Part I

Cyclic codes and channel codes

CHAPTER 3: CYCLIC, STREAM and CHANNEL CODES - SPECIAL DECODINGS

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4. Locally decodable codes can be seen as a theoretical extreme of coding theory with deep theoretical implications.

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In order to specify a binary cyclic code with 2^k codewords of length n it is sufficient to write down

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codeword of length n - a generator of the code C.

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(i) C **is a linear code;**

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(iv) Is Hamming code $Ham(2, 3)$ with the generator matrix

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(a) cyclic?

(b) or at least equivalent to a cyclic code?

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For some cases, for example for $n = 19$ and $F = GF(2)$, the above four trivial cyclic codes are the only cyclic codes.

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and it is cyclic because the right shifts have the following impacts

$$c_1 \rightarrow c_2, \qquad c_2 \rightarrow c_3, \qquad c_3 \rightarrow c_1 + c_3$$
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A codeword of a cyclic code is usually denoted by

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Definition Let $f(x)$ be a fixed polynomial in $F_q[x]$. Two polynomials $g(x)$, $h(x)$ are said to be congruent modulo $f(x)$, notation

$$g(x) \equiv h(x) \pmod{f(x)},$$

if $g(x) - h(x)$ is divisible by $f(x)$.

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The word starting with 2^{124} zeros and followed by one 1 has the polynomial representation:

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In the alphabet $\{0, 1, 2\}$ $2x^2$ represents the string 002

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- G is closed under *; that is if $a, b \in G$, so is $a * b$.
- The operation * is associative, that is $a * (b * c) = (a * b) * c$, for any $a, b, c \in G$.
- G has an identity e element such that $e * a = a * e = a$ for any $a \in G$.
- Every element $a \in G$ has an inverse $a^{-1} \in G$, such that $a * a^{-1} = a^{-1} * a = e$.

A group G is called an **Abelian group** if the operation $*$ is commutative, that is $a * b = b * a$ for any $a, b \in G$.

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- What happens if we consider only matrices with determinants not equal zero?

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A non-zero element g is a **primitive element** of a field F if all non-zero elements of F are powers of g.

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For any polynomial $f(x)$, the set of all polynomials in $F_q[x]$ of degree less than $deg(f(x))$, with addition and multiplication modulo $f(x)$, forms a **ring** **denoted** $F_q[x]/f(x)$.

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If $f(x)$ is not reducible, then it is said to be **irreducible** in $F_q[x]$.

Theorem The ring $F_q[x]/f(x)$ is a field if $f(x)$ is irreducible in $F_q[x]$.

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multiplication of $p(w)$ by x in R_n corresponds to a single cyclic shift of w. Indeed,

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is in C by (i) because summons are cyclic shifts of $a(x)$.

(2) Let (i) and (ii) hold

- Taking $r(x)$ to be a scalar the conditions (i) and (ii) imply linearity of C.
- Taking $r(x) = x$ the conditions (i) and (ii) imply cyclicity of C.

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Theorem A binary code C of words of length n is cyclic if and only if it satisfies two conditions

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$$C = \{000, 110, 101, 011\}$$

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for a cyclic code C, then g is called the generator polynomial for the code C.

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Problem: Find all binary cyclic codes of length 3.

Solution: Make decomposition

$$x^3 - 1 = \quad \underbrace{(x-1)(x^2+x+1)}_{\text{both factors are irreducible in GF(2)}}$$

Therefore, we have the following generator polynomials and cyclic codes of length 3.

Generator polynomials	Code in R_3	Code in $V(3,2)$
1	R_3	$V(3,2)$
$x+1$	$\{0, 1+x, x+x^2, 1+x^2\}$	$\{000, 110, 011, 101\}$
x^2+x+1	$\{0, 1+x+x^2\}$	$\{000, 111\}$
$x^3-1\ (=0)$	$\{0\}$	$\{000\}$

DESIGN of GENERATOR MATRICES for CYCLIC CODES

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Hence

$$\begin{aligned} q(x)g(x) &= (q_0 + q_1 x + \ldots + q_{n-r-1} x^{n-r-1})g(x) \\ &= q_0 g(x) + q_1 x g(x) + \ldots + q_{n-r-1} x^{n-r-1} g(x). \end{aligned}$$

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EXAMPLE - II

In order to determine all binary cyclic codes of length 7, consider decomposition

$$x^7 - 1 = (x - 1)(x^3 + x + 1)(x^3 + x^2 + 1)$$

Since we want to determine binary codes, all computations should be modulo 2 and therefor all minus signs can be replaced by plus signs. Therefore

$$x^7 + 1 = (x + 1)(x^3 + x + 1)(x^3 + x^2 + 1)$$

Therefore generators for 2^3 binary cyclic codes of length 7 are

$$1, \quad a(x) = x + 1, \quad b(x) = x^3 + x + 1, \quad c(x) = x^3 + x^2 + 1$$
$$a(x)b(x), \quad a(x)c(x), \quad b(x)c(x), \quad a(x)b(x)c(x) = x^7 + 1$$

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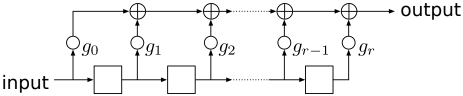

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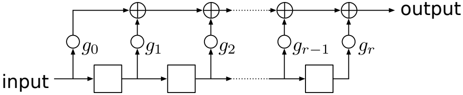

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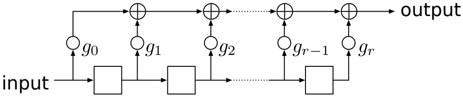

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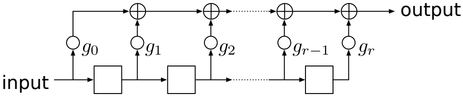

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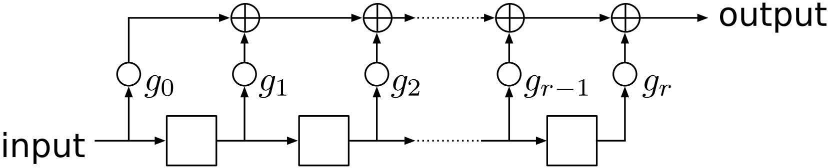

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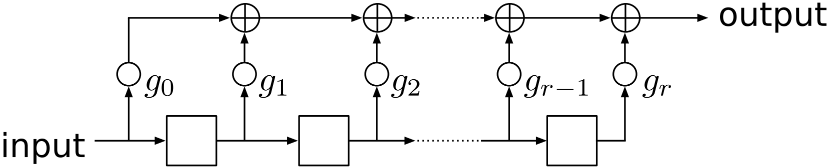

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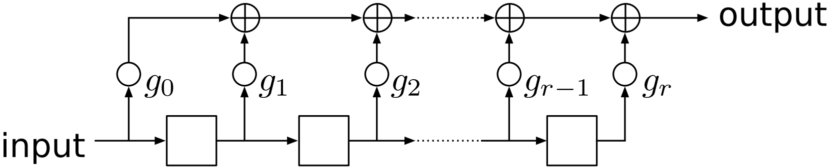

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and therefore the input to the shift register is the word

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MULTIPLICATION of POLYNOMIALS by SHIFT-REGISTERS

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EXAMPLES of CYCLIC CODES

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0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
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G_{24} is $(24, 12, 8)$-code and the weights of all codewords are multiples of 4. G_{23} is obtained from G_{24} by deleting last symbols of each codeword of G_{24}. G_{23} is $(23, 12, 7)$-code. It is a perfect code.

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This code can be constructed via factorization of $x^{23} - 1$.

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Golay codes are named to honour Marcel J. E. Golay - from 1949.

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A **Polynomial code**, with codewords of length n, **generated by a (generator) polynomial** $g(x)$ of degree $m < n$ over a GF(q) is the code whose codewords are represented exactly by those polynomials of degree less than n that are divisible by $g(x)$.

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Example: For the binary polynomial code with $n = 5$ and $m = 2$ generated by the polynomial $g(x) = x^2 + x + 1$ all codewords are of the form:

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where

$$a(x) \in \{0, 1, x, x + 1, x^2, x^2 + 1, x^2 + x, x^2 + x + 1\}$$

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what results in the code with codewords

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Special cases of Reed-Muller codes are Hadamard code and Reed-Solomon code.

BCH CODES and REED-SOLOMON CODES

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Comments: For BCH codes there exist efficient variations of syndrome decoding. A Reed-Solomon code is a special primitive BCH code.

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Reed-Solomon codes found many important applications from deep-space travel to consumer electronics.

They are very useful especially in those applications where one can expect that errors occur in bursts - such as ones caused by solar energy.

CHANNELS (STREAMS) CODING

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Moreover, the theorem says that probability of a decoding error can be made to decrease exponentially as the block length N of the coding scheme goes to infinity.

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A breakthrough came when D. Forney, in his PhD thesis in 1972, showed that so called concatenated codes could be used to achieve exponentially decreasing error probabilities at all data rates less than the Shannon channel capacity, with decoding complexity increasing only polynomially with the code length.

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The code rate express the amount of redundancy in the code - the lower is the code rate, the more redundancy is in the codewords.

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By the **noisy-channel Shannon coding theorem**, the channel capacity of a given channel is the limiting code rate (in units of information per unit time) that can be achieved with arbitrary small error probability.

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The joint distribution $P_{X,Y}(x,y)$ is then defined by

$$P_{X,Y}(x,y) = P_{Y|X}(y|x)P_X(x),$$

where $P_X(x)$ is the marginal distribution.

CHANNEL CAPACITY - FORMAL DEFINITION

Let X and Y be random variables representing the input and output of a channel.

Let $P_{Y|X}(y|x)$ be the conditional probability distribution function of Y given X, which can be seen as an inherent fixed probability of the communication channel.

The joint distribution $P_{X,Y}(x,y)$ is then defined by

$$P_{X,Y}(x,y) = P_{Y|X}(y|x)P_X(x),$$

where $P_X(x)$ is the marginal distribution.

The channel capacity is then defined by

$$C = \sup_{P_X(x)} I(X,Y)$$

where

$$I(X,Y) = \sum_{y \in Y} \sum_{x \in X} P_{X,Y}(x,y) \log \left(\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)} \right)$$

is the mutual distribution - a measure of variables mutual distribution.

SHANNON NOISY CHANEL THEOREM

For every discrete memoryless channel, the channel capacity

$$C = \sup_{P_X} I(X, Y)$$

has the following properties:

1. For every $\varepsilon > 0$ and $R < C$, for large enough N there exists a code of length N and code rate R and a decoding algorithm, such that the maximal probability of the block error is $\leq \varepsilon$.

2. If a probability of the block error p_b is acceptable, code rates up to $R(p_b)$ are achievable, where

$$P(p_b) = \frac{C}{1 - H_2(p_b)}$$

and $H_2(p_b)$ is the binary entropy function.

3. For any p_b code rates greater than $R(p_b)$ are not achievable.

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For example,

$$G_1 = [x^2 + 1, x^2 + x + 1]$$

is the generator matrix for a $(2, 1)$ convolution code, denoted CC_1, and

$$G_2 = \begin{pmatrix} 1 + x & 0 & x + 1 \\ 0 & 1 & x \end{pmatrix}$$

is the generator matrix for a $(3, 2)$ convolution code denoted CC_2

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That is the output streams C_0 and C_1 are obtained by convoluting the input stream with polynomials of G_1.

ENCODING

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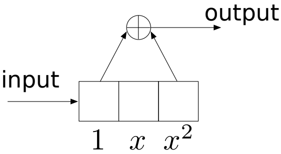

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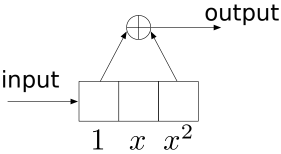

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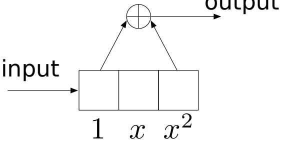

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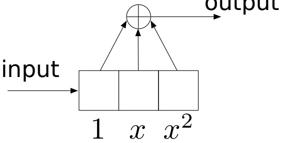

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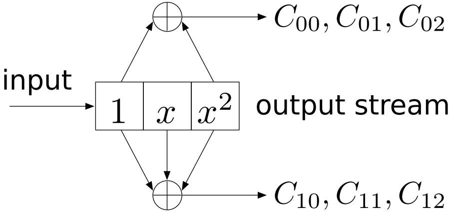

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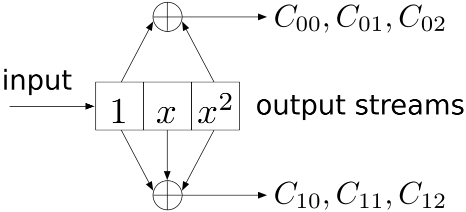

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Given $(x, y) \in \{-1, 1\} \times R$, the noise $y - x$ is distributed according to the Gaussian distribution of zero mean and standard derivation σ of the channel

$$Pr(y|x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2\sigma^2}}$$

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Concatenated codes and Turbo codes, discussed later, have such a Shannon capacity approaching property.

CONCATENATED CODES - I

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The basic idea of concatenated codes is extremely simple. A given message is first encoded by the first (outer) code C_1 (C_{out}) and C_1-output is then encoded by the second code C_2 (C_{in}). To decode, at first C_2 decoding and then C_1 decoding are used.

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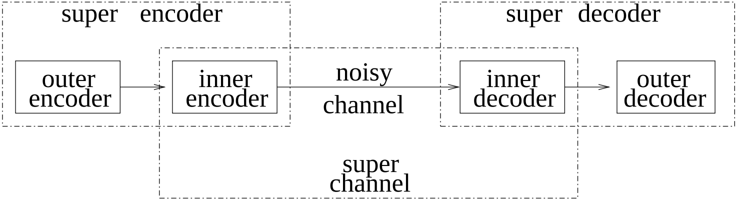

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In 1965 concatenated codes were considered as unfeasible. However, already in 1970s technology has advanced sufficiently and they became standardize by NASA for space applications.

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A code concatenated codes C_{out} and C_{in} maps a message

$$m = (m_1, m_2, \ldots, m_K),$$

as follows: At first C_{out} encoding is applied to get

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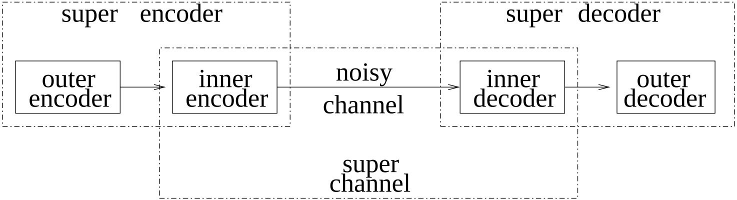

- **Outer code:** - (n_2, k_2) code
- **Inner code**: - (n_1, k_1) binary code
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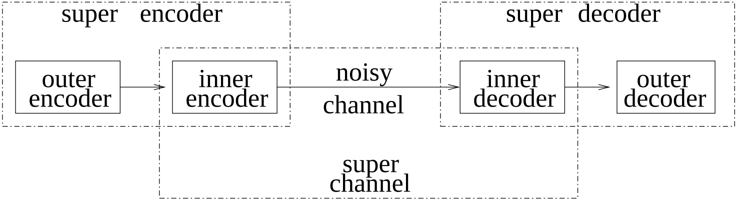

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- **length** of such a concatenated code is $n_1 n_2$
- **dimension** of such a concatenated code is $k_1 k_2$
- if **minimal distances** of both codes are d_1 and d_2, then resulting concatenated code has minimal distance $\geq d_1 d_2$.

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- In such a case we can use an exponential time but optimal maximum likelihood decoder for the inner code.

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- Concatenated codes are used also on Compact Disc.
- The best concatenated codes for many applications were based on outer Reed-Solomon codes and inner Viterbi-decoded short constant length convolution codes.

EXAMPLE from SPACE EXPLORATION

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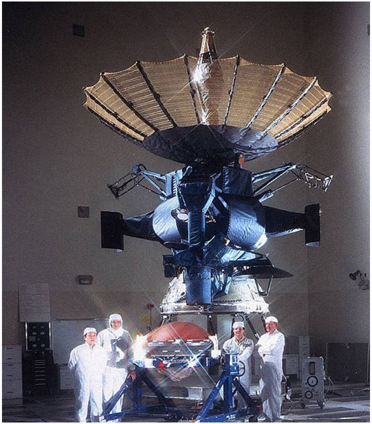

At the very beginning of the Galileo mission to explore Jupiter and its moons in 1989 it was discovered that primary antenna (deployed in the figure on the top) failed to deploy,

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After all reparations and new encodings it was possible to send up to 1000 b/s.

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After all reparations and new encodings it was possible to send up to 1000 b/s. Mission was rescued.

GALILEO MISSION - SOLUTION

The primary antenna was designed to send 100, 000 b/s. Spacecraft had also another antenna, but that was capable to send only 10 b/s. The whole mission looked as being a disaster.
A heroic engineering effort was immediately undertaken in the mission center to design the most powerful concatenated code conceived up to that time, and to program it into the spacecraft computer.

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Nowadays when so called iterative decoding is used concatenation of even very simple codes can yield superb performance.

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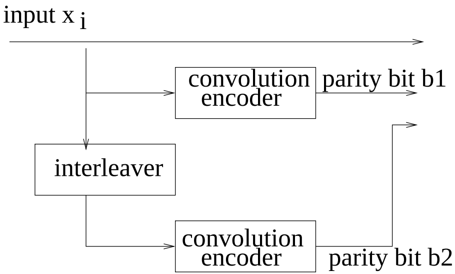
A Turbo encoder is formed from the parallel composition of two (convolution) encoders separated by an interleaver.



EXAMPLES of TURBO and CONVOLUTION ENCODERS

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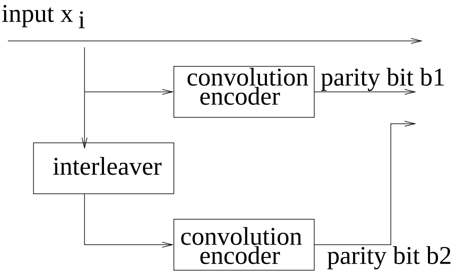

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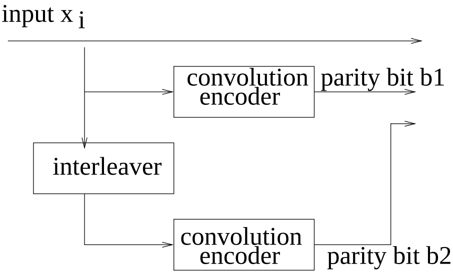
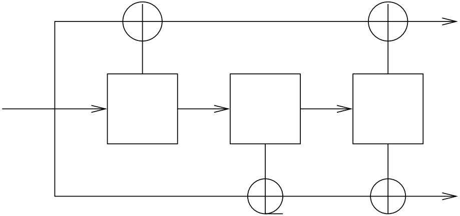


and a convolution encoder



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However, after the inverse permutation the output actually wll be

$$c.n.j.200k.$$

which is quite easy to decode correctly!!!!

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- Literature: M.C. Valenti and J.Sun: Turbo codes - tutorial, Handbook of RF and Wireless Technologies, 2004 - reachable by Google.

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A decibel is a relative measure. If E is the actual energy and E_{ref} is the theoretical lower bound, then the relative energy increase in decibels is

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- For code rate $\frac{1}{2}$ the relative increase in energy consumption is about 4.8 dB for convolution codes and 0.98 for Turbo codes.

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- Permutations performed by interleaver can often by specified by simple polynomials that make one-to-one mapping of some sets $\{0, 1, \ldots, q-1\}$.

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- A turbo code can be seen as a refinement of concatenated codes plus an iterative algorithm for decoding.

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List decoding seems to be a stronger error-correcting mode than unique decoding.

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For Reed-Solomon codes there is a list decoding up to $1 - \sqrt{2R}$ errors.

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Theorem let $q \geq 2$, $0 \leq p \leq 1 - 1/q$ and $\varepsilon \geq 0$ then for large enough block length n if the code rate $R \leq 1 - H_q(p) - \varepsilon$, then there exists a $(p, O(1/\varepsilon))$-list decodable code.
[$H_q(p) = p\log_q(q-1) - p\log_q p - (1-p)\log_q(1-p)$ is q-ary entropy function.]
Moreover, if $R > 1 - H_q(p) + \varepsilon$, then every (p, L)-list-decodable code has $L = q^{\Omega(n)}$

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APPENDIX - I.

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- Modern versions of concatenated Reed-Solomon/Viterbi decoder convolution coding were and are used on the Mars Pathfinder, Galileo, Mars exploration Rover and Cassini missions, where they performed within about 1-1.5dB of the ultimate limit imposed by the Shannon theorem.

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- New computation tools are developed - for example special types of parallelization,....

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Locally decodable codes have a variety of applications in cryptography and theory of fault-tolerant computation.

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Moreover, this can be done by picking at random only three bits of the received message and combining them in a right way.