Part I

Linear codes

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Many practically important linear codes have also an efficient decoding.

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Comment. To design linear codes we will use Galois fields $GF(q)$ with q being prime. One can also use Galois fields $GF(q^k), k > 1$, but their structure and operations are defined in a more complex way, see the Appendix.

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Encoding (code) is called systematic if for any $m \in M \subset \Sigma^*$

$$e(m) = mc_m \;\text{ for some }\; c_m \in \Sigma^*$$

SYSTEMATIC CODES I

A code is called systematic if its encoder transmit a message (an input dataword) w into a codeword of the form wc_w, or (w, c_w). That is if the codeword for the message w consists of two parts: the message w itself (called also information part) and a redundancy part c_w

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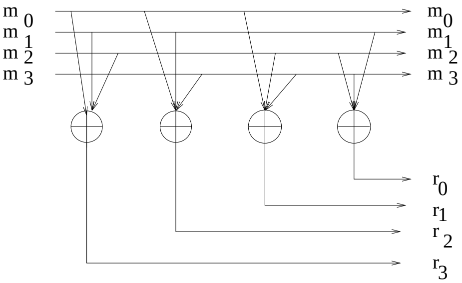
Nowadays most of the stream codes that are used in practice are systematic.

An example of a systematic encoder, that produces so called extended Hamming $(8, 4, 1)$ code is in the following figure.



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Linear codes are also called "group codes".

EXERCISE

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(b) multiplication of a row by a non-zero scalar

(c) addition of one row to another

(d) permutation of columns

(e) multiplication of a column by a non-zero scalar

Proof Operations (a) - (c) just replace one basis by another. Last two operations convert a generator matrix to one of an equivalent code.

EQUIVALENCE of LINEAR CODES II

Theorem Let G be a generator matrix of an $[n, k]$-code. Rows of G are then linearly independent .By operations (a) - (e) the matrix G can be transformed into the form: $[I_k|A]$ where I_k is the $k \times k$ identity matrix, and A is a $k \times (n-k)$ matrix.

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$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \rightarrow$$

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$$0 = \sum_{i=1}^k u_{1,i} w_i - \sum_{i=1}^k u_{2,i} w_i = \sum_{i=1}^k (u_{1,i} - u_{2,i}) w_i$$

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And, therefore, since w_i are linearly independent, $u_1 = u_2$.

LINEAR CODES as SYSTEMATIC CODES

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Since to each linear $[n, k]$-code C there is a generator matrix of the form $G = [I_k|A]$ an encoding of a dataword w with G has the form

$$wG = w \cdot wA$$

Each linear code is therefore equivalent to a systematic code.

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In practice, this decoding method is too slow and requires too much memory.

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If $p = 0.01$, then $P_{corr} = 0.9897$

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Lemma Suppose C is an $[n, k]$-code having a generator matrix G. Then for $v \in F_q^n$

$$v \in C^{\perp} \Leftrightarrow vG^{\top} = 0,$$

where $G^{\top}$ denotes the transpose of the matrix G.

DUAL CODES

Inner product of two vectors (words)

$$u = u_1 \dots u_n, \quad v = v_1 \dots v_n$$

in F_q^n is an element of $GF(q)$ defined (using modulo q operations) by

$$u \cdot v = u_1 v_1 + \dots + u_n v_n.$$

Example *In* $F_2^4 : 1001 \cdot 1001 = 0$
In $F_3^4 : 2001 \cdot 1210 = 2$
$1212 \cdot 2121 = 2$

If $u \cdot v = 0$ then words (vectors) u and v are called **orthogonal words**.

Properties If $u, v, w \in F_q^n, \lambda, \mu \in GF(q)$, then
$u \cdot v = v \cdot u, (\lambda u + \mu v) \cdot w = \lambda(u \cdot w) + \mu(v \cdot w)$.

Given a linear $[n,k]$-code C, then the dual code of C, denoted by $C^\perp$, is defined by

$$C^\perp = \{v \in F_q^n \mid v \cdot u = 0 \text{ for all } u \in C\}.$$

Lemma Suppose C is an $[n,k]$-code having a generator matrix G. Then for $v \in F_q^n$

$$v \in C^\perp \Leftrightarrow vG^\top = 0,$$

where $G^\top$ denotes the transpose of the matrix G. Proof Easy.

PARITE CHECKS versus ORTHOGONALITY

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Answer: All words of S have at the end the same symbol as at the beginning.

EXAMPLE

For the $[n, 1]$-repetition (binary) code C, with the generator matrix

$$G = (1, 1, \ldots, 1)$$

the dual code $C^{\perp}$ is $[n, n-1]$-code with the generator matrix $G^{\perp}$, described by

$$G^{\perp} = \begin{pmatrix} 1 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 1 & 0 & \ldots & 0 \\ & \ldots & & & & \\ 1 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

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The rows of a parity check matrix are parity checks on codewords. They actually say that certain linear combinations of elements of every codeword are zeros modulo 2.

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When preparing a "syndrome decoding" it is sufficient to store only two columns: one for **coset leaders** and one for **syndromes**.

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An important family of simple linear codes that are easy to encode and decode, are so-called Hamming codes.

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Properties of binary Hamming codes Coset leaders are precisely words of weight ≤ 1. The syndrome of the word $0 \ldots 010 \ldots 0$ with 1 in j-th position and 0 otherwise is the transpose of the j-th column of H.

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Decoding algorithm for the case the columns of H are arranged in the order of increasing binary numbers the columns represent.

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Hamming codes were originally used to deal with errors in long-distance telephon calls.

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G_{24} is $(24, 12, 8)$-code and the weights of all codewords are multiples of 4. G_{23} is obtained from G_{24} by deleting last symbols of each codeword of G_{24}. G_{23} is $(23, 12, 7)$-code.

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Rows of the last 11 columns are cyclic permutations of the first row which has 1 at those positions that are squares modulo 11, that is

$$0, 1, 3, 4, 5, 9.$$

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- Matrix Q_r is obtained by considering all combinations of r rows of $G_{1,m}$ and by obtaining products of these rows/vectors, component by component. The result of each of such a multiplication constitues a row of Q_r.

EXAMPLE

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Codes $R(m-r-1,m)$ and $R(r,m)$ are dual codes.

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Plotkin bound implies that q-nary error-correcting codes with $d \geq n(1 - 1/q)$ have only polynomially many codewords and hence are not very interesting.

SHORTENING and PUNCTURING of LINEAR CODES

If C is a q-ary linear $[n, k, d]$-code, then

$D = \{(x_1, \ldots, x_{n-1}) | (x_1, \ldots, x_{n-1}, 0) \in C\}$. is a linear code - a shortening of the code C.

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When $d = 1$, then E is an $[n-1, k, 1]$ code, if C has no codeword of weight 1 whose nonzero entry is in last coordinate; otherwise, if $k > 1$, then E is an $[n-1, k-1, d^*]$ code with $d^* > 1$

REED-SOLOMON CODES

An important example of MDS-codes are q-ary Reed-Solomon codes RSC(k, q), for $k \leq q$.

They are codes a generator matrix of which has rows labelled by polynomials X^i, $0 \leq i \leq k-1$, columns labeled by elements $0, 1, \ldots, q-1$ and the element in the row labelled by a polynomial p and in the column labelled by an element u is $p(u)$.

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Reed-Solomon codes are used in digital television, satellite communication, wireless communication, barcodes, compact discs, DVD,... They are very good to correct burst errors - such as ones caused by solar energy.

SOCCER GAMES BETTING SYSTEM

Ternary Golay code with parameters (11, 729, 5) can be used to bet for results of 11 soccer games with potential outcomes 1 (if home team wins), 2 (if guest team wins) and 3 (in case of a draw).

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In case one has to bet for 13 games, then one can usually have two games with pretty sure outcomes and for the rest one can use the above ternary Golay code.

APPENDIX

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Parity-check matrices for LDPC codes are often (pseudo)-randomly generated, subject to sparsity constrains. Such LDPC codes are proven to be good with a high probability.

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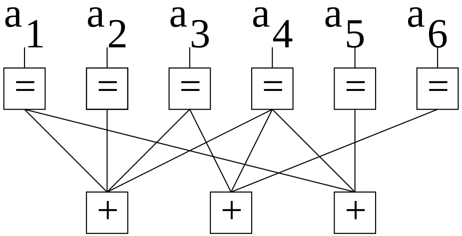
An $[n,k]$ LDPC code can be represented by a bipartite graph between a set of n top "variable-nodes (v-nodes)" and a set of bottom $(n-k)$ "parity check nodes (c-nodes)".



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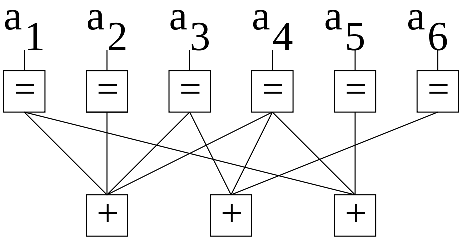

The corresponding parity check matrix has $n-k$ rows and n columns and i-th column has 1 in the j-th row exactly in case if i-th v-node is connected to j-th c-node.

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

TANNER GRAPHS - CONTINUATION

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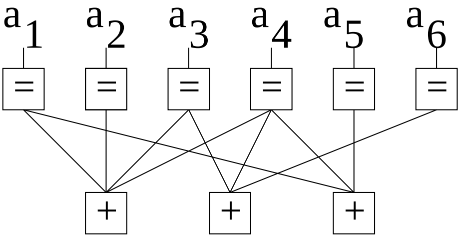
The LDPC-code with the Tanner bipartite graph for $(6, 3)$ LDPC-code.



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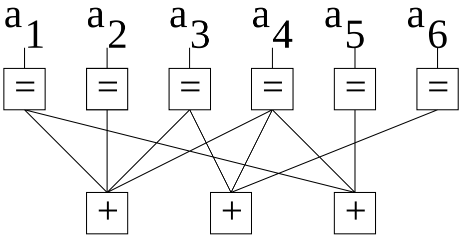

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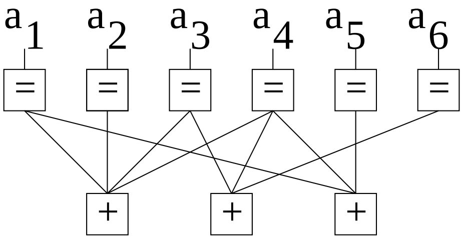

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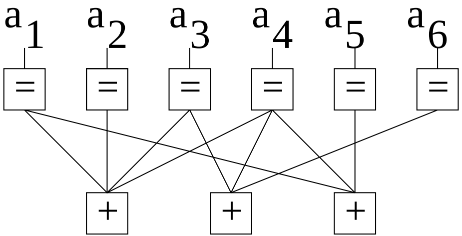

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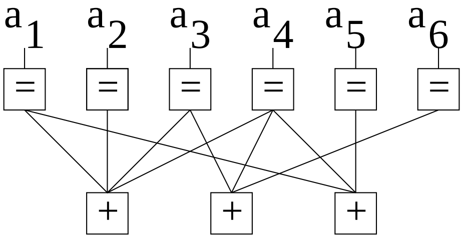

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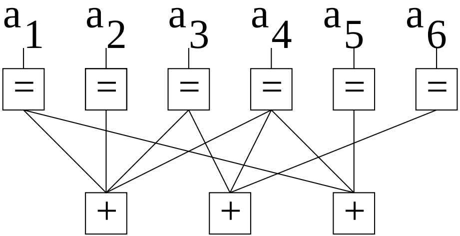

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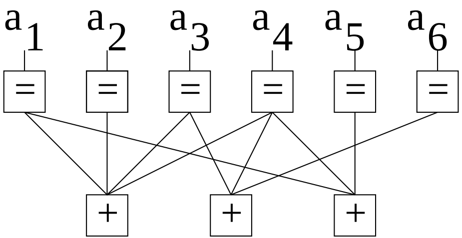

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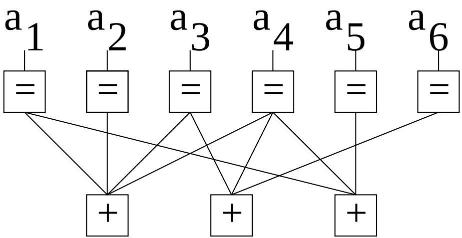

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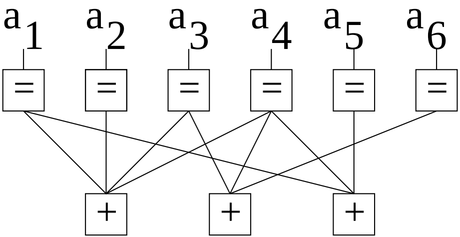

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Using so called **iterative belief propagation techniques**, LDPC codes can be decoded in time linear to their block length.

DESIGN of LDPC codes

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- Some good LDPC codes were designed through randomly chosen parity check matrices.
- Some LDPC codes are based on Reed-Solomon codes, such as the RS-LDPC code used in the 10-gigabit Ethernet standard.

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- LDPC is also used for 10Gbase-T Ethernet, which sends data at 10 gigabits per second over twisted-pair cables.
- Since 2009 LDPC codes are also part of of the Wi-Fi 802.11 standard as an optional part of 802.11n, in the High Throughput PHY specification.