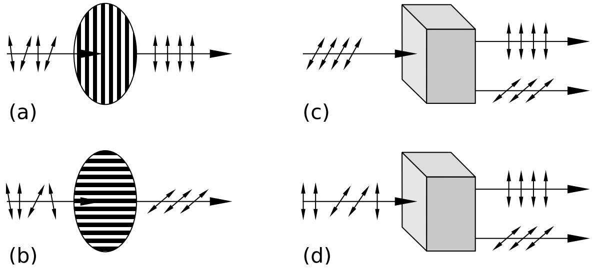


Figure: Photon polarizers and measuring devices

For any polarizations there are generators that produce photons only of a given polarizations. For example, calcite crystals, shown in Fig, a and b can do the job.

Fig. c – a calcite crystal that makes θ-polarized photons to be horizontally (vertically) polarized with probability $cos^2\theta(sin^2\theta)$.

Fig. d – a calcite crystal can be used to separate horizontally and vertically polarized photons.

QUANTUM GENERATION of CLASSICAL KEYS – PROLOGUE

Very basic setting Alice tries to send a quantum system to Bob and an eavesdropper tries to learn, or to change, as much as possible, without being detected.

Eavesdroppers have this time especially hard time, because quantum states cannot be copied and cannot be measured without causing, in general, a disturbance.

Key problem: If Alice prepares a quantum system in a specific way, unknown fully to the eavesdropper Eve, and sends it to Bob

then the question is how much information can Eve extract of that quantum system and how much it costs in terms of the disturbance of the system.

Three special cases

1. Eve has no information about the state $|\psi\rangle$ Alice sends.
2. Eve knows that $|\psi\rangle$ is one of the states of an orthonormal basis $\{|\phi_i\rangle\}_{i=1}^n$.
3. Eve knows that $|\psi\rangle$ is one of the states $|\phi_1\rangle, \ldots, |\phi_n\rangle$ that **are not** mutually orthonormal and that p_i is the probability that $|\psi\rangle = |\phi_i\rangle$.

BB84 QUANTUM GENERATION of CLASSICAL RANDOM KEY

Quantum key generation protocol BB84 (due to Bennett and Brassard), for generation of a key of length n, has several phases:

Preparation phase

Alice is assumed to have four transmitters of photons in one of the following four polarizations 0, 45, 90 and 135 degrees

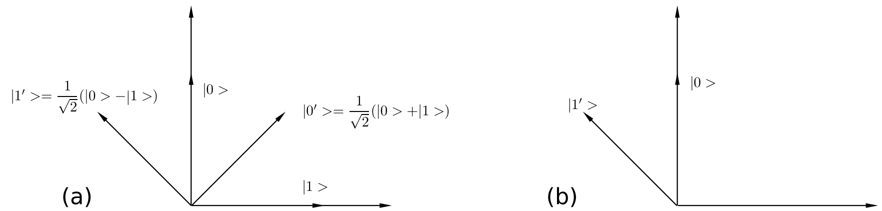


Figure 8: Polarizations of photons for BB84 and B92 protocols

Expressed in a more general form, Alice uses for encoding states from the set $\{|0\rangle, |1\rangle, |0'\rangle, |1'\rangle\}$.
Bob has a detector that can be set up to distinguish between rectilinear polarizations (0 and 90 degrees) or can be quickly reset to distinguish between diagonal polarizations (45 and 135 degrees).

BB84 QUANTUM KEY GENERATION PROTOCOL III

An example of an encoding – decoding process is in the Figure 10.

Raw key extraction

Bob makes public the sequence of bases he used to measure the photons he received – but not the results of the measurements – and Alice tells Bob, through a classical channel, in which cases he has chosen the same basis for measurement as she did for encoding. The corresponding bits then form the basic raw key.

1	0	0	0	1	1	0	0	0	1	1	Alice's random sequence
$\|1\rangle$	$\|0'\rangle$	$\|0\rangle$	$\|0'\rangle$	$\|1\rangle$	$\|1'\rangle$	$\|0'\rangle$	$\|0\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1'\rangle$	Alice's polarizations
0	1	1	1	0	0	1	0	0	1	0	Bob's random sequence
B	D	D	D	B	B	D	B	B	D	B	Bob's observable
1	0	R	0	1	R	0	0	0	R	R	outcomes

Figure 10: Quantum transmissions in the BB84 protocol – R stands for the case that the result of the measurement is random.

BB84 QUANTUM KEY GENERATION PROTOCOL III

Test for eavesdropping

Alice and Bob agree on a sequence of indices of the raw key and make the corresponding bits of their raw keys public.

Case 1. Noiseless channel. If the subsequences chosen by Alice and Bob are not completely identical eavesdropping is detected. Otherwise, the remaining bits are taken as creating the final key.

Case 2. Noisy channel. If the subsequences chosen by Alice and Bob contains more errors than the admitable error of the channel (that has to be determined from channel characteristics), then eavesdropping is assumed. Otherwise, the remaining bits are taken as the next result of the raw key generation process.

Error correction phase

In the case of a noisy channel for transmission it may happen that Alice and Bob have different raw keys after the key generation phase.

A way out is to use a special error correction techniques and at the end of this stage both Alice and Bob share identical keys.

BB84 QUANTUM KEY GENERATION PROTOCOL IV

Privacy amplification phase

One problem remains. Eve can still have quite a bit of information about the key both Alice and Bob share. Privacy amplification is a tool to deal with such a case.

Privacy amplification is a method how to select a short and very secret binary string s from a longer but less secret string s'. The main idea is simple. If $|s| = n$, then one picks up n random subsets $S_1, \ldots, S_n$ of bits of s' and let s_i, the i-th bit of S, be the parity of S_i. One way to do it is to take a random binary matrix of size $|s| \times |s'|$ and to perform multiplication Ms'^T, where s'^T is the binary column vector corresponding to s'.

The point is that even in the case where an eavesdropper knows quite a few bits of s', she will have almost no information about s.

More exactly, if Eve knows parity bits of k subsets of s', then if a random subset of bits of s' is chosen, then the probability that Eve has any information about its parity bit is less than $\frac{2^{-(n-k-1)}}{\ln 2}$.

EXPERIMENTAL CRYPTOGRAPHY

Successes

1. Transmissions using optical fibers to the distance of 200 km.
2. Open air transmissions to the distance 144 km at day time (from one pick of Canary Islands to another).
3. Next goal: earth to satellite transmissions.

All current systems use optical means for quantum state transmissions

Problems and tasks

1. No single photon sources are available. Weak laser pulses currently used contains in average 0.1 - 0.2 photons.
2. Loss of signals in the fiber. (Current error rates: 0,5 - 4%)
3. To move from the experimental to the developmental stage.

QUANTUM TELEPORTATION - BASIC SETTING

Quantum teleportation allows to transmit unknown quantum information to a very distant place in spite of impossibility to measure or to broadcast information to be transmitted.

Alice and Bob share two particles in the EPR-state

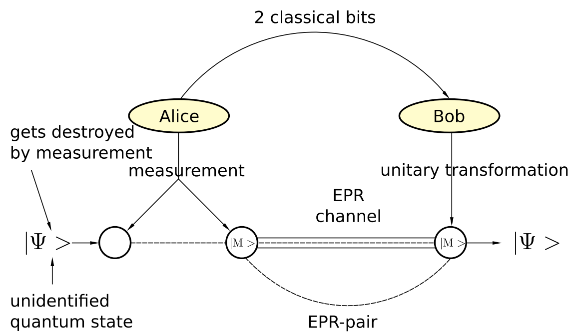
$$|EPR_{pair}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

and then Alice receives another particle in an unknown qubit state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Alice then measure her two particles in the Bell basis.

QUANTUM TELEPORTATION - BASIC SETTING I



$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad |EPR - pair\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Total state

$$|\psi\rangle|EPR - pair\rangle = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$$

Alice measures her two qubits with respect to the "Bell basis":

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \qquad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

QUANTUM TELEPORTATION II

Since the total state of all three particles is:

$$|\psi\rangle|EPR - pair\rangle = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$$

and can be expressed also as follows:

$$|\psi\rangle|EPR - pair\rangle = |\Phi^+\rangle\frac{1}{\sqrt{2}}(\alpha|0\rangle + \beta|1\rangle) + |\Psi^+\rangle\frac{1}{\sqrt{2}}(\beta|0\rangle + \alpha|1\rangle) + |\Phi^-\rangle\frac{1}{\sqrt{2}}(\alpha|0\rangle - \beta|1\rangle) + |\Psi^-\rangle\frac{1}{\sqrt{2}}(-\beta|0\rangle + \alpha|1\rangle)$$

then the Bell measurement of the first two particles projects the state of Bob's particle into a "small modification" $|\psi_1\rangle$ of the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$,

$$|\Psi_1\rangle = \text{either } |\Psi\rangle \text{ or } \sigma_x|\Psi\rangle \text{ or } \sigma_z|\Psi\rangle \text{ or } \sigma_x\sigma_z|\psi\rangle$$

The unknown state $|\psi\rangle$ can therefore be obtained from $|\psi_1\rangle$ by applying one of the four operations

$$\sigma_x, \sigma_y, \sigma_z, I$$

and the result of the Bell measurement provides two bits specifying which of the above four operations should be applied.

These four bits Alice needs to send to Bob using a classical channel (by email, for example).

QUANTUM TELEPORTATION III.

If the first two particles of the state

$$|\psi\rangle|EPR - pair\rangle = |\Phi^+\rangle\frac{1}{\sqrt{2}}(\alpha|0\rangle + \beta|1\rangle) + |\Psi^+\rangle\frac{1}{\sqrt{2}}(\beta|0\rangle + \alpha|1\rangle) + |\Phi^-\rangle\frac{1}{\sqrt{2}}(\alpha|0\rangle - \beta|1\rangle) + |\Psi^-\rangle\frac{1}{\sqrt{2}}(-\beta|0\rangle + \alpha|1\rangle)$$

are measured with respect to the Bell basis then Bob's particle gets into the mixed state

$$\left(\frac{1}{4}, \alpha|0\rangle + \beta|1\rangle\right) \oplus \left(\frac{1}{4}, \alpha|0\rangle - \beta|1\rangle\right) \oplus \left(\frac{1}{4}, \beta|0\rangle + \alpha|1\rangle\right) \oplus \left(\frac{1}{4}, \beta|0\rangle - \alpha|1\rangle\right)$$

to which corresponds the density matrix

$$\frac{1}{4}\binom{\alpha^*}{\beta^*}(\alpha, \beta) + \frac{1}{4}\binom{\alpha^*}{-\beta^*}(\alpha, -\beta) + \frac{1}{4}\binom{\beta^*}{\alpha^*}(\beta, \alpha) + \frac{1}{4}\binom{\beta^*}{-\alpha^*}(\beta, -\alpha) = \frac{1}{2}I$$

The resulting density matrix is identical to the density matrix for the mixed state

$$\left(\frac{1}{2}, |0\rangle\right) \oplus \left(\frac{1}{2}, |1\rangle\right)$$

Indeed, the density matrix for the last mixed state has the form

$$\frac{1}{2}\binom{1}{0}(1, 0) + \frac{1}{2}\binom{0}{1}(0, 1) = \frac{1}{2}I$$

QUANTUM TELEPORTATION – COMMENTS

- Alice can be seen as dividing information contained in $|\psi\rangle$ into
  - quantum information – transmitted through EPR channel
  - classical information – transmitted through a classical channel
- In a quantum teleportation an unknown quantum state $|\phi\rangle$ can be disassembled into, and later reconstructed from, two classical bit-states and an maximally entangled pure quantum state.
- Using quantum teleportation an unknown quantum state can be teleported from one place to another by a sender who does need to know – for teleportation itself – neither the state to be teleported nor the location of the intended receiver.
- The teleportation procedure can not be used to transmit information faster than light

  but

  it can be argued that quantum information presented in unknown state is transmitted instantaneously (except two random bits to be transmitted at the speed of light at most).
- EPR channel is irreversibly destroyed during the teleportation process.

WHY IS QUANTUM INFORMATION PROCESSING SO IMPORTANT

- QIPC is believed to lead to new Quantum Information Processing Technology that could have broad impacts.
- Several areas of science and technology are approaching such points in their development where they badly need expertise with storing, transmission and processing of particles.
- It is increasingly believed that new, quantum information processing based, understanding of (complex) quantum phenomena and systems can be developed.
- Quantum cryptography seems to offer new level of security and be soon feasible.
- QIPC has been shown to be more efficient in interesting/important cases.

UNIVERSAL SETS of QUANTUM GATES

The main task at quantum computation is to express solution of a given problem P as a unitary matrix U and then to construct a circuit C_U with elementary quantum gates from a universal sets of quantum gates to realize U.

A simple universal set of quantum gates consists of gates.

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \sigma_z^{\frac{1}{4}} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{\pi}{4} i} \end{pmatrix}$$

FUNDAMENTAL RESULTS

The first really satisfactory results, concerning universality of gates, have been due to Barenco et al. (1995)

Theorem 0.1 CNOT gate and all one-qubit gates form a universal set of gates.

The proof is in principle a simple modification of the RQ-decomposition from linear algebra. Theorem 0.1 can be easily improved:

Theorem 0.2 CNOT gate and elementary rotation gates

$$R_\alpha(\theta) = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} \sigma_\alpha \qquad \text{for } \alpha \in \{x, y, z\}$$

form a universal set of gates.

QUANTUM ALGORITHMS

Quantum algorithms are methods of using quantum circuits and processors to solve algorithmic problems.

On a more technical level, a design of a quantum algorithm can be seen as a process of an efficient decomposition of a complex unitary transformation into products of elementary unitary operations (or gates), performing simple local changes.

The four main features of quantum mechanics that are exploited in quantum computation:

- Superposition;
- Interference;
- Entanglement;
- Measurement.

EXAMPLES of QUANTUM ALGORITHMS

Deutsch problem: Given is a black-box function f: $\{0,1\} \rightarrow \{0,1\}$, how many queries are needed to find out whether f is constant or balanced:
Classically: 2
Quantumly: 1

Deutsch-Jozsa Problem: Given is a black-box function $f : \{0,1\}^n \rightarrow \{0,1\}$ and a promise that f is either constant or balanced, how many queries are needed to find out whether f is constant or balanced.
Classically: n
Quantumly 1

Factorization of integers: all classical algorithms are exponential.
Peter Shor developed polynomial time quantum algorithm

Search of an element in an unordered database of n elements:
Classically n queries are needed in the worst case
Lov Grover showed that quantumly $\sqrt{n}$ queries are enough

FACTORIZATION on QUANTUM COMPUTERS

In the following we present the basic idea behind a polynomial time algorithm for quantum computers to factorize integers.

Quantum computers works with superpositions of basic quantum states on which very special (unitary) operations are applied and and very special quantum features (non-locality) are used.

Quantum computers work not with bits, that can take on any of two values 0 and 1, but with qubits (quantum bits) that can take on any of infinitely many states $\alpha|0\rangle + \beta|1\rangle$, where α and β are complex numbers such that $|\alpha|^2 + |\beta|^2 = 1$.

REDUCTIONS

Shor's polynomial time quantum factorization algorithm is based on an understanding that factorization problem can be reduced

1. first on the problem of solving a simple modular quadratic equation;
2. second on the problem of finding periods of functions $f(x) = a^x \bmod n$.

FIRST REDUCTION

Lemma If there is a polynomial time deterministic (randomized) [quantum] algorithm to find a nontrivial solution of the modular quadratic equations

$$a^2 \equiv 1 \pmod{n},$$

then there is a polynomial time deterministic (randomized) [quantum] algorithm to factorize integers.

Proof. Let $a \neq \pm 1$ be such that $a^2 \equiv 1 \pmod{n}$. Since

$$a^2 - 1 = (a+1)(a-1),$$

if n is not prime, then a prime factor of n has to be a prime factor of either $a+1$ or $a-1$. By using Euclid's algorithm to compute

$$gcd(a+1, n) \quad \text{and} \quad gcd(a-1, n)$$

we can find, in $O(\lg n)$ steps, a prime factor of n.

SECOND REDUCTION

The second key concept is that of the **period** of functions

$$f_{n,x}(k) = x^k \bmod n.$$

Period is the smallest integer r such that

$$f_{n,x}(k+r) = f_{n,x}(k)$$

for any k, i.e. the smallest r such that

$$x^r \equiv 1 \pmod{n}.$$

AN ALGORITHM TO SOLVE EQUATION $x^2 \equiv 1 \pmod{n}$.

1. *Choose randomly $1 < a < n$.*
2. *Compute $gcd(a, n)$. If $gcd(a, n) \neq 1$ we have a factor.*
3. *Find period r of function $a^k \bmod n$.*
4. *If r is odd or $a^{r/2} \equiv \pm 1 \pmod{n}$, then go to step 1; otherwise stop.*

If this algorithm stops, then $a^{r/2}$ is a non-trivial solution of the equation

$$x^2 \equiv 1 \pmod{n}.$$

EXAMPLE

Let $n = 15$. Select $a < 15$ such that $gcd(a, 15) = 1$.
{The set of such a is $\{2, 4, 7, 8, 11, 13, 14\}$}

Choose $a = 11$. Values of $11^x \bmod 15$ are then

$$11, 1, 11, 1, 11, 1$$

which gives $r = 2$.

Hence $a^{r/2} = 11 \pmod{15}$. Therefore

$$gcd(15, 12) = 3, \qquad gcd(15, 10) = 5$$

For $a = 14$ we get again $r = 2$, but in this case

$$14^{2/2} \equiv -1 \pmod{15}$$

and the following algorithm fails.

1. *Choose randomly $1 < a < n$.*
2. *Compute $gcd(a, n)$. If $gcd(a, n) \neq 1$ we have a factor.*
3. *Find period r of function $a^k \bmod n$.*
4. *If r is odd or $a^{r/2} \equiv \pm 1 \pmod{n}$, then go to step 1; otherwise stop.*

EFFICIENCY of REDUCTION

Lemma If $1 < a < n$ satisfying $gcd(n, a) = 1$ is selected in the above algorithm randomly and n is not a power of prime, then

$$Pr\{r \text{ is even and } a^{r/2} \not\equiv \pm 1\} \geq \frac{9}{16}.$$

1. *Choose randomly $1 < a < n$.*
2. *Compute $gcd(a, n)$. If $gcd(a, n) \neq 1$ we have a factor.*
3. *Find period r of function $a^k \bmod n$.*
4. *If r is odd or $a^{r/2} \equiv \pm 1 \pmod{n}$, then go to step 1; otherwise stop.*

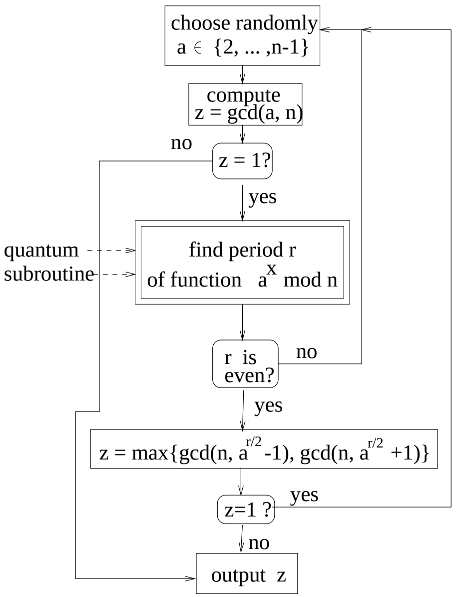
Corollary If there is a polynomial time randomized [quantum] algorithm to compute the period of the function

$$f_{n,a}(k) = a^k \bmod n,$$

then there is a polynomial time randomized [quantum] algorithm to find non-trivial solution of the equation $a^2 \equiv 1 \pmod{n}$ (and therefore also to factorize integers).

A GENERAL SCHEME for Shor's ALGORITHM

The following flow diagram shows the general scheme of Shor's quantum factorization algorithm



SHOR's QUANTUM FACTORIZATION ALGORITHM I.

1. For given $n, q = 2^d, a$ create states

$$\frac{1}{\sqrt{q}}\sum_{x=0}^{q-1}|n,a,q,x,\mathbf{0}\rangle \text{ and } \frac{1}{\sqrt{q}}\sum_{x=0}^{q-1}|n,a,q,x,a^x \bmod n\rangle$$

2. By measuring the last register the state collapses into the state

$$\frac{1}{\sqrt{A+1}}\sum_{j=0}^{A}|n,a,q,jr+l,y\rangle \text{ or, shortly } \frac{1}{\sqrt{A+1}}\sum_{j=0}^{A}|jr+l\rangle,$$

where A is the largest integer such that $l + Ar \leq q$, r is the period of a^x mod n and l is the offset.

$$\sqrt{\frac{r}{q}}\sum_{j=0}^{\frac{q}{r}-1}|jr+l\rangle$$

3. By applying quantum Fourier transformation we get then the state

$$\frac{1}{\sqrt{r}}\sum_{j=0}^{r-1}e^{2\pi ilj/r}|j\frac{q}{r}\rangle.$$

4. By measuring the resulting state we get $c = \frac{jq}{r}$ and if $gcd(j,r) = 1$, what is very likely, then from c and q we can determine the period r.

SHOR's QUANTUM FACTORIZATION ALGORITHM II.

Indeed, since

$$c = \frac{jq}{r}$$

for randomly chosen j and still unknown period r and very likely $gcd(j,r) = 1$
we have

$$\frac{c}{j} = \frac{q}{r}$$

and therefore

$$r = \frac{q}{gcd(c,q)}$$