Part III

Cyclic codes

CHAPTER 3: CYCLIC CODES and CHANNEL CODES

Cyclic codes are special linear codes of large interest and importance because

- They **posses a rich algebraic structure** that can be utilized in a variety of ways.
- They **have extremely concise specifications**.
- Their encodings **can be efficiently implemented** using simple shift registers.
- Many of the practically very important codes are cyclic.

Channel codes are used to **encode streams of data** (bits). Some of them, as **Turbo codes**, **reach theoretical Shannon bound concerning efficiency**, and are currently used very often.

IMPORTANT NOTE

In order to specify a non-linear binary code with 2^k codewords of length n one may need to write down

$$2^k$$

codewords of length n.

In order to specify a linear binary code of the dimension k with 2^k codewords of length n it is sufficient to write down

$$k$$

codewords of length n.

In order to specify a binary cyclic code with 2^k codewords of length n it is sufficient to write down

$$1$$

codeword of length n.

BASIC DEFINITION AND EXAMPLES

Definition A code C is cyclic if

(i) C is a linear code;

(ii) any cyclic shift of a codeword is also a codeword, i.e. whenever $a_0, \dots a_{n-1} \in C$, then also $a_{n-1}a_0 \dots a_{n–2} \in C$ and $a_1 a_2 \dots a_{n-1} a_0 \in C$.

Example

(i) Code $C = \{000, 101, 011, 110\}$ is cyclic.

(ii) Hamming code $Ham(3, 2)$: with the generator matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

is equivalent to a cyclic code.

(iii) The binary linear code $\{0000, 1001, 0110, 1111\}$ is not cyclic, but it is equivalent to a cyclic code.

(iv) Is Hamming code $Ham(2, 3)$ with the generator matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

(a) cyclic?

(b) or at least equivalent to a cyclic code?

FREQUENCY of CYCLIC CODES

Comparing with linear codes, cyclic codes are quite scarce. For example, there are 11 811 linear [7,3] binary codes, but only two of them are cyclic.

Trivial cyclic codes. For any field F and any integer $n \geq 3$ there are always the following cyclic codes of length n over F:

- No-information code - code consisting of just one all-zero codeword.
- Repetition code - code consisting of all codewords (a, a, . . . ,a) for $a \in F$.
- Single-parity-check code - code consisting of all codewords with parity 0.
- No-parity code - code consisting of all codewords of length n

For some cases, for example for $n = 19$ and $F = GF(2)$, the above four trivial cyclic codes are the only cyclic codes.

AN EXAMPLE of a CYCLIC CODE

The code with the generator matrix

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

has, in addition to the codeword 0000000, the following codewords

$$c_1 = 1011100 \qquad c_2 = 0101110 \qquad c_3 = 0010111$$

$$c_1 + c_2 = 1110010 \qquad c_1 + c_3 = 1001011 \qquad c_2 + c_3 = 0111001$$

$$c_1 + c_2 + c_3 = 1100101$$

and it is cyclic because the right shifts have the following impacts

$$c_1 \rightarrow c_2, \qquad c_2 \rightarrow c_3, \qquad c_3 \rightarrow c_1 + c_3$$

$$c_1 + c_2 \rightarrow c_2 + c_3, \qquad c_1 + c_3 \rightarrow c_1 + c_2 + c_3, \qquad c_2 + c_3 \rightarrow c_1$$

$$c_1 + c_2 + c_3 \rightarrow c_1 + c_2$$

POLYNOMIALS over GF(q)

A codeword of a cyclic code is usually denoted

$$a_0 a_1 \dots a_{n-1}$$

and to each such a codeword the polynomial

$$a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1}$$

will be associated – am ingenious idea!!.

NOTATION: $F_q[x]$ will denote the set of all polynomials $f(x)$ over $GF(q)$.

$deg(f(x))$ = the largest m such that x^m has a non-zero coefficient in $f(x)$.

Multiplication of polynomials If $f(x)$, $g(x) \in Fq[x]$, then

$$deg(f(x)g(x)) = deg(f(x)) + deg(g(x)).$$

Division of polynomials For every pair of polynomials $a(x)$, $b(x) \neq 0$ in $F_q[x]$ there exists a unique pair of polynomials $q(x)$, $r(x)$ in $F_q[x]$ such that

$$a(x) = q(x)b(x) + r(x), deg(r(x)) < deg(b(x)).$$

Example Divide $x^3 + x + 1$ by $x^2 + x + 1$ in $F_2[x]$.

Definition Let $f(x)$ be a fixed polynomial in $F_q[x]$. Two polynomials $g(x)$, $h(x)$ are said to be congruent modulo $f(x)$, notation

$$g(x) \equiv h(x) \pmod{f(x)},$$

if $g(x) - h(x)$ is divisible by $f(x)$.

RINGS of POLYNOMIALS

For any polynomial $f(x)$, the set of all polynomials in $F_q[x]$ of degree less than $deg(f(x))$, with addition and multiplication modulo $f(x)$, forms a **ring denoted** $F_q[x]/f(x)$.

Example **Calculate $(x+1)^2$ in $F_2[x]/(x^2+x+1)$.** It holds

$$(x+1)^2 = x^2 + 2x + 1 \equiv x^2 + 1 \equiv x \pmod{x^2+x+1}.$$

How many elements has $F_q[x]/f(x)$?

Result $|F_q[x]/f(x)| = q^{deg(f(x))}$.

Example Addition and multiplication tables for $F_2[x]/(x^2+x+1)$

+	0	1	x	1+x
0	0	1	x	1+x
1	1	0	1+x	x
x	x	1+x	0	1
1+x	1+x	x	1	0

•	0	1	x	1+x
0	0	0	0	0
1	0	1	x	1+x
x	0	x	1+x	1
1+x	0	1+x	1	x

Definition A polynomial $f(x)$ in $F_q[x]$ is said to be reducible if $f(x) = a(x)b(x)$, where $a(x),\ b(x) \in F_q[x]$ and

$$deg(a(x)) < deg(f(x)), \qquad deg(b(x)) < deg(f(x)).$$

If $f(x)$ is not reducible, then it is said to be irreducible in $F_q[x]$.

Theorem The ring $F_q[x]/f(x)$ is a field if $f(x)$ is irreducible in $F_q[x]$.

FIELD $R_n, R_n = F_q[x]/(x^n - 1)$

Computation modulo $x^n - 1$ in the field $R_n = F_q[x]/(x^n - 1)$

Since $x^n \equiv 1 \pmod{(x^n - 1)}$ we can compute $f(x) \bmod (x^n - 1)$ by replacing, in $f(x)$, $x^n by 1$, x^{n+1} by x, x^{n+2} by x^2, x^{n+3} by x^3, ...

Replacement of a word

$$w = a_0 a_1 \ldots a_{n-1}$$

by a polynomial

$$p(w) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$$

is of large importance because

multiplication of $p(w)$ by x in R_n corresponds to a single cyclic shift of w

$$x(a_0 + a_1 x + \ldots a_{n-1} x^{n-1}) = a_{n-1} + a_0 x + a_1 x^2 + \ldots + a_{n-2} x^{n-1}$$

An ALGEBRAIC CHARACTERIZATION of CYCLIC CODES

Theorem A binary code C of words of length n is cyclic if and only if it satisfies two conditions

(i) $a(x), b(x) \in C \Rightarrow a(x) + b(x) \in C$

(ii) $a(x) \in C, r(x) \in R_n \Rightarrow r(x)a(x) \in C$

Proof

(1) Let C be a cyclic code. C is linear $\Rightarrow$

(i) holds.

(ii)

$$Let\ a(x) \in C, r(x) = r_0 + r_1 x + \ldots + r_{n-1}x^{n-1}$$
$$r(x)a(x) = r_0 a(x) + r_1 x a(x) + \ldots + r_{n-1}x^{n-1}a(x)$$

is in C by (i) because summands are cyclic shifts of $a(x)$.

(2) Let (i) and (ii) hold

- Taking $r(x)$ to be a scalar the conditions imply linearity of C.
- Taking $r(x) = x$ the conditions imply cyclicity of C.

CONSTRUCTION of CYCLIC CODES

Notation For any $f(x) \in R_n$, we can define

$$\langle f(x)\rangle = \{r(x)f(x) \mid r(x) \in R_n\}$$

(with multiplication modulo $x^n - 1$) to be a set of polynomials - a code.

Theorem For any $f(x) \in R_n$, the set $\langle f(x)\rangle$ is a cyclic code (generated by f).

Proof We check conditions (i) and (ii) of the previous theorem.

(i) If $a(x)f(x) \in \langle f(x)\rangle$ and also $b(x)f(x) \in \langle f(x)\rangle$, then

$$a(x)f(x) + b(x)f(x) = (a(x) + b(x))f(x) \in \langle f(x)\rangle$$

(ii) If $a(x)f(x) \in \langle f(x)\rangle$, $r(x) \in R_n$, then

$$r(x)(a(x)f(x)) = (r(x)a(x))f(x) \in \langle f(x)\rangle$$

Example let $C = \langle 1 + x^2\rangle$, $n = 3$, $q = 2$.
In order to determine C we have to compute $r(x)(1 + x^2)$ for all $r(x) \in R_3$.

$$R_3 = \{0, 1, x, 1 + x, x^2, 1 + x^2, x + x^2, 1 + x + x^2\}.$$

Result

$$C = \{0, 1 + x, 1 + x^2, x + x^2\}$$
$$C = \{000, 110, 101, 011\}$$

CHARACTERIZATION THEOREM for CYCLIC CODES

We show that **all cyclic codes C have the form $C = \langle f(x) \rangle$ for some $f(x) \in R_n$**.

Theorem Let C be a non-zero cyclic code in R_n. Then

- there exists a unique monic polynomial $g(x)$ of the smallest degree such that
- $C = \langle g(x) \rangle$
- $g(x)$ is a factor of $x^n - 1$.

Proof

(i) Suppose $g(x)$ and $h(x)$ are two monic polynomials in C of the smallest degree. Then the polynomial $g(x) - h(x) \in C$ and it has a smaller degree and a multiplication by a scalar makes out of it a monic polynomial. Therefore the assumption $g(x) \neq h(x)$ leads to a contradiction.

(ii) If $a(x) \in C$, ten for some $q(x)$ and $r(x)$

Then

$$a(x) = q(x)g(x) + r(x), \qquad (deg\ r(x) < deg\ g(x)).$$

and

$$r(x) = a(x) - q(x)g(x) \in C.$$

By minimality

$$r(x) = 0$$

and therefore $a(x) \in \langle g(x) \rangle$.

CHARACTERIZATION THEOREM for CYCLIC CODES - continuation

(iii) Clearly, for some $q(x)$ and $r(x)$

$$x^n - 1 = q(x)g(x) + r(x) \quad \text{with} \quad deg\ r(x) < deg\ g(x)$$

and therefore

$$r(x) \equiv -q(x)g(x) \pmod{x^n - 1} \quad \text{and}$$
$$r(x) \in C \Rightarrow r(x) = 0 \Rightarrow g(x) \quad \text{is a factor of } x^n - 1.$$

GENERATOR POLYNOMIALS

Definition If

$$C = \langle g(x) \rangle,$$

holds for a cyclic code C, then g is called the generator polynomial for the code C.

HOW TO DESIGN CYCLIC CODES?

The last claim of the previous theorem gives **a recipe to get all cyclic codes of the given length n in GF(q)**

Indeed, all we need to do is to find all factors (**in GF(q)**) of

$$x^n - 1.$$

Problem: Find all binary cyclic codes of length 3.

Solution: Since

$$x^3 - 1 = \underbrace{(x-1)(x^2+x+1)}_{\text{both factors are irreducible in GF(2)}}$$

we have the following generator polynomials and codes.

Generator polynomials	Code in R_3	Code in $V(3,2)$
1	R_3	$V(3,2)$
$x+1$	$\{0, 1+x, x+x^2, 1+x^2\}$	$\{000, 110, 011, 101\}$
x^2+x+1	$\{0, 1+x+x^2\}$	$\{000, 111\}$
$x^3-1\ (=0)$	$\{0\}$	$\{000\}$

DESIGN of GENERATOR MATRICES for CYCLIC CODES

Theorem Suppose C is a cyclic code of codewords of length n with the generator polynomial

$$g(x) = g_0 + g_1 x + \ldots + g_r x^r.$$

Then $dim\ (C) = n - r$ and a generator matrix G_1 for C is

$$G_1 = \begin{pmatrix} g_0 & g_1 & g_2 & \ldots & g_r & 0 & 0 & 0 & \ldots & 0 \\ 0 & g_0 & g_1 & g_2 & \ldots & g_r & 0 & 0 & \ldots & 0 \\ 0 & 0 & g_0 & g_1 & g_2 & \ldots & g_r & 0 & \ldots & 0 \\ \ldots & \ldots & & & & & & & & \ldots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & g_0 & \ldots & g_r \end{pmatrix}$$

Proof

(i) All rows of G1 are linearly independent.

(ii) The $n - r$ rows of G represent codewords

$$g(x), xg(x), x^2 g(x), \ldots, x^{n-r-1} g(x) \quad (*)$$

(iii) It remains to show that every codeword in C can be expressed as a linear combination of vectors from (*).

Indeed, if $a(x) \in C$, then

$$a(x) = q(x)g(x).$$

Since $deg\ a(x) < n$ we have $deg\ q(x) < n - r$.
Hence

$$\begin{aligned} q(x)g(x) &= (q_0 + q_1 x + \ldots + q_{n-r-1} x^{n-r-1}) g(x) \\ &= q_0 g(x) + q_1 x g(x) + \ldots + q_{n-r-1} x^{n-r-1} g(x). \end{aligned}$$

EXAMPLE

The task is to determine all ternary codes of length 4 and generators for them.
Factorization of $x^4 - 1$ over $GF(3)$ has the form

$$x^4 - 1 = (x-1)(x^3 + x^2 + x + 1) = (x-1)(x+1)(x^2+1)$$

Therefore there are $2^3 = 8$ divisors of $x^4 - 1$ and each generates a cyclic code.

Generator polynomial	Generator matrix
1	I_4
$x - 1$	$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$
$x + 1$	$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$
$x^2 + 1$	$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$
$(x-1)(x+1) = x^2 - 1$	$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$
$(x-1)(x^2+1) = x^3 - x^2 + x - 1$	$\begin{bmatrix} -1 & 1 & -1 & 1 \end{bmatrix}$
$(x+1)(x^2+1)$	$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$
$x^4 - 1 = 0$	$\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$

COMMENTS

The last matrix is not, however, formally a generator matrix - the corresponding code is empty. On the previous slide "generator polynomials" $x - 1$, $x^2 - 1$ and $x^3 - x^2 + x + 1$ are formally not in R_n becasue only allowable coefficients are $0, 1, 2$.

A good practice is, however, to use also coefficients -2, and -1 as ones that are equal, modulo 3, to 1 and 2 and they can be replace in such a way also in matrices to be fully correct formally.

CHECK POLYNOMIALS and PARITY CHECK MATRICES for CYCLIC CODES

Let C be a cyclic $[n, k]$-code with the generator polynomial $g(x)$ (of degree $n-k$). By the last theorem $g(x)$ is a factor of $x^n - 1$. Hence

$$x^n - 1 = g(x)h(x)$$

for some $h(x)$ of degree k. ($h(x)$ is called the check polynomial of C.)

Theorem Let C be a cyclic code in R_n with a generator polynomial $g(x)$ and a check polynomial $h(x)$. Then an $c(x) \in R_n$ is a codeword of C if and only if $c(x)h(x) \equiv 0$ –(this and next congruences are all modulo $x^n - 1$).

Proof Note, that $g(x)h(x) = x^n - 1 \equiv 0$

(i) $c(x) \in C \Rightarrow c(x) = a(x)g(x)$ for some $a(x) \in R_n$

$$\Rightarrow c(x)h(x) = a(x)\underbrace{g(x)h(x)}_{\equiv 0} \equiv 0.$$

(ii) $c(x)h(x) \equiv 0$

$$c(x) = q(x)g(x) + r(x), deg\ r(x) < n - k = deg\ g(x)$$
$$c(x)h(x) \equiv 0 \Rightarrow r(x)h(x) \equiv 0 \pmod{x^n - 1}$$

Since $deg\ (r(x)h(x)) < n - k + k = n$, we have $r(x)h(x) = 0$ in $F[x]$ and therefore

$$r(x) = 0 \Rightarrow c(x) = q(x)g(x) \in C.$$

POLYNOMIAL REPRESENTATION of DUAL CODES

Since $dim\ (\langle h(x)\rangle) = n - k = dim(C^{\perp})$ we might easily be fooled to think that the check polynomial $h(x)$ of the code C generates the dual code $C^{\perp}$.

Reality is "slightly different":

Theorem Suppose C is a cyclic $[n, k]$-code with the check polynomial

$$h(x) = h_0 + h_1 x + \ldots + h_k x^k,$$

then

(i) a parity-check matrix for C is

$$H = \begin{pmatrix} h_k & h_{k-1} & \ldots & h_0 & 0 & \ldots & 0 \\ 0 & h_k & \ldots & h_1 & h_0 & \ldots & 0 \\ \ldots & \ldots & & & & & \\ 0 & 0 & \ldots & 0 & h_k & \ldots & h_0 \end{pmatrix}$$

(ii) $C^{\perp}$ is the cyclic code generated by the polynomial

$$\overline{h}(x) = h_k + h_{k-1}x + \ldots + h_0 x^k$$

i.e. by the reciprocal polynomial of $h(x)$.

POLYNOMIAL REPRESENTATION of DUAL CODES

Proof A polynomial $c(x) = c_0 + c_1x + \ldots + c_{n-1}x^{n-1}$ represents a code from C if $c(x)h(x) = 0$. For $c(x)h(x)$ to be 0 the coefficients at $x^k, \ldots, x^{n-1}$ must be zero, i.e.

$$\begin{aligned} c_0h_k + c_1h_{k-1} + \ldots + c_kh_0 &= 0 \\ c_1h_k + c_2h_{k-1} + \ldots + c_{k+1}h_0 &= 0 \\ &\ldots \\ c_{n-k-1}h_k + c_{n-k}h_{k-1} + \ldots + c_{n-1}h_0 &= 0 \end{aligned}$$

Therefore, any codeword $c_0c_1 \ldots c_{n-1} \in C$ is orthogonal to the word $h_kh_{k-1} \ldots h_000 \ldots 0$ and to its cyclic shifts.

Rows of the matrix H are therefore in $C^\perp$. Moreover, since $h_k = 1$, these row vectors are linearly independent. Their number is $n - k = dim\ (C^\perp)$. Hence H is a generator matrix for $C^\perp$, i.e. a parity-check matrix for C.

In order to show that $C^\perp$ is a cyclic code generated by the polynomial

$$\overline{h}(x) = h_k + h_{k-1}x + \ldots + h_0x^k$$

it is sufficient to show that $\overline{h}(x)$ is a factor of $x^n - 1$.

Observe that $\overline{h}(x) = x^kh(x^{-1})$and since $h(x^{-1})g(x^{-1}) = (x^{-1})^n - 1$

we have that $x^kh(x^{-1})x^{n-k}g(x^{-1}) = x^n(x^{-n} - 1) = 1 - x^n$

and therefore $\overline{h}(x)$ is indeed a factor of $x^n - 1$.

ENCODING with CYCLIC CODES I

Encoding using a cyclic code can be done by a multiplication of two polynomials - a message (codeword) polynomial and the generating polynomial for the code.

Let C be a cyclic $[n, k]$-code over a Galoi field with the generator polynomial

$$g(x) = g_0 + g_1 x + \ldots + g_{r-1} x^{r-1} \text{ of degree } r = n - k.$$

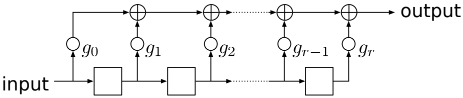
If a message vector m is represented by a polynomial $m(x)$ of degree k and m is encoded, using the generator matrix G induced by $g(x)$, by

$$m \Rightarrow c = mG,$$

then the following relation between $m(x)$ and $c(x)$ holds

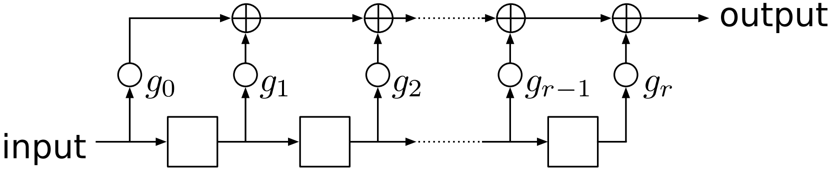
$$c(x) = m(x)g(x).$$

Such an encoding can be realized by the shift register shown in Figure below, where input is the k-bit message to be encoded followed by $n - k$ 0' and the output will be the encoded message.



Shift-register encodings of cyclic codes. Small circles represent multiplication by the corresponding constant, $\bigoplus$ nodes represent modular addition, squares are shift elements

EXAMPLE



Shift-register encodings of cyclic codes. Small circles represent multiplication by the corresponding constant, $\bigoplus$ nodes represent modular addition, squares are delay elements

HAMMING CODES as CYCLIC CODES I

Definition (Again!) Let r be a positive integer and let H be an $r \times (2^r - 1)$ matrix whose columns are all distinct non-zero vectors of $GF(r)$. Then the code having H as its parity-check matrix is called binary **Hamming code** denoted by $Ham\ (r, 2)$.

It can be shown that:

Theorem The binary Hamming code $Ham\ (r, 2)$ is equivalent to a cyclic code.

Definition If $p(x)$ is an irreducible polynomial of degree r such that x is a primitive element of the field $F[x]/p(x)$, then $p(x)$ is called a primitive polynomial.

Theorem If $p(x)$ is a primitive polynomial over $GF(2)$ of degree r , then the cyclic code $\langle p(x) \rangle$ is the code $Ham\ (r, 2)$.

HAMMING CODES as CYCLIC CODES II

Example Polynomial x^3+x+1 is irreducible over $GF(2)$ and x is primitive element of the field $F_2[x]/(x^3+x+1)$.

$$F_2[x]/(x^3+x+1) =$$

$$\{0, 1, x, x^2, x^3 = x+1, x^4 = x^2+x, x^5 = x^2+x+1, x^6 = x^2+1\}$$

The parity-check matrix for a cyclic version of *Ham* $(3, 2)$

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

PROOF of THEOREM

The binary Hamming code *Ham* $(r, 2)$ is equivalent to a cyclic code.
It is known from algebra that if $p(x)$ is an irreducible polynomial of degree r, then the ring $F_2[x]/p(x)$ is a field of order 2^r.
In addition, every finite field has a primitive element. Therefore, there exists an element α of $F_2[x]/p(x)$ such that

$$F_2[x]/p(x) = \{0, 1, \alpha, \alpha^2, \ldots, \alpha^{2r-2}\}.$$

Let us identify an element $a_0 + a_1 + \ldots a_{r-1}x^{r-1}$ of $F_2[x]/p(x)$ with the column vector

$$(a_0, a_1, \ldots, a_{r-1})^\top$$

and consider the binary $r \times (2^r - 1)$ matrix

$$H = [1\ \alpha\ \alpha^2 \ldots \alpha^{2^r-2}].$$

Let now C be the binary linear code having H as a parity check matrix.
Since the columns of H are all distinct non-zero vectors of $V(r, 2)$, $C = Ham\ (r, 2)$.
Putting $n = 2^r - 1$ we get

$$\begin{aligned} C &= \{f_0 f_1 \ldots f_{n-1} \in V(n,2) | f_0 + f_1\alpha + \ldots + f_{n-1}\alpha^{n-1} = 0\} \qquad (1) \\ &= \{f(x) \in R_n | f(\alpha) = 0 \text{ in } F_2[x]/p(x)\} \qquad (2) \end{aligned}$$

If $f(x) \in C$ and $r(x) \in R_n$, then $r(x)f(x) \in C$ because

$$r(\alpha)f(\alpha) = r(\alpha) \bullet 0 = 0$$

and therefore, by one of the previous theorems, this version of *Ham* $(r, 2)$ is cyclic.

EXAMPLES of CYCLIC CODES

GOLAY CODES - DESCRIPTION

Golay codes G_{24} and G_{23} were used by Voyager I and Voyager II to transmit color pictures of Jupiter and Saturn. Generation matrix for G_{24} has the form

$$G = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

G_{24} is $(24, 12, 8)$-code and the weights of all codewords are multiples of 4. G_{23} is obtained from G_{24} by deleting last symbols of each codeword of G_{24}. G_{23} is $(23, 12, 7)$-code.

GOLAY CODE II

Golay code G_{23} is a $(23, 12, 7)$-code and can be defined also as the cyclic code generated by the codeword

11000111010100000000000

and all its cyclic shifts. This code can be constructed via factorization of $x^{23} - 1$.
In his search for perfect codes Golay observed that

$$\sum_{j=0}^{3} \binom{23}{j} = 2^{23-12} = 2^{11}$$

Observe that an $(n, M, 2t + 1)$-code is perfect if

$$M \sum_{i=0}^{t} \binom{n}{i} (q-1)^{i} = q^{n}.$$

Golay code G_{24} was used in NASA Deep Space Missions - in sonds Voyager 1 and Voyager 2. It was also used in the US-government standards for automatic link establishment in High Frequency radio systems.

Golay codes are named to honour Marcel J. E. Golay - from 1949.

POLYNOMIAL CODES

Polynomial code generated by a (generator) polynomial $g(x)$ of degree $m < n$ over a GF(q) is the code whose codewords are represented exactly by those polynomials of degree less than n that are divisible by $g(x)$.

Example Binary polynomial code with $n = 5$ and $m = 2$ generated by the polynomial $g(x) = x^2 + x + 1$ has codewords

$$a(x)g(x)$$

where

$$a(x) \in \{0, 1, x, x + 1, x^2, x^2 + 1, x^2 + x, x^2 + x + 1\}$$

what results in the code with codewords

$$00000, 00111, 01110, 01001,$$

$$11100, 11011, 10010, 10101.$$

BCH CODES and REED-SOLOMON CODES

To the most important cyclic codes for applications belong BCH codes and Reed-Solomon codes.

Definition A polynomial p is said to be minimal for a complex number x in $GF(q)$ if $p(x) = 0$ and p is irreducible over $GF(q0$.

Definition A cyclic code of codewords of length n over $GF(p^r)$, p is a prime, is called BCH code[1] of distance d if its generator $g(x)$ is the least common multiple of the minimal polynomials for

$$\omega^l, \omega^{l+1}, \ldots, \omega^{l+d-2}$$

for some l, where

ω is the primitive n-th root of unity.

If $n = q^m - 1$ for some m, then the BCH code is called primitive.

Definition A Reed-Solomon code is a primitive BCH code with $n = q - 1$.

Properties:

- Reed-Solomon codes are self-dual.

[1] BHC stands for Bose and Ray-Chaudhuri and Hocquenghem who discovered these codes.

BCH CODES

Let q be a prime, m and integer. Consider GF(q^m) and $n = q^m - 1$.

Let ω_n be the primitive nth root of unity in GF(q^m).

For all $i < d$ let $m_i(x)$ be the minimal polynomial of ω_n^i with coefficients in GF(q). BCH codes are a special case of polynomial codes. The generator polynomial of a simplified BCH code of the minimal distance d is defined as the least common multiple of

$$g(x) = lcm(m_1(x), m_2(x), \dots, m_{d-1}(x)).$$

For BCH codes there exists nice variations of syndrome decoding. They were invented in 1959 by Hocquenghem and, independently, in 1960 by Bose and Ray-Chaudhuri.

REED-SOLOMON CODES - basic idea

A message of k symbols is encoded by viewing these symbols as coefficients of a polynomial of degree $k-1$ over a finite field of order N, evaluating this polynomial at more than k distinct points and sending the outcomes to the receiver.

Having more than k points of the polynomial allows to determine exactly, through the Lagrangian interpolation, the original polynomial (message).

Variations of Reed-Solomon codes are obtained by specifying ways distinct points are generated and error-correction is performed.

Reed-Solomon codes found many important applications from deep-space travel to consumer electronics.

They are very useful especially in those applications where one can expect that errors occur in bursts - such as ones caused by solar energy.

REED-SOLOMON CODES - I

Reed-Solomon codes RSC(k,q), for $k \leq q$. are codes generator matrix of which has rows labeled by polynomials X^i, $0 \leq i \leq k-1$, columns are labelled by elements $0, 1, \ldots, q-1$ and the element in a row labeled by a polynomial p and in a column labeled by an element u is $p(u)$.

Each RSC(k,q) code is $[q, k, q-k+1]$ code **Example** Generator matrix for RSC$(3,5)$

code is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 4 & 1 \end{pmatrix}$$

An interesting property of Reed-Solomon codes:

$$\text{RSC}(k,q)^{\perp} = \text{RSC}(q-k,q).$$

Reed-Solomon codes were used in digital television, satellite communication, wireless communication, barcodes, compact discs, DVD,...

REED-SOLOMON CODES - HISTORY and APPLICATIONS

- Reed-Solomon (RS) codes are non-binary cyclic codes.
- They were invented by Irving S. Reed and Gustave Solomon in 1960.
- Efficient decoding algorithm for them was invented by Elwyn Berlekamp and James Massey in 1969.
- Using Reed-Solomon codes one can show that it is sufficient to inject $2e$ additional symbols into a message in order to be able to correct e errors.
- Reed-Solomon codes can be decoded efficiently using so-called list decoding method (described next).
- In 1977 RS codes have been implemented in Voyager space program
- The first commercial application of RS codes in mass-consumer products was in 1982.

UNIQUE versus LIST DEODING

In the **unique decoding** model for error-correction, considered so far, the task is for a received (corrupted) message w_c to find the closest codeword w to w_c.

This error-correction task/model is not sufficiently good in case when the number of error is potentially large.

In the **list decoding** model the task is for a received (corrupted) message w_c and a given ϵ to output (list of) all codewords with the distance at most ε from w_c.

List decoding is considered to be successful in case the outputted list contains the codeword that was sent.

It has turned out that for a variety of important codes, say for Reed-Solomon codes, there are efficient algorithms for list decoding that allow to correct a large variety of errors.

EFFICIENCY of LIST DEODING

With list decoding the error-correction performance doubles.

It has been shown, non-constructively, that codes of the rate R exist that can be list decoded up to a fraction of errors approaching $1 - R$.

The quantity $1 - R$ is referred to as the **list decoding capacity**.

For Reed-Solomon codes there is list decoding up to $1 - \sqrt{2R}$ errors.

CHANNEL (STREAM) CODING

Channel coding is concerned with encoding efficiently streams of data and sending them at the highest possible rate over a given communication channel and then obtaining the original data reliably, at the receiver side, by decoding the received data efficiently.

Shannon's channel coding theorem says that over many common channels there exist data coding schemes that are able to transmit data reliably at all rates smaller than a certain threshold, called nowadays the **Shannon channel capacity of a given channel**.

Moreover, the probability of a decoding error can be made to decrease exponentially as the block length N of the coding scheme goes to infinity.

However, the complexity of a "naive" optimum decoding scheme increases exponentially with N - therefore such an optimum decoder rapidly becomes infeasible. A breakthrough

came when D. Forney, in his PhD thesis in 1972, showed that concatenated codes could be used to achieve exponentially decreasing error probabilities at all data rates less than the capacity, with decoding complexity increasing only polynomially with the code block length.

CHANNEL (STREAMS) CODING I.

The task of channel coding is to encode streams of data in such a way that if they are sent over a noisy channel errors can be detected and/or corrected by the receiver.

In case no receiver-to-sender communication is allowed we speak about **forward error correction**.

An important parameter of a channel code is **code rate**

$$r = \frac{k}{n}$$

in case k bits are encoded by n bits.

The code rate expressed the amount of redundancy in the code - the lower is the rate, the more redundant is the code.

CHANNEL (STREAM) CODING II

Design of a channel code is always a tradeoff between **energy efficiency** and **bandwidth efficiency**.

Codes with lower code rate can usually correct more errors. Consequently, the communication system can operate

- with a lower transmit power;
- transmit over longer distances;
- tolerate more interference;
- use smaller antennas;
- transmit at a higher data rate.

These properties make codes with lower code rate energy efficient.

On the other hand such codes require larger bandwidth and decoding is usually of higher complexity.

The selection of the code rate involves a tradeoff between energy efficiency and bandwidth efficiency.

Central problem of channel encoding: encoding is usually easy, but decoding is usually hard.

CONVOLUTION CODES

Our first example of channel codes are convolution codes.

Convolution codes have simple encoding and decoding, are quite a simple generalization of linear codes and have encodings as cyclic codes.

An (n, k) convolution code (CC) is defined by an $k \times n$ generator matrix, entries of which are polynomials over F_2.

For example,

$$G_1 = [x^2 + 1, x^2 + x + 1]$$

is the generator matrix for a $(2, 1)$ convolution code CC_1 and

$$G_2 = \begin{pmatrix} 1 + x & 0 & x + 1 \\ 0 & 1 & x \end{pmatrix}$$

is the generator matrix for a $(3, 2)$ convolution code CC_2

ENCODING of FINITE POLYNOMIALS

An (n,k) convolution code with a k x n generator matrix G can be used to encode a k-tuple of plain-polynomials (polynomial input information)

$$I = (I_0(x), I_1(x), \ldots, I_{k-1}(x))$$

to get an n-tuple of crypto-polynomials

$$C = (C_0(x), C_1(x), \ldots, C_{n-1}(x))$$

As follows

$$C = I \cdot G$$

EXAMPLES

EXAMPLE 1

$$\begin{aligned}(x^3+x+1)\cdot G_1 &= (x^3+x+1)\cdot(x^2+1, x^2+x+1)\\ &= (x^5+x^2+x+1, x^5+x^4+1)\end{aligned}$$

EXAMPLE 2

$$(x^2+x, x^3+1)\cdot G_2 = (x^2+x, x^3+1)\cdot\begin{pmatrix}1+x & 0 & x+1\\ 0 & 1 & x\end{pmatrix}$$

ENCODING of INFINITE INPUT STREAMS

The way infinite streams are encoded using convolution codes will be Illustrated on the code CC_1.

An input stream $I = (I_0, I_1, I_2, \dots)$ is mapped into the output stream $C = (C_{00}, C_{10}, C_{01}, C_{11} \dots)$ defined by

$$C_0(x) = C_{00} + C_{01}x + \dots = (x^2 + 1)I(x)$$

and

$$C_1(x) = C_{10} + C_{11}x + \dots = (x^2 + x + 1)I(x).$$

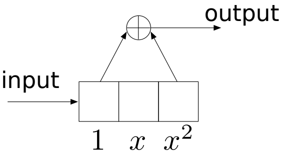
The first multiplication can be done by the first shift register from the next figure; second multiplication can be performed by the second shift register on the next slide and it holds

$$C_{0i} = I_i + I_{i+2}, \quad C_{1i} = I_i + I_{i-1} + I_{i-2}.$$

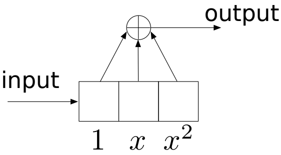
That is the output streams C_0 and C_1 are obtained by convolving the input stream with polynomials of G_1.

ENCODING

The first shift register



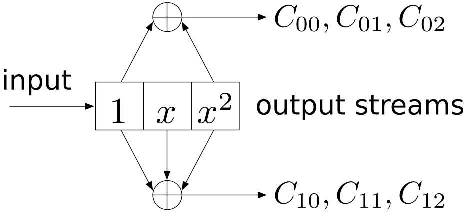
will multiply the input stream by $x^2 + 1$ and the second shift register



will multiply the input stream by $x^2 + x + 1$.

ENCODING and DECODING

The following shift-register will therefore be an encoder for the code CC_1



For decoding of convolution codes so called

Viterbi algorithm

Is used.

APPENDIX I

APPENDIX I

SHANNON CHANNEL CAPACITY

For every combination of bandwidth (W), channel type , signal power (S) and received noise power (N), there is a theoretical upper bound, called **channel capacity** or **Shannon capacity**, on the data transmission rate R for which error-free data transmission is possible.

For so-called Additive White Gaussian Noise (AWGN) channels, that well capture deep space channels, this limit is (so-called Shannon-Hartley theorem):

$$R < W \log\left(1 + \frac{S}{N}\right) \quad \{\text{bits per second}\}$$

Shannon capacity sets a limit to the energy efficiency of the code.

Till 1993 channel code designers were unable to develop codes with performance close to Shannon capacity limit, that is Shannon capacity approaching codes, and practical codes required about twice as much energy as theoretical minimum predicted.

Therefore there was a big need for better codes with performance (arbitrarily) close to Shannon capacity limits.

Concatenated codes and Turbo codes have such a Shannon capacity approaching property.

CONCATENATED CODES

Let $C_{in} : A^k \longrightarrow A^n$ be an $[n, k, d]$ code over alphabet A.

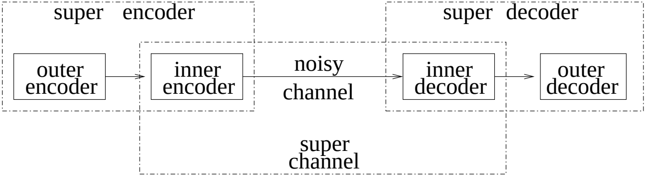
Let $C_{out} : B^K \longrightarrow B^N$ be an $[N, K, D]$ code over alphabet B with $|B| = |A|^k$ symbols.

Concatenation of C_{out} (as outer code) with C_{in} (as inner code), denoted $C_{out} \circ C_{in}$ is the $[nN, kK, dD]$ code

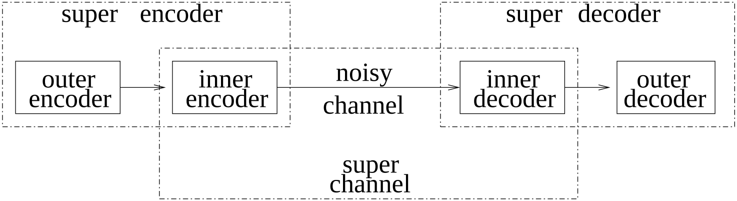
$$C_{out} \circ C_{in} : A^{kK} \longrightarrow A^{nN}$$

that maps an input message $m = (m_1, m_2, \ldots, m_K)$ to a codeword $(C_{in}(m_1^{'}), C_{in}(m_2^{'}), \ldots, C_{in}(m_N^{'}))$, where

$$(m_1^{'}, m_2^{'}, \ldots, m_N^{'}) = C_{out}(m_1, m_2, \ldots, m_K)$$



CONCATENATED CODES



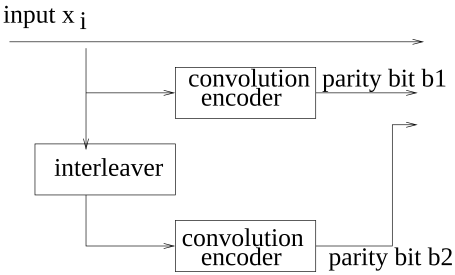
Of the key importance is the fact that if C_{in} is decoded using the *maximum-likelihood principle* (thus showing an exponentially decreasing error probability with increasing length) and C_{out} is a code with length $N = 2^n r$ that can be decoded in polynomial time in N, then the concatenated code can be decoded in polynomial time with respect to $n2^{nr}$ and has exponentially decreasing error probability even if C_{in} has exponential decoding complexity.

APPLICATIONS

- Concatenated codes started to be used for deep space communication starting with Voyager program in 1977 and stayed so until the invention of Turbo codes and LDPC codes.
- Concatenated codes are used also on Compact Disc.
- The best concatenated codes for many applications were based on outer Reed-Solomon codes and inner Viterbi-decoded short constant length convolution codes.

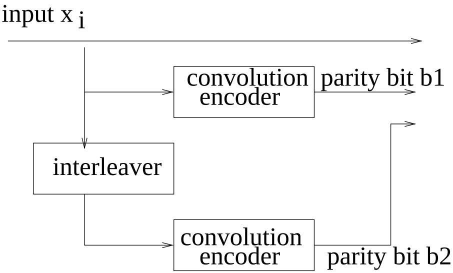
TURBO CODES

Turbo codes were introduced by Berrou, Glavieux and Thitimajshima in 1993.
A Turbo code is formed from the parallel composition of two (convolution) codes separated by an interleaver (that permutes blocks of data in a fixed (pseudo)-random way).
A Turbo encoder is formed from the parallel composition of two (convolution) encoders separated by an interleaver.

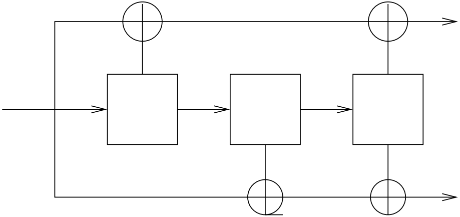


EXAMPLES of TURBO and CONVOLUTION ENCODERS

A Turbo encoder



and a convolution encoder



DECODING and PERFORMANCE of TURBO CODES

- A soft-in-soft-out decoding is used - the decoder gets from the analog/digital demodulator a soft value of each bit - probability that it is 1 and produces only a soft-value for each bit.
- The overall decoder uses decoders for outputs of two encoders that also provide only soft values for bits and by exchanging information produced by two decoders and from the original input bit, the main decoder tries to increase, by an iterative process, likelihood for values of decoded bits and to produce finally hard outcome - a bit 1 or 0.
- Turbo codes performance can be very close to theoretical Shannon limit.
- This was, for example the case for UMTS (the third Generation Universal Mobile Telecommunication System) Turbo code having a less than 1.2-fold overhead. in this case the interleaver worked with block of 40 bits.
- Turbo codes were incorporated into standards used by NASA for deep space communications, digital video broadcasting and both third generation cellular standards.
- Literature: M.C. Valenti and J.Sun: Turbo codes - tutorial, Handbook of RF and Wireless Technologies, 2004 - reachable by Google.

REACHING SHANNON LIMIT

- Though Shannon developed his capacity bound already in 1940, till recently code designers were unable to come with codes with performance close to theoretical limit.
- In 1990 the gap between theoretical bound and practical implementations was still at best about 3dB

  A decibel is a relative measure. If E is the actual energy and E_{ref} is the theoretical lower bound, then the relative energy increase in decibels is

$$10 \log_{10} \frac{E}{E_{ref}}$$

  Since $\log_{10} 2 = 0.3$ a two-fold relative energy increase equals $3dB$.
- For code rate $\frac{1}{2}$ the relative increase in energy consumption is about 4.8 dB for convolution codes and 0.98 for Turbo codes.

APPENDIX II

TURBO CODES - SUMMARY

- Turbo codes encoding devices are usually built from two (usually identical) recursive systematic convolution encoders , linked together by nonuniform interleaver (permutation) devices.
- Soft decoding is an iterative process in which each component decoder takes advantage of the work of other at the previous step, with the aid of the original concept of intrinsic information.
- For sufficiently large size of interleavers , the correcting performance of turbo codes, as shown by simulations, appears to be close to the theoretical shannon limit.
- Permutations performed by interleaver can often by specified by simple polynoms that make one-to-one mapping of some sets $\{0, 1, \dots, q-1\}$.

WHY ARE TURBO CODES SO GOOD?

- Turbo codes are linear codes.
- A "good" linear code is one that has mostly high-weight codewords.
- High-weight codewords are desirable because they are more distinct and the decoder can more easily distinguish among them.
- A big advantage of Turbo encoders is that they reduce the number of low-weight codewords because their output is the sum of the weights of the input and two parity output bits.
- A turbo code can be seen as a refinment of concatenated codes plus an iterative algorithm for decoding.

LIST DECODING - MATHEMATICAL FORMULATION

Let C be a q-nary linear $[n, k, d]$ error correcting code.

For a given q-nary input word w of length n and a given error bound e output a list of codewords of C whose Hamming distance from w is at most e

We are, naturally, interested only in polynomial, in n, algorithms able to do that.

(p, L)-list decodability Let C be a q-nary code of codewords of length n; $0 \leq p \leq 1$ and $L > 1$ an integer. If for every q-nary word w of length n the number of codewords of C

withing hamming distance pn from w is at most L, then the code C is said to be (p, L)-list-decodable.

Theorem let $q \geq 2$, $0 \leq p \leq 1 - 1/q$ and $\varepsilon \geq 0$ then for large enough block length n if the code rate $R \leq 1 - H_q(p) - \varepsilon$, then there exists a $(p, O(1/\varepsilon)$-list decodable code. $[H_q(p)$ is q-ary entropy function.]

LIST DECODING POTENTIAL

- The concept of list decoding was proposed by Peter Elias in 1950s.
- It has been shown, nonconstructively, that codes of rate R exist that can be list decoded up to a fraction of errors approaching $(1 - R)$.
- The quantity $(1 - R)$ is usually called list decoding capacity
- In 2006 Guruswami and Atri Rudra gave explicit codes that achieve list decoding capacity.
- Their codes are called folded Reed-Solomon codes and they are actually nothing but plain Reed-Solomon codes but viewed as codes over a larger alphabet by careful bundling of codeword symbols.
- List decoding can be seen as frmalizing the notion of error-correction when the number of errors is potentially very large. In such a case the received word can actually be closer to other codewords than the transmitted one.
- Algorithms developed for list decoding of several code families found interesting applications in computational complexity theory and in cryptography (for example in construction of hard-core predicates, extractors and pseudo-random generators).

APPLICATIONS

- Reed-Solomon codes have been widely used in mass storage systems to correct the burst errors caused by media deffects.
- Special types of Reed-Solomon codes have been used to overcome unreliable nature of data transmission over erasure channels.
- Several bar-code systems use Reed-Solomon codesto allow correct reading even if a portion of the bar code is damaged.
- Reed-Solomon codes were used to encode pictures sent vy the Voyager space probe.
- Modern versions of concatenated Reed-Solomon/Viterbi decoder convolution coding were and are used on the Mars Pathfinder, Galileo, Mars exploration Rover and Cassini missions, where they performed within about 1-1.5dB of the ultimate limit imposed by the shannon capacity.

APPENDIX III

GROUPS

A **group** G is a set of elements and an operation, call it *, with the following properties:

- G is closed under *; that is if $a, b \in G$, so is $a * b$.
- The operation * is associative $(a * (b * c) = (a * b) * c$, for any $a, b, c \in G$.
- G has an identity e element so that $e * a = a * e = a$ for any $a \in G$.
- Every element $a \in G$ has an inverse $a^{-1} \in G$, so that $a * a^{-1} = a^{-1} * a = e$.

A group G is called **Abelian group** if the operation $*$ is commutative $(a * b = b * a$ for any $a, b \in G)$.

Example Which of the following sets is an (Abelian) group:

- The set of real numbers with $*$ being: (a) addition; (b) multiplication.
- The set of matrices of degree n and an operations (a) addition; (b) multiplication.
- What happens if we consider only matrices with determinants not equal zero?

RINGS and FIELDS

A **ring** R is a set with two operations $+$ (addition) and $\cdot$ (multiplication) , with the following properties:

- R is closed under $+$ and $\cdot$.
- R is an Abelian group under $+$ (with the unity element for addition called **zero**).
- The associative law for multiplication holds.
- R has an identity element 1 for multiplication
- The distributive law holds $(a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in R$.

A ring is called **commutative ring** if multiplication is commutative

A **field** F is a set with two operations $+$ (addition) and $\cdot$ (multiplication) , with the following properties:

- F is a commutative ring.
- Non-zero elements of F form an Abelian group under multiplication.

A non-zero element g is a **primitive element** of a field F if all non-zero elements of F are powers of g.

FINITE FIELDS

Finite field are very well understood.

Theorem If p is a prime, then the integers mod p, $GF(p)$, constitute a field. Every finite field F contains a subfield that is $GF(p)$, up to relaabeling, for some prime p and $p \cdot \alpha = 0$ for every $\alpha \in F$.

If a field F contains the prime field $GF(p)$, then p is called the **characteristic** of F.

Theorem (1) Every finite field F has p^m elements for some prime p and some m. (2) For any prime p and any integer m there is a unique (up to isomorphism) field of p^m elements $GF(p^m)$. (3) If $f(x)$ is an irreducible polynomial of degree m in $F_p[x]$, then the set of polynomials in $F_p[x]$ with additions and multiplications modulo $f(x)$ is a field with p^m elements.

FINITE FIELDS $GF(p^k), k > 1$

There are two important ways GF(4), the Galois field of four elements, is realized.
1. It is easy to verify that such a field is the set

$$\mathrm{GF}(4) = \{0, 1, \omega, \omega^2\}$$

with operations $+$ and $\cdot$ satisfying laws

- $0 + x = x$ for all x;
- $x + x = 0$ for all x;
- $1 \cdot x = x$ for all x;
- $\omega + 1 = \omega^2$

2. Let $\mathbf{Z}_2[x]$ be the set of polynomials whose coefficients are integers mod 2. GF(4) is also $\mathbf{Z}_2[x] \pmod{x^2 + x + 1}$ therefore the set of polynomials

$$0, 1, x, x + 1$$

where addition and multiplication are $\pmod{x^2 + x + 1}$.

3. Let p be a prime and $\mathbf{Z}_p[x]$ be the set of polynomials with coefficients mod p. If $p(x)$ is a irreducible polynomial $\bmod p$ of degree n, then $\mathbf{Z}_p[x] \pmod{p(x)}$ is a $\mathrm{GF}(p^n)$ with p^n elements.