Bisimilarity is Decidable in the Union of Normed BPA and Normed BPP Processes

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Abstract

We compare the classes of behaviours (transition systems) which can be generated by normed BPA_τ and normed BPP_τ processes. We exactly classify the intersection of these two classes, i.e., the class of transition systems which can be equivalently (up to bisimilarity) described by the syntax of normed BPA_τ and normed BPP_τ processes. We provide such a characterization for classes of normed BPA and normed BPP processes as well.

Next we show that it is decidable in polynomial time whether for a given normed BPA_τ (or BPP_τ) process Δ there is some normed BPP_τ (or BPA_τ) process Δ' such that Δ is bisimilar to Δ'. Moreover, if the answer is positive then the process Δ' can be effectively constructed. Simplified versions of the algorithms mentioned above for normed BPA and normed BPP are given too.

As an immediate (but important) consequence we also obtain decidability of bisimilarity in the union of normed BPA_τ and normed BPP_τ processes.

1 Introduction

We study a relationship between the classes of transition systems which are generated by normed BPA_τ [BK88] and normed BPP_τ [Chr93] processes. We also examine such a relationship between their respective subclasses, namely normed BPA and normed BPP processes. BPA processes form type 2 class of prefix processes in Chomsky hierarchy for processes as given in [Sti96] and [Mol96], whereas BPP form type 2 class of commutative processes in this hierarchy.

BPA processes can be seen as simple sequential programs (they are equipped with a binary sequential operator). This class of processes has been intensively studied by many researchers. Baeten, Bergstra and Klop proved in [BBK87] that bisimilarity is decidable for normed BPA processes. Much simpler proofs of this were later given in [Cau88,HS91,Gro91]. In [HS91] Hüttel and Stirling

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used a tableau decision method and gave also sound and complete equational theory. Hirshfeld, Jerrum and Moller demonstrated in [HJM94a] that the problem is decidable in polynomial time. The decidability result was later extended to the whole class of BPA processes by Christensen, Hüttel and Stirling in [CHS92].

If we replace the binary sequential operator with the parallel operator, we obtain BPP processes. They can thus be seen as simple parallel programs. Christensen, Hirshfeld and Moller proved in [CHM93] that bisimilarity is decidable for BPP processes. A polynomial decision algorithm for normed BPP processes was presented in [HJM94b] by Hirshfeld, Jerrum and Moller.

If we allow a parallel operator not to specify just merge but also an internal communication between two BPP processes resulting in a special action τ, we obtain the class of BPP_τ processes [Chr93]. In order to compare this class with its sequential counterpart we employ the class of BPA_τ processes [BK88]. Decidability and complexity results just mentioned hold for these classes as well.

An interesting problem is, what is the exact relationship between BPA_τ and BPP_τ processes (and between their subclasses BPA and BPP), i.e., what is the relationship between sequencing and parallelism (possibly allowing simple communication). We answer these questions for normed subclasses of processes just mentioned. In the sequel we denote these subclasses by nBPA_τ, nBPP_τ, nBPA and nBPP, respectively.

Our paper is organized as follows. First we recall some basic definitions and properties of BPA_τ, BPA, BPP_τ, BPP and regular processes which are relevant to the subject of our paper. In Section 3 we give an exact characterization of those transition systems which can be equivalently (up to bisimilarity) described by the syntax of nBPA_τ and nBPP_τ processes. Next we show that if we restrict ourselves to nBPA and nBPP processes we obtain much simpler (and hopefully nice) characterization of those behaviours which are common to these subclasses. In Section 4 we demonstrate it is decidable whether for a given nBPA_τ, nBPP_τ, nBPA or nBPP process Δ there is some (unspecified) nBPP_τ, nBPA_τ, nBPP or nBPA process Δ' such that $\Delta \sim \Delta'$, respectively. These algorithms are polynomial. We also show that if the answer to the previous question is positive, then the process Δ' can be effectively constructed. Hence, as an important consequence we also obtain decidability of bisimulation equivalence in the union of nBPA_τ and nBPP_τ processes. We conclude with remarks to related work and future research.

In many constructions of our paper we use the fact that regularity is decidable for nBPA_τ and nBPP_τ processes in polynomial time (a process is regular if it is bisimilar to a process with finitely many states). Regularity of BPA processes was examined for the first time by Mauw and Mulder in [MM94], but their notion of regularity is different from the usual one. Kučera showed in [Kuč95] that the result of Mauw and Mulder can be used to decide regularity of nBPA processes (nBPA_τ case is an easy consequence) and that regular-

$$\frac{}{a \xrightarrow{a} \epsilon} \qquad \frac{E \xrightarrow{a} E'}{E.F \xrightarrow{a} E'.F} \qquad \frac{E \xrightarrow{a} E'}{E+F \xrightarrow{a} E'} \qquad \frac{F \xrightarrow{a} F'}{E+F \xrightarrow{a} F'}$$

$$\frac{E \xrightarrow{a} E'}{E\|F \xrightarrow{a} E'\|F} \qquad \frac{F \xrightarrow{a} F'}{E\|F \xrightarrow{a} E\|F'} \qquad \frac{E \xrightarrow{a} E'}{E|F \xrightarrow{a} E'|F} \qquad \frac{F \xrightarrow{a} F'}{E|F \xrightarrow{a} E|F'}$$

$$\frac{E \xrightarrow{a} E' \quad F \xrightarrow{\overline{a}} F'}{E|F \xrightarrow{\tau} E'|F'} \ (a \neq \tau) \qquad \frac{E \xrightarrow{a} E'}{X \xrightarrow{a} E'} \ (X \stackrel{def}{=} E \in \Delta)$$

Fig. 1. SOS rules

ity of nBPP and nBPP$_\tau$ processes is also decidable. These algorithms are polynomial.

2 Basic definitions, preliminary knowledge

2.1 BPA$_\tau$, BPA, BPP$_\tau$ and BPP processes

Let $\Lambda = \{a, b, c, \ldots\}$ be a countably infinite set of *atomic actions* such that for each $a \in \Lambda$ there is a corresponding *dual* action $\overline{a}$ with the convention that $\overline{\overline{a}} = a$. Let $Act = \Lambda \cup \{\tau\}$ where $\tau \notin \Lambda$ is a special (silent) action. Let $Var = \{X, Y, Z, \ldots\}$ be a countably infinite set of *variables* such that $Var \cap Act = \emptyset$. The classes of recursive BPA$_\tau$, BPA, BPP$_\tau$ and BPP expressions are defined by the following abstract syntax equations:

$$\begin{array}{ll}
E_{BPA_\tau} & ::= \epsilon \ | \ a \ | \ \tau \ | \ X \ | \ E_{BPA_\tau}.E_{BPA_\tau} \ | \ E_{BPA_\tau} + E_{BPA_\tau} \\
E_{BPA} & ::= \epsilon \ | \ a \ | \ X \ | \ E_{BPA}.E_{BPA} \ | \ E_{BPA} + E_{BPA} \\
E_{BPP_\tau} & ::= \epsilon \ | \ a \ | \ \tau \ | \ X \ | \ aE_{BPP_\tau} \ | \ E_{BPP_\tau}|E_{BPP_\tau} \ | \ E_{BPP_\tau} + E_{BPP_\tau} \\
E_{BPP} & ::= \epsilon \ | \ a \ | \ X \ | \ aE_{BPP} \ | \ E_{BPP}\|E_{BPP} \ | \ E_{BPP} + E_{BPP}
\end{array}$$

Here a ranges over Λ and X ranges over Var. The symbol ϵ denotes the empty expression.

As usual, we restrict our attention to guarded expressions. A BPA$_\tau$, BPA, BPP$_\tau$ or BPP expression E is *guarded* if every variable occurrence in E is within the scope of an atomic action.

A *guarded* BPA$_\tau$, BPA, BPP$_\tau$ or BPP process is defined by a finite family Δ of recursive process equations

$$\Delta = \{X_i \stackrel{def}{=} E_i \quad | \quad 1 \leq i \leq n\}$$

where X_i are distinct elements of Var and E_i are guarded BPA$_\tau$, BPA, BPP$_\tau$ or BPP expressions, respectively, containing variables from $\{X_1, \ldots, X_n\}$. The set of variables which appear in Δ is denoted by $Var(\Delta)$.

The variable X_1 plays a special role (X_1 is sometimes called *the leading variable*—it is a root of a labelled transition system, defined by the process Δ and SOS rules of Figure 1.

Presented rules should be considered modulo *structural congruence* which is the smallest congruence relation over process expressions such that the following laws hold:

- associativity and ϵ as a unit for sequential composition (the '.' operator).
- associativity, commutativity and ϵ as a unit for operators of pure merge '$\|$', parallel composition '$|$' and nondeterministic choice '$+$'.

Nodes of the transition system generated by Δ are BPA$_\tau$, BPA, BPP$_\tau$ or BPP expressions, which are often called *states of* Δ, or just "states" when Δ is understood from the context. We also define the relation $\stackrel{w}{\rightarrow}^*$, where $w \in Act^*$, as the reflexive and transitive closure of $\stackrel{a}{\rightarrow}$ (we often write $E \rightarrow^* F$ instead of $E \stackrel{w}{\rightarrow}^* F$ if w is irrelevant). Given two states E, F, we say that F is *reachable from* E, if $E \rightarrow^* F$. States of Δ which are reachable from X_1 are said to be *reachable*.

Remark 2.1 *Processes are often identified with their leading variables. Furthermore, if we assume a fixed process Δ, we can view any process expression E (not necessarily guarded) whose variables are defined in Δ as a process too; we simply add a new leading equation $X \stackrel{def}{=} E'$ to Δ, where X is a variable from Var such that $X \not\in Var(\Delta)$ and E' is a process expression which is obtained from E by substituting each variable in E with the right-hand side of its corresponding defining equation in Δ (E' must be guarded now). All notions originally defined for processes can be used for process expressions in this sense too.*

2.1.1 Bisimulation

The equivalence between process expressions (states) we are interested in here is *bisimilarity* [Par81], defined as follows:

Definition 2.2 (bisimilarity) *A binary relation R over process expressions is a* bisimulation *if whenever $(E, F) \in R$ then for each $a \in Act$*

- *if $E \stackrel{a}{\rightarrow} E'$, then $F \stackrel{a}{\rightarrow} F'$ for some F' such that $(E', F') \in R$*
- *if $F \stackrel{a}{\rightarrow} F'$, then $E \stackrel{a}{\rightarrow} E'$ for some E' such that $(E', F') \in R$*

Processes Δ and Δ' are bisimilar, *written $\Delta \sim \Delta'$, if their leading variables are related by some bisimulation.*

2.1.2 Normed processes

An important subclass of BPA$_\tau$, BPA, BPP$_\tau$ and BPP processes can be obtained by an extra restriction of *normedness*. A variable $X \in Var(\Delta)$ is *normed* if there is $w \in Act^*$ such that $X \stackrel{w}{\rightarrow}^* \epsilon$. In that case we define the *norm* of X, written $|X|$, to be the length of the shortest such w. In case of BPP$_\tau$ processes we also require that no τ action which appears in w is a result of a communication on dual actions in the sense of operational semantics given in Figure 1. This is necessary if we want the norm to be additive over the '$|$'

operator (τ can still occur in w—remember it can be used as an action prefix). A process Δ is *normed* if all variables of $Var(\Delta)$ are normed. The norm of Δ is then defined to be the norm of X_1.

Remark 2.3 *As normed processes are intensively studied in this paper, we emphasise some properties of the norm:*

- *Note the norm of a normed process is easy to compute by the following rules:* $|a| = 1$, $|E + F| = \min\{|E|, |F|\}$, $|E.F| = |E| + |F|$, $|\, E|F\,| = |E| + |F|$, $|\, E\|F\,| = |E| + |F|$ *and if* $X_i \stackrel{def}{=} E_i$ *and* $|E_i| = n$, *then* $|X_i| = n$.
- *Bisimilar processes must have the same norm.*

2.1.3 Greibach normal form

Any BPA_τ, BPA, BPP_τ or BPP process Δ can be effectively presented in so-called 3-Greibach normal form (see [BBK87] and [Chr93]).

Definition 2.4 (GNF for $\mathbf{BPA}_\tau$ and BPA processes) *A BPA_τ (or BPA) process Δ is said to be in* Greibach normal form (GNF) *if all its defining equations are of the form*

$$X \stackrel{def}{=} \sum_{j=1}^{n} a_j \alpha_j$$

where $n \in N$, $a_j \in Act$ (or $a_j \in \Lambda$) and $\alpha_j \in Var(\Delta)^*$. If $Length(\alpha_j) \leq 2$ for each j, $1 \leq j \leq n$, then Δ is said to be in 3-GNF. Moreover, we also require that for each $Y \in Var(\Delta)$ there is a reachable state $\alpha \in Var(\Delta)^*$ such that α begins with Y.

Before the definition of GNF for BPP_τ and BPP process we need to introduce the set $Var(\Delta)^\otimes$ of all finite multisets over $Var(\Delta)$. Each multiset of $Var(\Delta)^\otimes$ denotes a BPP_τ or BPP expression by combining its elements in parallel using the '$|$' or '$\|$' operator, respectively.

Definition 2.5 (GNF for $\mathbf{BPP}_\tau$ and BPP processes) *A BPP_τ (or BPP) process Δ is said to be in* Greibach normal form (GNF) *if all its defining equations are of the form*

$$X \stackrel{def}{=} \sum_{j=1}^{n} a_j \alpha_j$$

where $n \in N$, $a_j \in Act$ (or $a_j \in \Lambda$) and $\alpha_j \in Var(\Delta)^\otimes$. If $card(\alpha_j) \leq 2$ for each j, $1 \leq j \leq n$, then Δ is said to be in 3-GNF. Moreover, we also require that for each $Y \in Var(\Delta)$ there is a reachable state $\alpha \in Var(\Delta)^\otimes$ such that $Y \in \alpha$.

From now on we assume that all BPA_τ, BPA, BPP_τ and BPP processes we work with are presented in GNF. This justifies also the assumption that all reachable states of a BPA_τ or BPA process Δ are elements of $Var(\Delta)^*$ and all reachable states of a BPP_τ or BPP process Δ' are elements of $Var(\Delta')^\otimes$.

Remark 2.6 *In the rest of this paper we let Greek letters $\alpha, \beta, \ldots$ range over reachable states of a BPA_τ, BPA, BPP_τ or BPP process Δ in GNF. Occasionally we also use the notation α^i with the following meaning:*

$$\alpha^i = \underbrace{\alpha.\alpha \cdots \alpha}_{i} \quad \textit{if } \alpha \textit{ is a state of some } BPA_\tau \textit{ or BPA process in GNF}$$

$$\alpha^i = \underbrace{\alpha|\alpha \cdots |\alpha}_{i} \quad \textit{if } \alpha \textit{ is a state of some } BPP_\tau \textit{ process in GNF}$$

$$\alpha^i = \underbrace{\alpha\|\alpha \cdots \|\alpha}_{i} \quad \textit{if } \alpha \textit{ is a state of some BPP process in GNF}$$

2.2 Regular processes

Many proofs in this paper take advantage of the fact that regularity of nBPA_τ, nBPA, nBPP_τ and nBPP processes is decidable in polynomial time. Regularity of BPA processes was examined for the first time by Mauw and Mulder in [MM94], but their notion of regularity is different from the usual one. Kučera showed in [Kuč95] that the result of Mauw and Mulder can be used to decide regularity of nBPA (and thus also nBPA_τ) processes and that regularity of nBPP and nBPP_τ processes is decidable as well. These algorithms are polynomial. The next definition explains what is meant by the notion of regularity and introduces standard normal form for regular processes.

Definition 2.7 *A process Δ is* regular *if there is a process Δ' with finitely many states such that $\Delta \sim \Delta'$. A regular process Δ is said to be in normal form if all its equations are of the form*

$$X \stackrel{def}{=} \sum_{j=1}^{n} a_j X_j$$

where $n \in N$, $a_j \in Act$ and $X_j \in Var(\Delta)$.

It is easy to see that a process is regular iff it can reach only finitely many states up to bisimilarity. In [Mil89] it is shown that regular processes can be represented in the normal form just defined. Thus a process Δ is regular iff there is a regular process Δ' in normal form such that $\Delta \sim \Delta'$. Now we present several propositions which concern regularity of nBPA_τ, nBPA, nBPP_τ and nBPP processes. Proofs can be found in [Kuč95].

Proposition 2.8 *Let Δ be a $nBPA_\tau$, nBPA, $nBPP_\tau$ or nBPP process. The problem whether Δ is regular is decidable in polynomial time. Moreover, if Δ is regular then a regular process Δ' in normal form such that $\Delta \sim \Delta'$ can be effectively constructed.*

Definition 2.9 (growing variable) *Let Δ be a $nBPA_\tau$, $nBPP_\tau$, nBPA or nBPP process. A variable $Y \in Var(\Delta)$ is* growing *if $Y \rightarrow^* Y.\alpha$, $Y \rightarrow^* Y|\alpha$, $Y \rightarrow^* Y.\alpha$ or $Y \rightarrow^* Y\|\alpha$ where $Length(\alpha) \geq 1$, respectively.*

Proposition 2.10 *A $nBPA_\tau$, nBPA, $nBPP_\tau$ or nBPP process Δ in GNF is non-regular iff $Var(\Delta)$ contains a growing variable.*

3 The characterization of $\mathbf{nBPA}_\tau \cap \mathbf{nBPP}_\tau$

In this section we give an exact characterization of those normed processes which can be equivalently defined by BPA_τ and BPP_τ syntax.

Definition 3.1 ($\mathbf{nBPA}_\tau \cap \mathbf{nBPP}_\tau$) *The semantical intersection of $nBPA_\tau$ and $nBPP_\tau$ processes is defined as follows:*

$$\begin{aligned} nBPA_\tau \cap nBPP_\tau = &\{\Delta \in nBPA_\tau, \mid \exists \Delta' \in nBPP_\tau \text{ such that } \Delta \sim \Delta'\} \cup \\ &\{\Delta \in nBPP_\tau, \mid \exists \Delta' \in nBPA_\tau \text{ such that } \Delta \sim \Delta'\} \end{aligned}$$

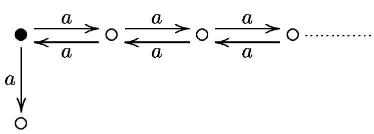
The class $\mathrm{nBPA}_\tau \cap \mathrm{nBPP}_\tau$ is clearly nonempty because each normed finite-state process belongs to $\mathrm{nBPA}_\tau \cap \mathrm{nBPP}_\tau$. But $\mathrm{nBPA}_\tau \cap \mathrm{nBPP}_\tau$ contains also processes with infinitely many states—assume the following process:

(1) $X \stackrel{def}{=} a(X|X) + a$

X is a nBPP_τ process with infinitely many states. If we replace the '$|$' operator with the '.' operator, we obtain a bisimilar nBPA_τ process:

(2) $\overline{X} \stackrel{def}{=} a(\overline{X}.\overline{X}) + a$

Clearly $X \sim \overline{X}$ because transition systems generated by those processes are even isomorphic:



Now we modify the process X slightly:

(3) $X \stackrel{def}{=} a(X|X) + a + \overline{a}$

Although the process (3) does not differ from the process (1) too much, it is not hard to prove that there is *no* nBPA_τ process bisimilar to (3).

Now we prove that each nBPP_τ processes from $\mathrm{nBPA}_\tau \cap \mathrm{nBPP}_\tau$ can be represented in a special normal form, denoted $\mathrm{INF_{BPP}}$ (**I**ntersection **N**ormal **F**orm for nBPP_τ processes). Before the definition of $\mathrm{INF_{BPP}}$ we first introduce the notion of *reduced* processes:

Definition 3.2 (reduced processes) *Let Δ be a $nBPA_\tau$ or $nBPP_\tau$ process in GNF. We say that Δ is* reduced *if its variables are pairwise non-bisimilar.*

As bisimilarity is decidable for nBPA_τ and nBPP_τ processes in polynomial time (see [HJM94a], [HJM94b]), each nBPA_τ or nBPP_τ process can be effectively transformed into a bisimilar reduced process in polynomial time.

Definition 3.3 ($\mathbf{INF_{BPP}}$) *Let Δ be a reduced $nBPP_\tau$ process in GNF.*

(i) A variable $Z \in Var(\Delta)$ is simple *if all summands in the def. equation for Z are of the form aZ^i, where $a \in Act$ and $i \in N \cup \{0\}$. Moreover, at*

least one of those summands must be of the form aZ^k where $a \in Act$ and $k \geq 2$. Finally, the def. equation for Z must not contain two summands of the form $b, \overline{b}$, where $b \in Act$.*

(ii) The process Δ is said to be in INF_{BPP} if whenever $a\alpha$ is a summand in a def. equation from Δ such that $Length(\alpha) \geq 2$, then $\alpha = Z^i$ for some simple variable Z and $i \geq 2$.

Note that if Z is a simple variable, then $|Z| = 1$ because Z could not be normed otherwise.

Example 3.4 *The following process as well as process (1) are in INF_{BPP}, while the processes (3) is not:*

$$X \stackrel{def}{=} aY + b(Z|Z) + b + \overline{b}$$
$$Y \stackrel{def}{=} cY + bX + a(Z|Z|Z)$$
$$Z \stackrel{def}{=} a(Z|Z) + \overline{a}(Z|Z|Z) + b + \overline{a}$$

Remark 3.5 *The set of all reachable states of a process Δ in INF_{BPP} looks as follows:*

$$Var(\Delta) \cup \{Z^i \mid Z \in Var(\Delta) \text{ is a simple variable and } i \in N \cup \{0\}\}$$

Proposition 3.6 *Each process Δ in INF_{BPP} belongs to $nBPA_\tau \cap nBPP_\tau$.*

Proof. We show how to construct a nBPA_τ process $\overline{\Delta}$ which is bisimilar to Δ. First we need to define the notion of a *closed* simple variable—a simple variable $Z \in Var(\Delta)$ is closed if the following condition holds: If the def. equation for Z contains two summands of the form $bZ^i, \overline{b}Z^j$, then it also contains a summand τZ^{i+j-1} (note that Z is simple, hence the case $i = j = 0$ is impossible).

The set of variables of $\overline{\Delta}$ looks as follows: for each $V \in Var(\Delta)$ we fix a fresh variable $\overline{V}$. Moreover, for each simple non-closed variable $Z \in Var(\Delta)$ we also fix a fresh variable $\overline{Z}_C$. Now we can start to transform Δ into $\overline{\Delta}$. For each equation $Y \stackrel{def}{=} \sum_{i=1}^n a_i\alpha_i$ of Δ we add the equation $\overline{Y} \stackrel{def}{=} \sum_{i=1}^n a_i\mathcal{T}(\alpha_i)$ to $\overline{\Delta}$, where $\mathcal{T}$ is defined as follows:

(i) $\mathcal{T}(V) = \overline{V}$, where $V \in Var(\Delta)$.

(ii) $\mathcal{T}(Z^i) = \overline{Z}^i$, where $i \geq 2$ and $Z \in Var(\Delta)$ is a closed simple variable.

(iii) $\mathcal{T}(Z^i) = \overline{Z}_C^{\,i-1}.\overline{Z}$, where $i \geq 2$ and $Z \in Var(\Delta)$ is a non-closed simple variable.

The defining equation for $\overline{Z}_C$, where $Z \in Var(\Delta)$ is a non-closed simple variable, is constructed using following rules:

(i) if aZ^i is a summand in the def. equation for Z, then $a\overline{Z}_C^{\,i}$ is a summand in the def. equation for $\overline{Z}_C$ in $\overline{\Delta}$.

(ii) if bZ^i, $\overline{b}Z^j$ are summands in the def. equation for Z, then $\tau\overline{Z}_C^{\,i+j-1}$ is a summand in the def. equation for $\overline{Z}_C$ in $\overline{\Delta}$.

The fact $\Delta \sim \overline{\Delta}$ is easy to check. $\square$

Example 3.7 *If we apply the transformation algorithm to the process from Example 3.4, we obtain the following bisimilar nBPA$_\tau$ process:*

$$\begin{aligned}
\overline{X} &\stackrel{def}{=} a\overline{Y} + b(\overline{Z_C}.\overline{Z}) + b + \overline{b} \\
\overline{Y} &\stackrel{def}{=} c\overline{Y} + b\overline{X} + a(\overline{Z_C}.\overline{Z_C}.\overline{Z}) \\
\overline{Z} &\stackrel{def}{=} a(\overline{Z_C}.\overline{Z}) + \overline{a}(\overline{Z_C}.\overline{Z_C}.\overline{Z}) + b + \overline{a} \\
\overline{Z_C} &\stackrel{def}{=} a(\overline{Z_C}.\overline{Z_C}) + \overline{a}(\overline{Z_C}.\overline{Z_C}.\overline{Z_C}) + b + \overline{a} + \tau(\overline{Z_C}.\overline{Z_C}.\overline{Z_C}.\overline{Z_C}) + \tau\overline{Z_C}
\end{aligned}$$

Now we prove that each nBPP$_\tau$ process from nBPA$_\tau \cap$ nBPP$_\tau$ is bisimilar to a process in INF$_{\text{BPP}}$. Several auxiliary definitions and lemmas are needed:

Definition 3.8 (Assoc set) *Let Δ be a reduced nBPP$_\tau$ process in 3-GNF. For each growing variable $Y \in Var(\Delta)$ we define the set $Assoc(Y) \subseteq Var(\Delta)$ in the following way:*

$$\begin{aligned}
Assoc(Y) = &\{P \in Var(\Delta),\ Y \rightarrow^* P\} \cup \\
&\{P \in Var(\Delta),\ P|Y \text{ is a reachable state of } \Delta\}
\end{aligned}$$

A variable $L \in Var(\Delta)$ is lonely *if $L \notin Assoc(Y)$ for any growing variable $Y \in Var(\Delta)$.*

Lemma 3.9 *Let Δ be a reduced nBPP$_\tau$ process in 3-GNF such that $\Delta \in$ nBPA$_\tau \cap$ nBPP$_\tau$. Let $Y \in Var(\Delta)$ be a growing variable. Then there is exactly one variable $Z_Y \in Var(\Delta)$ such that:*

- *Z_Y is non-regular and $|Z_Y| = 1$*
- *if $P \in Assoc(Y)$ then $P \sim Z_Y^{|P|}$ and Z_Y is reachable from P*
- *if $a\alpha$ is a summand in the defining equation for Z_Y in Δ, then $\alpha \sim Z_Y^{|\alpha|}$*

Proof. As Y is growing, $Y \rightarrow^* Y|\beta$ where $\beta \in Var(\Delta)^\otimes$, $\beta \neq \emptyset$. As Δ is normed and in GNF, there is $Z_Y \in Var(\Delta)$, $|Z_Y| = 1$ such that $\beta \rightarrow^* Z_Y$. Hence $Y \rightarrow^* Y|\beta^i \rightarrow^* Y|Z_Y^i$ for any $i \in N$ (note that Z_Y is reachable from Y). From this and the definition of *Assoc* set we can easily conclude that if $P \in Assoc(Y)$ then the state $P|Z_Y^i$ is reachable for any $i \in N$.

As $\Delta \in$ nBPA$_\tau \cap$ nBPP$_\tau$, there is a nBPA$_\tau$ process Δ' in GNF such that $\Delta \sim \Delta'$. Let $n = |P|$, $m = \max\{|A|,\ A \in Var(\Delta')\}$. The state $P|Z_Y^{n.m}$ is a reachable state of Δ and therefore there is $\gamma \in Var(\Delta')^*$ such that $P|Z_Y^{n.m} \sim \gamma$. Bisimilar states must have the same norm, hence γ is a sequence of at least $n+1$ variables — $\gamma = A_1.A_2 \ldots A_{n+1}.\delta$ where $\delta \in Var(\Delta')^*$. As $|P| = n$, there is $s \in Act^*$, $Length(s) = n$ such that $P \stackrel{s}{\rightarrow}^* \epsilon$ — hence $P|Z_Y^{n.m} \stackrel{s}{\rightarrow}^* Z_Y^{n.m}$. The state $A_1.A_2 \ldots A_{n+1}.\delta$ must be able to match the norm reducing sequence of actions s. As $Length(s) = n$, at most the first n variables of $A_1.A_2 \ldots A_{n+1}.\delta$ can contribute to the sequence s, i.e., $A_1.A_2 \ldots A_{n+1}.\delta \stackrel{s}{\rightarrow}^* \eta.A_{n+1}.\delta$ where $\eta \in Var(\Delta')^*$. As Δ' is normed, there is $t \in Act^*$, $Length(t) = |\eta|$ such that $\eta.A_{n+1}.\delta \stackrel{t}{\rightarrow}^* A_{n+1}.\delta$. The state $Z_Y^{n.m}$ can match the norm reducing sequence

t only by removing $Length(t)$ copies of Z_Y:

$$\begin{array}{ccc} P|Z_Y^{n.m} & \sim & A_1 \dots A_{n+1}.\delta \\ \big\downarrow s & & \big\downarrow s \\ Z_Y^{n.m} & \sim & \eta.A_{n+1}.\delta \\ \big\downarrow t & & \big\downarrow t \\ Z_Y^{n.m-|\eta|} & \sim & A_{n+1}.\delta \end{array}$$

Now let $k = Length(s) + Length(t)$ (i.e., $k = |A_1 \dots A_n|$). Clearly $k \le n.m$. As $|Z_Y| = 1$, there is $p \in Act^*$, $Length(p) = k$ such that $P|Z_Y^{n.m} \xrightarrow{p}{}^* P|Z_Y^{n.m-k}$. The norm reducing sequence p must be matched by $A_1.A_2 \dots A_{n+1}.\delta$. As $Length(p) = k = |A_1 \dots A_n|$, we have $A_1.A_2 \dots A_{n+1}.\delta \xrightarrow{p}{}^* A_{n+1}.\delta$ and thus $P|Z_Y^{n.m-k} \sim A_{n+1}.\delta$. By transitivity of $\sim$ we now obtain $P|Z_Y^{n.m-k} \sim Z_Y^{n.m-|\eta|}$, hence $P \sim Z_Y^{|P|}$.

As the variable Y is non-regular and $Y \sim Z_Y^{|Y|}$, the variable Z_Y is also non-regular. Moreover, Z_Y is a unique variable with the property $P \sim Z_Y^{|P|}$ for each $P \in Assoc(Y)$ because Δ is reduced.

A similar argument can be used to prove that Z_Y is reachable from each $P \in Assoc(Y)$. As P is normed, $P \to^* P'$ where $|P'| = 1$. As $P \sim Z_Y^{|P|}$, $P' \sim Z_Y$ and hence $P' = Z_Y$.

It remains to check that if $a\alpha$ is a summand of the defining equation for Z_Y in Δ then $\alpha \sim Z_Y^{|\alpha|}$. But each variable $V \in \alpha$ belongs to $Assoc(Y)$ (because $Y \to^* Z_Y \to^* V$) and thus $V \sim Z_Y^{|V|}$. Hence $\alpha \sim Z_Y^{|\alpha|}$. □

Remark 3.10 *In the rest of this paper the symbol Z_Y, where $Y \in Var(\Delta)$ is a growing variable, always denotes the unique variable of Lemma 3.9.*

Lemma 3.11 *Let Δ be a reduced nBPP$_\tau$ process in 3-GNF such that $\Delta \in$ nBPA$_\tau \cap$ nBPP$_\tau$. Let $A|B$ be a reachable state of Δ such that $A \in Assoc(Y)$ and $B \in Assoc(Q)$ for some growing variables $Y, Q \in Var(\Delta)$. Then $Z_Y = Z_Q$.*

Proof. As Δ is reduced, it suffices to prove that $Z_Y \sim Z_Q$. As $A \in Assoc(Y)$, $A \to^* Z_Y$ (see Lemma 3.9). Similarly, $B \to^* Z_Q$ and hence $Z_Y|Z_Q$ is a reachable state of Δ. As Z_Q is non-regular, it can reach a state of an arbitrary norm—for each $i \in N$ there is $\alpha_i \in Var(\Delta)^{\otimes}$ such that $Z_Q \to^* \alpha_i$ and $|\alpha_i| = i$. Clearly $\alpha_i \sim Z_Q^i$ because each variable from α_i belongs to $Assoc(Q)$. Hence $Z_Y|\alpha_i \sim Z_Y|Z_Q^i$.

As $\Delta \in$ nBPA$_\tau \cap$ nBPP$_\tau$, there is a nBPA$_\tau$ process Δ' in GNF such that $\Delta \sim \Delta'$. Let $m = \max\{|V|,\ V \in Var(\Delta')\}$. The state $Z_Y|\alpha_m$ is a reachable state of Δ and therefore there is $\gamma \in Var(\Delta')^*$ such that $Z_Y|\alpha_m \sim \gamma$ and hence also $Z_Y|Z_Q^m \sim \gamma$. Moreover, γ is a sequence of at least two variables.

Now we can use a similar construction as in the proof of Lemma 3.9 and conclude that $Z_Y|Z_Q^j \sim Z_Q^{j+1}$ for some $j \in N$. This implies $Z_Y \sim Z_Q$. □

Lemma 3.12 *Let Δ be a reduced nBPP$_\tau$ process in 3-GNF such that $\Delta \in$ nBPA$_\tau \cap$ nBPP$_\tau$. Let $L|A$ be a reachable state of Δ such that L is a lonely variable. Then A is a regular process (see Remark 2.1).*

Proof. Let us assume that A is not regular. Then $A \to^* Y$, where $Y \in Var(\Delta)$ is a growing variable (see Proposition 2.10). But then $L|A \to^* L|Y$, thus $L \in Assoc(Y)$—we have a contradiction. $\square$

Proposition 3.13 *Let Δ be a nBPP$_\tau$ process from nBPA$_\tau \cap$ nBPP$_\tau$. Then there is a process Δ' in INF$_{BPP}$ such that $\Delta \sim \Delta'$.*

Proof. We can assume (w.l.o.g.) that Δ is reduced and in 3-GNF. The process Δ' can be obtained by the following transformation of defining equations of Δ (which can also add completely new variables and equations): if $X \stackrel{def}{=} \sum_{j=1}^{m} a_j\alpha_j$ is a defining equation from Δ, then $X \stackrel{def}{=} \sum_{j=1}^{m} \mathcal{T}(a_j\alpha_j)$ is added to Δ', where $\mathcal{T}$ is defined as follows:

- if $card(\alpha_j) \leq 1$ then $\mathcal{T}(a_j\alpha_j) = a_j\alpha_j$
- if $card(\alpha_j) = 2$ (i.e., $\alpha_j = A|B$) then there are three possibilities:

(i) $A \in Assoc(Y)$ and $B \in Assoc(Q)$ for some growing variables $Y, Q \in Var(\Delta)$. Then $A \sim Z_Y^{|A|}$ and $B \sim Z_Q^{|B|}$ (see Lemma 3.9). As $A|B$ is a reachable state, we can conclude (with a help of Lemma 3.11) that $Z_Y = Z_Q$, hence $A|B \sim Z_Y^{|A|+|B|}$. Thus $\mathcal{T}(a(A|B)) = a(Z_Y^{|A|+|B|})$.

(ii) $A \in Assoc(Y)$ for some growing variable $Y \in Var(\Delta)$ and B is lonely. But then $A \sim Z_Y^{|A|}$ and as Z_Y is not regular, A is not regular either. As the state $A|B$ is reachable and B is lonely, it contradicts Lemma 3.12. Hence this case is in fact impossible (as well as the case when A is lonely and $B \in Assoc(Q)$).

(iii) A and B are lonely. Then A and B are regular (due to Lemma 3.12) and therefore the state $A|B$ is also regular. Each regular process can be represented in normal form (see Definition 2.7). Let $\Delta_{A|B}$ be a regular process in normal form which is bisimilar to $A|B$. We can assume (w.l.o.g.) that $Var(\Delta_{A|B}) \cap Var(\Delta') = \emptyset$. $\mathcal{T}$ adds all equations from $\Delta_{A|B}$ to Δ' and $\mathcal{T}(a(A|B)) = a.N$ where N is the leading variable of $\Delta_{A|B}$.

The transformation $\mathcal{T}$ preserves bisimilarity—hence $\Delta \sim \Delta'$. It remains to check that Δ' is in INF$_{\text{BPP}}$. Clearly each summand of each defining equation from Δ' is of the form which is admitted by INF$_{\text{BPP}}$. If aZ^j is a summand of a defining equation in Δ' such that $j \geq 2$, then $Z = Z_Y$ for some growing variable $Y \in Var(\Delta)$. Let $a\alpha$ be a summand in the original defining equation for Z_Y in Δ. We need to show that each such summand must have been tranformed into $aZ_Y^{|\alpha|}$ by $\mathcal{T}$. But it is obvious as each variable from α belongs to $Assoc(Y)$. If α is composed of a single variable V, then $V = Z_Y$ because $V \sim Z_Y$ (due to Lemma 3.9) and Δ is reduced. Moreover, at least one summand in the defining equation for Z_Y in Δ' is of the form aZ_Y^l where $l \geq 2$, because Z_Y would be regular otherwise. To complete the proof we need to show that the defining equation for Z_Y in Δ' cannot contain two summands of the form $b, \bar{b}$. Assume the converse. As $\Delta' \in \mathrm{nBPA}_\tau \cap \mathrm{nBPP}_\tau$, there is a nBPA_τ process Δ_2 in GNF such that $\Delta' \sim \Delta_2$. As Z_Y^i is a reachable state of Δ' for each $i \in N \cup \{0\}$ (see Remark 3.5), there is $\alpha_i \in \mathit{Var}(\Delta_2)^*$ such that $Z_Y^i \sim \alpha_i$ for each i. Moreover, we can assume (w.l.o.g.) that each α_i is of maximal *Length*, i.e., if $\alpha_i \sim \beta$ for some $\beta \in \mathit{Var}(\Delta_2)^*$, then $\mathit{Length}(\alpha_i) \geq \mathit{Length}(\beta)$. Let k be the minimal number with the property $\mathit{Length}(\alpha_k) \geq 2$. Clearly $\mathit{Length}(\alpha_k) = 2$, because otherwise we could easily obtain a contradiction with the minimality of k. Hence $\alpha_k = P.Q$ for some $P, Q \in \mathit{Var}(\Delta_2)$. As $Z_Y^k \xrightarrow{b} Z_Y^{k-1}$, we also have $P.Q \xrightarrow{b} \gamma$ for some $\gamma \sim \alpha_{k-1}$. By the definition of α_i and k, γ must be composed of a single variable. The only such state which can be entered by $P.Q$ in one step is Q, hence $\alpha_{k-1} \sim Q$. As the defining equation for Z_Y contains two summands $b, \bar{b}$, we also have a transition $Z_Y^k \xrightarrow{\tau} Z_Y^{k-2}$. But $P.Q$ cannot reach a state which is bisimilar to α_{k-2} in one step, because α_{k-2} is (again by the definition of α_i and k) composed of at most one variable which must be different from Q because $\alpha_{k-1} \not\sim \alpha_{k-2}$. Hence $\alpha_k \not\sim Z_Y^k$ and we have a contradiction. □

Propositions 3.6 and 3.13 give us the classification of $\mathrm{nBPA}_\tau \cap \mathrm{nBPP}_\tau$ in terms of nBPP_τ syntax:

Theorem 3.14 *The class $nBPA_\tau \cap nBPP_\tau$ contains exactly (up to bisimilarity) $nBPP_\tau$ processes in INF_{BPP}.*

The class $\mathrm{nBPA}_\tau \cap \mathrm{nBPP}_\tau$ can also be characterized using nBPA_τ syntax. To do this, we introduce a special normal form for nBPA_τ processes:

Definition 3.15 ($\mathbf{INF_{BPA}}$) *Let Δ be a reduced $nBPA_\tau$ process in GNF.*

(i) Let $X, Y \in \mathit{Var}(\Delta)$ be non-regular variables. We say that Y is a communication closure *(C-closure) of X if the following conditions hold:*
- *All summands in the def. equation for X are either of the form a where $a \in Act$, or $a(Y^i.X)$ where $a \in Act$ and $i \in N \cup \{0\}$. Moreover, at least one summand is of the form $a(Y^k.X)$ where $k \geq 1$.*
- *All summands in the def. equation for Y are of the form aY^i, where $a \in Act$ and $i \in N \cup \{0\}$.*
- *aY^i is a summand in the def. equation for Y iff one of the following conditions holds:*

  (a) $i = 0$ and a is a summand in the def. equation for X.

  (b) $i \geq 1$ and $a(Y^{i-1}.X)$ is a summand in the def. equation for X.

  (c) $a = \tau$ and there are two summands of the form $b\alpha_1, \bar{b}\alpha_2$ in the def. equation for X such that $i = \mathit{Length}(\alpha_1) + \mathit{Length}(\alpha_2) - 1$ (note that this condition ensures that def. equations for X, Y do not contain two summands of the form $b, \bar{b}$).

(ii) The process Δ is said to be in INF_{BPA} if whenever $a\alpha$ is a summand in a def. equation from Δ such that $\mathit{Length}(\alpha) \geq 2$, then $\alpha = X^i.Y$ for some $i \in N$ and $X, Y \in \mathit{Var}(\Delta)$ such that Y is a C-closure of X (note that X, Y need not be different).

Note that if Y is a C-closure of X, then $|Y| = |X| = 1$. Another interesting property of X and Y is presented in the following remark.

Remark 3.16 *It is easy to check that if Y is a C-closure of X, then $Y^i.X \sim \overline{X}^{i+1}$ where $\overline{X}$ is a nBPP$_\tau$ process composed of a single variable whose def. equation is obtained from the def. equation for X by substituting '.' with '|' and replacing each occurence of X and Y with $\overline{X}$.*

Theorem 3.17 *The class nBPA$_\tau$ $\cap$ nBPP$_\tau$ contains exactly (up to bisimilarity) nBPA$_\tau$ processes in INF$_{BPA}$.*

Proof. Each nBPA$_\tau$ process in INF$_{\text{BPA}}$ belongs to nBPA$_\tau$ $\cap$ nBPP$_\tau$, as a bisimilar nBPP$_\tau$ process can be easily constructed by an algorithm which is inverse to the algorithm presented in the proof of Proposition 3.6 (see Remark 3.16). The fact that for each nBPA$_\tau$ process of nBPA$_\tau$ $\cap$ nBPP$_\tau$ there is a bisimilar nBPA$_\tau$ process in INF$_{\text{BPA}}$ follows directly from Proposition 3.6 and Proposition 3.13 (note that the algorithm presented in the proof of Proposition 3.6 returns a nBPA$_\tau$ process which is almost in INF$_{\text{BPA}}$—the only "problem" is that it can contain different bisimilar variables). □

Our results apply to nBPA and nBPP processes as well. So far we have investigated the intersection of nBPA$_\tau$ and nBPP$_\tau$. It was desirable to work with this unrestricted syntax, because we could also examine when it is possible to simulate "real" communications in a nBPP$_\tau$ process by a sequential nBPA$_\tau$ process. However, the characterization of nBPA $\cap$ nBPP is much simpler and therefore we present it explicitly.

Definition 3.18 (INF) *Let Δ be a reduced nBPA (or nBPP) process in GNF.*

(i) A variable $Z \in Var(\Delta)$ is simple *if all summands in the def. equation for Z are of the form aZ^i, where $a \in Act$ and $i \in N \cup \{0\}$. Moreover, at least one of those summands must be of the form aZ^k where $a \in Act$ and $k \geq 2$.*

(ii) The process Δ is said to be in INF if whenever $a\alpha$ is a summand in a def. equation from Δ such that $Length(\alpha) \geq 2$ (or $card(\alpha) \geq 2$), then $\alpha = Z^i$ for some simple variable Z and $i \geq 2$.

Note that nBPA (or nBPP) processes in INF have a nice property—a bisimilar nBPP (or nBPA) process can be obtained just by replacing the '.' operator with the '‖' operator (or by replacing the '‖' operator with the '.' operator).

Theorem 3.19 *The class nBPA $\cap$ nBPP contains exactly (up to bisimilarity) nBPA (or nBPP) processes in INF.*

4 Deciding whether $\Delta \in$ nBPA$_\tau$ $\cap$ nBPP$_\tau$

In this section we prove that the problem whether a given nBPA$_\tau$ or nBPP$_\tau$ process Δ belongs to nBPA$_\tau$ $\cap$ nBPP$_\tau$ is decidable in polynomial time. The technique is essentially similar in both cases—we check if each summand of each defining equation of Δ whose form is not admitted by $\mathrm{INF}_{\mathrm{BPA}}$ or $\mathrm{INF}_{\mathrm{BPP}}$ can be in principal transformed so that requirements of $\mathrm{INF}_{\mathrm{BPA}}$ or $\mathrm{INF}_{\mathrm{BPP}}$ are fulfilled. We also present simplified versions of our algorithms which work for nBPA and nBPP processes.

Next we show how to modify presented algorithms so that they become constructive. Unfortunately, these algorithms are no longer polynomial. We start with some definitions:

Definition 4.1 ($\mathbf{S}(\Delta), \mathbf{R}(\Delta)$ and $\mathbf{G}(\Delta)$ sets) *Let Δ be a nBPA$_\tau$ or nBPP$_\tau$ process in GNF.*

- *The set $S(\Delta) \subseteq Var(\Delta)$ is composed of all variables V such that $|V| = 1$, V is non-regular and if $a\alpha$ is a summand in the defining equation for V in Δ, then $\alpha \sim V^{|\alpha|}$.*
- *The set $R(\Delta) \subseteq Var(\Delta)$ contains all regular variables of Δ.*
- *The set $G(\Delta) \subseteq Var(\Delta)$ contains all growing variables of Δ.*

The sets $S(\Delta)$, $R(\Delta)$ and $G(\Delta)$ can be constructed in polynomial time because bisimilarity and regularity are decidable in polynomial time for nBPA$_\tau$ and nBPP$_\tau$ processes (see [HJM94a], [HJM94b] and Proposition 2.8).

If Δ is a nBPA$_\tau$ (or nBPP$_\tau$) process from nBPA$_\tau \cap$ nBPP$_\tau$, then there is Δ' in $\mathrm{INF}_{\mathrm{BPA}}$ (or $\mathrm{INF}_{\mathrm{BPP}}$) such that $\Delta \sim \Delta'$. In case of nBPP$_\tau$ processes the set $S(\Delta)$ contains in fact variables which can be (potentially) bisimilar to simple variables of Δ'. In case of nBPA$_\tau$ processes the set $S(\Delta)$ contains variables which can be bisimilar to C-closures of variables from $Var(\Delta')$.

The three lemmas below together prove corectness of our algorithm which decides whether a given nBPP$_\tau$ process belongs to nBPA$_\tau \cap$ nBPP$_\tau$.

Lemma 4.2 *Let Δ be a reduced nBPP$_\tau$ process in 3-GNF and let $a(A|B)$ be a summand of a defining equation from Δ such that A is regular and B is non-regular. Then $\Delta \notin$ nBPA$_\tau \cap$ nBPP$_\tau$.*

Proof. Assume there is a nBPP$_\tau$ process Δ' in $\mathrm{INF}_{\mathrm{BPP}}$ such that $\Delta \sim \Delta'$. Let $n = \max\{|Y|,\ Y \in Var(\Delta')\}$. As B is non-regular, it can reach a state of an arbitrary norm—let $B \to^* \beta$ where $|\beta| > n$. Then $A|\beta$ is a reachable state of Δ and thus $A|\beta \sim \beta'$ for some reachable state β' of Δ'. As $|\,A|\beta\,| > n$, we can conclude that $\beta' = Z^{|\,A|\beta\,|}$ where $Z \in Var(\Delta')$ is a simple variable (see Remark 3.5). Hence $A \sim Z^{|A|}$ and as each simple variable is growing (see Definition 3.3), it contradicts regularity of A. □

Lemma 4.3 *Let Δ be a reduced nBPP$_\tau$ process in 3-GNF such that $\Delta \in$ nBPA$_\tau \cap$ nBPP$_\tau$. Let $a(A|B)$ be a summand of a defining equation from Δ such that A and B are non-regular. Then there is exactly one variable $Z \in S(\Delta)$ such that $A|B \sim Z^{|\,A|B\,|}$.*

Proof. Let Δ' be a nBPP$_\tau$ process in $\mathrm{INF}_{\mathrm{BPP}}$ such that $\Delta \sim \Delta'$. Let $n = \max\{|Y|,\ Y \in Var(\Delta')\}$. Using the same argument as in the proof of Lemma

4.2 we obtain $A \sim P^{|A|}$, $B \sim Q^{|B|}$ where $P, Q \in Var(\Delta')$ are simple variables. We show that $P = Q$. Let $A \rightarrow^* \alpha$ where $|\alpha| > n$. Then clearly $\alpha \sim P^{|\alpha|}$ and as $\alpha|B$ is a reachable state of Δ, $\alpha|B \sim R^{|\alpha|B|}$ where $R \in Var(\Delta')$ is a simple variable. To sum up, we have $\alpha|B \sim P^{|\alpha|}|Q^{|B|} \sim R^{|\alpha|B|}$. Hence $P \sim R \sim Q$ and thus $P = R = Q$ because Δ' is reduced. As e.g. P is a reachable state of Δ', there is a reachable state γ of Δ such that $P \sim \gamma$. As $|P| = 1$, we can conclude $\gamma = Z$ for some $Z \in Var(\Delta)$ which clearly belongs to $S(\Delta)$. Moreover, Z is unique because Δ is reduced. $\square$

Lemma 4.4 *Let Δ be a nBPP$_\tau$ process in GNF and let $X \in S(\Delta)$. If the defining equation for X contains two summands of the form $b, \bar{b}$, then $\Delta \notin$ nBPA$_\tau \cap$ nBPP$_\tau$.*

Proof. Assume there is a nBPP$_\tau$ process Δ' in INF$_{\mathrm{BPP}}$ such that $\Delta \sim \Delta'$. Using the same kind of argument as in the proof of Lemma 4.2 we obtain $X \sim Z$ for some simple variable $Z \in Var(\Delta')$. As the def. equation for X contains two summands of the form $b, \bar{b}$ and $X \sim Z$, the def. equation for Z must contain those summands too—hence Z is not simple and we have a contradiction. $\square$

The promised (constructive) algorithm which decides the membership to the class nBPA$_\tau \cap$ nBPP$_\tau$ for nBPP$_\tau$ processes is presented in Figure 2. Steps which are executed only by the constructive algorithm are shaded—if we omit everything on a grey background, we obtain a non-constructive polynomial algorithm. The abbreviation "NFR(Δ)" stands for the **N**ormal **F**orm of the **R**egular process Δ, which can be effectively constructed (see Proposition 2.8). We always assume that NFR(Δ) contains fresh variables which are not contained in any other process we are working with. When the command return is executed, the algorithm *halts* and returns the value which follows immediately after the keyword return.

The constructive algorithm is not polynomial because the construction of NFR is not polynomial—a regular nBPP$_\tau$ process in 3-GNF with n variables can generally reach exponentially many pairwise non-bisimilar states and each of these states requires a special variable.

Our algorithm for nBPP$_\tau$ processes works for pure nBPP processes as well. It suffices to replace the '$|$' operator with the '$\|$' operator in our description. As there are no communications in nBPP, the notion of dual action is no longer sensible—hence the second step of our algorithm can be removed in case of nBPP processes.

Now we provide an analogous algorithm for nBPA$_\tau$ processes. We start with some auxiliary definitions and lemmas.

Definition 4.5 (CL sets) *Let Δ be a nBPA$_\tau$ process in GNF. For each $Y \in S(\Delta)$ we define the set $CL(Y)$, composed of all $X \in Var(\Delta)$ which fulfil the following conditions:*

- *If $a\alpha$ is a summand in the def. equation for X such that $Length(\alpha) \geq 1$,*

Algorithm: A constructive test of the membership to $\mathrm{nBPA}_\tau \cap \mathrm{nBPP}_\tau$ for nBPP_τ processes.

Input: A reduced nBPP_τ process Δ in 3-GNF.

Output: **YES** and a nBPP_τ process Δ' in $\mathrm{INF_{BPP}}$ such that $\Delta \sim \Delta'$ if $\Delta \in \mathrm{nBPA}_\tau \cap \mathrm{nBPP}_\tau$.
NO otherwise.

(i) Construct the sets $S(\Delta)$, $R(\Delta)$ and $G(\Delta)$.

(ii) If there is $X \in S(\Delta)$ whose def. equation contains two summands of the form $b, \overline{b}$ then
  return **NO**;

(iii) If $G(\Delta) = \emptyset$ then
  Δ' :=NFR(Δ) ;
  return **YES** and Δ' ;

(iv) $\Delta' := \Delta$;

(v) for each summand of the form $a(A|B)$ in defining equations of Δ do
  if $A, B \in R(\Delta)$ then
    Construct NFR($A|B$) ;
    Replace the summand $a(A|B)$ with aN in Δ', where N is the leading variable of NFR($A|B$) ;
    $\Delta' := \Delta' \cup$NFR($A|B$) ;
  if ($A \in R(\Delta)$ and $B \notin R(\Delta)$) or ($A \notin R(\Delta)$ and $B \in R(\Delta)$) then
    return **NO**;
  if $A, B \notin R(\Delta)$ then
    if there exists $Z \in S(\Delta)$ such that $A|B \sim Z^{|A|B|}$
      then Replace the summand $a(A|B)$ with $a(Z^{|A|B|})$ in Δ' ;
      else return **NO**;

(vi) return **YES** and Δ' ;

Fig. 2. An algorithm which (constructively) decides the membership to $\mathrm{nBPA}_\tau \cap \mathrm{nBPP}_\tau$ for nBPP_τ processes.

then $\alpha \sim Y^{|\alpha|-1}.X$.

- *The def. equation for* Y *contains a summand bisimilar to* aY^k, $k \in N \cup \{0\}$, *iff one of the following conditions holds:*

(i) $\alpha = \epsilon$ *and the def. equation for* X *contains a summand 'a'.*

(ii) $\alpha \neq \epsilon$ *and the def. equation for* X *contains a summand which is bisimilar to* $a(Y^{k-1}.X)$.

(iii) $a = \tau$ *and the def. equation for* X *contains two summands of the form* $b\alpha_1$, $\bar{b}\alpha_2$ *such that* $k = \mathit{Length}(\alpha_1) + \mathit{Length}(\alpha_2) - 1$.

It is easy to see that the set $CL(Y)$ can be constructed in polynomial time for each $Y \in S(\Delta)$. The following lemma is due to D. Caucal (see [Cau88]):

Lemma 4.6 *Let* Δ, Δ' *be nBPA*$_\tau$ *processes in GNF and let* $\alpha, \beta \in \mathit{Var}(\Delta)$, $\alpha', \beta' \in \mathit{Var}(\Delta')$ *such that* $\beta \sim \beta'$ *and* $\alpha.\beta \sim \alpha'.\beta'$. *Then* $\alpha \sim \alpha'$

Lemma 4.7 *Let* Δ, Δ' *be nBPA*$_\tau$ *processes in GNF. Let* $A_1, \ldots, A_k \in \mathit{Var}(\Delta)$, $X, Y \in \mathit{Var}(\Delta')$ *such that* $|X| = |Y| = 1$ *and* $A_1.\cdots.A_k \sim Y^l.X$ *where* $l = |A_1.\cdots.A_k| - 1$. *Then* $A_k \sim Y^{|A_k|-1}.X$ *and* $A_i \sim Y^{|A_i|}$ *for* $1 \leq i < k$.

Proof. Clearly $A_k \sim Y^{|A_k|-1}.X$. Hence $A_1.\cdots.A_{k-1} \sim Y^{|A_1.\cdots.A_{k-1}|}$ (due to Lemma 4.6). The fact $A_i \sim Y^{|A_i|}$ for $1 \leq i < k$ can be proved by induction on k. If $k = 2$ then $A_1 \sim Y^{|A_1|}$ and our lemma holds. If $k > 2$, then clearly $A_{k-1} \sim Y^{|A_{k-1}|}$ and due to Lemma 4.6 we have $A_1.\cdots.A_{k-2} \sim Y^{|A_1.\cdots.A_{k-2}|}$. Now we can use induction hypothesis and conclude that $A_i \sim Y^{|A_i|}$ for $1 \leq i < (k-2)$. □

Lemma 4.8 *Let* Δ *be a reduced nBPA*$_\tau$ *process in 3-GNF such that* $\Delta \in \mathit{nBPA}_\tau \cap \mathit{nBPP}_\tau$. *Let* $Q.\alpha$ *be a reachable state of* Δ *such that* $Q \in G(\Delta)$, $\alpha \neq \epsilon$. *Then there are unique variables* $Y \in S(\Delta)$, $X \in CL(Y)$ *such that* $Q.\alpha \sim Y^{|Q.\alpha|-1}.X$.

Proof. As $\Delta \in \mathrm{nBPA}_\tau \cap \mathrm{nBPP}_\tau$, there is a nBPA$_\tau$ process Δ' in INF$_{\mathrm{BPA}}$ such that $\Delta \sim \Delta'$. Let $n = \max\{|A|,\ A \in \mathit{Var}(\Delta')\}$. As Q is growing, $Q \to^* Q.\gamma$ where $\gamma \neq \epsilon$. Hence the state $Q.\gamma^n.\alpha$ is a reachable state of Δ and therefore there is a reachable state δ of Δ' such that $Q.\gamma^n.\alpha \sim \delta$. As $|Q.\gamma^n.\alpha| > n$, we can conclude $\delta = R^{|Q.\gamma^n.\alpha|-1}.S$, where R is a C-closure of S (see Definition 3.15). Hence $Q.\gamma^n.\alpha \sim R^{|Q.\gamma^n.\alpha|-1}.S$ and due to Lemma 4.7 we have $\alpha \sim R^{|\alpha|-1}.S$ and $Q \sim R^{|Q|}$, thus $Q.\alpha \sim R^{|Q.\alpha|-1}.S$. Now it suffices to show that there are $Y \in S(\Delta)$, $X \in CL(Y)$ such that $Y \sim R$ and $X \sim S$. As Δ is normed, $Q \xrightarrow{s}{}^* Y$ where $|Y| = 1$ and s is a norm-decreasing sequence of actions. Then $Q.\alpha \xrightarrow{s}{}^* Y.\alpha$ and as $Q.\alpha \sim R^{|Q.\alpha|-1}.S$, the state $R^{|Q.\alpha|-1}.S$ must be able to match the sequence s and enter a state bisimilar to $Y.\alpha$. As s is norm-decreasing and $|R| = 1$, the only such state is $R^{|Y.\alpha|-1}.S$. Hence $Y.\alpha \sim R^{|Y.\alpha|-1}.S$ and due to Lemma 4.7 we have $Y \sim R$. The fact $Y \in S(\Delta)$ follows directly from Definition 3.15. As S is a reachable state of Δ', there is a variable $X \in S(\Delta)$ such that $X \sim S$. Clearly $X \in CL(Y)$ (see Definition 3.15). Variables X, Y are unique because Δ is reduced. □

It is worth noting that the variables X, Y of the previous lemma need not be different—if a nBPA$_\tau$ process Δ belongs to nBPA$_\tau \cap$ nBPP$_\tau$, then each $Y \in S(\Delta)$ belongs to $CL(Y)$.

To prove corectness of our algorithm which decides the membership to nBPA$_\tau \cap$ nBPP$_\tau$ for nBPA$_\tau$ processes we need some lemmas about summands:

Lemma 4.9 *Let Δ be a reduced nBPA$_\tau$ process in 3-GNF and let $a(A.B)$ be a summand of a defining equation from Δ such that A is non-regular and B is regular. Then $\Delta \notin$ nBPA$_\tau \cap$ nBPP$_\tau$.*

Proof. As $a(A.B)$ is a summand of a defining equation from Δ and Δ is normed and in GNF, there is a reachable state of the form $A.B.\beta$. As A is non-regular, $A \to^* Q.\alpha$ where $Q \in G(\Delta)$. Hence $Q.\alpha.B.\beta$ is a reachable state of Δ and due to Lemma 4.8 we have $Q.\alpha.B.\beta \sim Y^{|Q.\alpha.B.\beta|-1}.X$ for some $Y \in S(\Delta)$, $X \in CL(Y)$. With a help of Lemma 4.7 we obtain $B \sim Y^{|B|}$ or $B \sim Y^{|B|-1}.X$ (the latter possibility holds if $\beta = \epsilon$). As X, Y are growing, it contradicts regularity of B. $\Box$

Lemma 4.10 *Let Δ be a reduced nBPA$_\tau$ process in 3-GNF. Let $a(A.B)$ be a summand of a defining equation from Δ such that A is regular and B is non-regular. Then it is possible to replace the summand $a(A.B)$ with aN where $N \notin Var(\Delta)$ and to add a finite number of new equations in INF$_{BPA}$ to Δ such that the resulting process Δ_1 is bisimilar to Δ.*

Proof. As A is regular, it is possible to construct $\Delta_A := NFR(A)$ such that $Var(\Delta) \cap Var(\Delta_A) = \emptyset$. Now we modify defining equations of Δ_A slightly—each summand of the form a where $a \in Act$ is replaced with aB. The resulting system of equations is in INF$_{\text{BPA}}$. If we add the modified system Δ_A to Δ and replace the summand $a(A.B)$ with aN where N is the leading variable of Δ_A, we obtain a process Δ_1 which is clearly bisimilar to Δ. $\Box$

Lemma 4.11 *Let Δ be a reduced nBPA$_\tau$ process in 3-GNF and let $a(A.B)$ be a summand of a defining equation from Δ such that A and B are non-regular. Then*

(i) If $\Delta \in$ nBPA$_\tau \cap$ nBPP$_\tau$ then there are unique variables $Y \in S(\Delta)$, $X \in CL(Y)$ such that $B \sim Y^{|B|-1}.X$

(ii) Let $B \sim Y^{|B|-1}.X$ for some $Y \in S(\Delta)$ and $X \in CL(Y)$. If there is a sequence of transitions $A = A_0 \stackrel{a_0}{\to} A_1.\alpha_1 \stackrel{a_1}{\to} A_2.\alpha_2 \stackrel{a_2}{\to} \cdots \stackrel{a_k}{\to} A_k.\alpha_k$ such that $k \geq 0$, $A_k \in G(\Delta)$ and $A_k.\alpha_k \not\sim Y^{|A_k.\alpha_k|}$, then $\Delta \notin$ nBPA$_\tau \cap$ nBPP$_\tau$.

(iii) Let $B \sim Y^{|B|-1}.X$ for some $Y \in S(\Delta)$ and $X \in CL(Y)$. If for each sequence of transitions $A = A_0 \stackrel{a_0}{\to} A_1.\alpha_1 \stackrel{a_1}{\to} A_2.\alpha_2 \stackrel{a_2}{\to} \cdots \stackrel{a_k}{\to} A_k.\alpha_k$ such that $A_k \in G(\Delta)$ the state $A_k.\alpha_k$ is bisimilar to $Y^{|A_k.\alpha_k|}$, then the summand $a(A.B)$ can be replaced with aN where $N \notin Var(\Delta)$ and a finite number of new equations in INF$_{BPA}$ can be added to Δ such that the resulting process Δ_2 is bisimilar to Δ.

Proof.

(i) As A is non-regular, $A \rightarrow^* Q.\alpha$ where $Q \in G(\Delta)$. The proof can be easily completed with a help of Lemma 4.7 and Lemma 4.8.

(ii) This is a consequence of Lemma 4.7 and Lemma 4.8.

(iii) It suffices to realize that if $A = A_0 \stackrel{a_0}{\rightarrow} A_1.\alpha_1 \stackrel{a_1}{\rightarrow} A_2.\alpha_2 \stackrel{a_2}{\rightarrow} \cdots \stackrel{a_k}{\rightarrow} A_k.\alpha_k$ is a sequence of transitions such that $A_0, \ldots, A_{k-1} \notin G(\Delta)$ and $A_k \in G(\Delta)$, then $\mathit{Length}(A_i.\alpha_i) \leq \mathit{card}(\mathit{Var}(\Delta))$ for $0 \leq i \leq k-1$ (here we use the assumption that Δ is in 3-GNF. Naturally, $\mathit{Length}(A_i.\alpha_i)$ is bounded also in case of general GNF). As there are only finitely many sequences of variables of this bounded length, we can introduce a fresh variable for each of them. To construct the process Δ_2, we use a similar procedure as in the proof of Lemma 4.10. □

An existence of a sequence $A = A_0 \stackrel{a_0}{\rightarrow} A_1.\alpha_1 \stackrel{a_1}{\rightarrow} A_2.\alpha_2 \stackrel{a_2}{\rightarrow} \cdots \stackrel{a_k}{\rightarrow} A_k.\alpha_k$ such that $A_k \in G(\Delta)$ and $A_k.\alpha_k \not\sim Y^{|A_k.\alpha_k|}$ is decidable in polynomial time:

Lemma 4.12 *Let Δ be a reduced nBPA$_\tau$ process in 3-GNF. Let $A \in \mathit{Var}(\Delta)$ be a non-regular variable and let $Y \in S(\Delta)$. The problem whether A can reach a state of the form $Q.\alpha$ where $Q \in G(\Delta)$ and $Q.\alpha \not\sim Y^{|Q.\alpha|}$ is decidable in polynomial time.*

Proof. We divide the set $\mathit{Var}(\Delta)$ into two disjoint subsets of *successful* and *unsuccessful* variables. $P \in \mathit{Var}(\Delta)$ is unsuccessful if one of the following conditions holds:

- P is growing and $P \not\sim Y^{|P|}$.
- The defining equation for P in Δ contains a summand of the form $a(R.S)$ where R is non-regular and $S \not\sim Y^{|S|}$.

A variable is successful if it is not unsuccessful. Furthermore, we define the binary relation '$\Rightarrow$' on $\mathit{Var}(\Delta)$: $U \Rightarrow V$ iff U is successful and the defining equation for U in Δ contains a summand which is of one of the following forms:

- aV
- $a(V.W)$ where $W \in \mathit{Var}(\Delta)$
- $a(W.V)$ where $W \in \mathit{Var}(\Delta)$ is regular

Let '$\Rightarrow^*$' be the reflexive and transitive closure of '$\Rightarrow$'. It is not hard to prove that A can reach a state of the form $Q.\alpha$ where Q is growing and $Q.\alpha \not\sim Y^{|Q.\alpha|}$ iff $A \Rightarrow^* T$ for some unsuccessful variable T. As the relation '$\Rightarrow^*$' can be constructed in polynomial time, the proof is finished. □

An algorithm which decides the membership to nBPA$_\tau \cap$ nBPP$_\tau$ for nBPA$_\tau$ processes is presented in Figure 3. We use the same notation as in the case of nBPP$_\tau$.

There is a little shortcoming in the constructive variant of our algorithm for nBPA$_\tau$ processes—as the process Δ_2 of Lemma 4.11 need not be in 3-

Algorithm: A constructive test of the membership to $\mathrm{nBPA}_\tau \cap \mathrm{nBPP}_\tau$ for nBPA_τ processes.

Input: A reduced nBPA_τ process Δ in 3-GNF.

Output: **YES** and a nBPA_τ process Δ' in $\mathrm{INF}_{\mathrm{BPA}}$ such that $\Delta \sim \Delta'$ if $\Delta \in \mathrm{nBPA}_\tau \cap \mathrm{nBPP}_\tau$.
NO otherwise.

(i) Construct the sets $S(\Delta)$, $R(\Delta)$, $G(\Delta)$ and for each $Y \in S(\Delta)$ construct the set $CL(Y)$.

(ii) If $(G(\Delta) = \emptyset)$ then
  $\Delta' :=\mathrm{NFR}(\Delta)$;
  return **YES** and Δ' ;

(iii) $\Delta' := \Delta$;

(iv) for each summand of the form $a(A.B)$ in defining equations of Δ do
  if $A, B \in R(\Delta)$ then
    Construct $\mathrm{NFR}(A.B)$;
    Replace the summand $a(A.B)$ with aN in Δ', where N is the leading variable of $\mathrm{NFR}(A.B)$;
    $\Delta' := \Delta' \cup \mathrm{NFR}(A.B)$;
  if $A \notin R(\Delta)$ and $B \in R(\Delta)$ then
    return **NO**;
  if $A \in R(\Delta)$ and $B \notin R(\Delta)$ then
    Construct the process Δ_1 of Lemma 4.10 ;
    $\Delta' := \Delta_1$;
  if $A, B \notin R(\Delta)$ then
    if there exist $Y \in S(\Delta), X \in CL(Y)$ such that $B \sim Y^{|B|-1}.X$
      then if A can reach a state $Q.\alpha$ where $Q \in G(\Delta)$ and $Q.\alpha \not\sim Y^{|Q.\alpha|}$
        then return **NO**;
        else Construct the process Δ_2 of Lemma 4.11 ;
          $\Delta' := \Delta_2$;
      else return **NO**;

(v) return **YES** and Δ' ;

Fig. 3. An algorithm which (constructively) decides the membership to $\mathrm{nBPA}_\tau \cap \mathrm{nBPP}_\tau$ for nBPA_τ processes.

GNF, the process Δ' need not remain in 3-GNF either. But each lemma about summands (Lemma 4.9, 4.10, 4.11) is formulated for nBPA$_\tau$ process in 3-GNF. Naturally, it is not a problem to prove analogous lemmas about processes in general GNF, but we think that readability is more important feature than technical accuracy (note there is a similar problem in the constructive variant of our algorithm for nBPP$_\tau$ processes).

In case of nBPA processes our algorithm must be slightly modified (and simplified). This is a consequence of the fact that a nBPA process Δ belongs to nBPA $\cap$ nBPP iff it can be represented in INF—and INF is a little diffrent from INF$_{\text{BPA}}$ (see Definitions 3.18 and 3.15). Lemma 4.9 and Lemma 4.10 are valid also for nBPA processes. Instead of Lemma 4.11 we can prove the following (in a similar way):

Lemma 4.13 *Let Δ be a reduced nBPA process in 3-GNF and let $a(A.B)$ be a summand of a defining equation from Δ such that A and B are non-regular. Then*

(i) If $\Delta \in nBPA \cap nBPP$ then there is a unique variable $Z \in S(\Delta)$ such that $B \sim Z^{|B|}$

(ii) Let $B \sim Z^{|B|}$ for some $Z \in S(\Delta)$. If there is a sequence of transitions $A = A_0 \overset{a_0}{\to} A_1.\alpha_1 \overset{a_1}{\to} A_2.\alpha_2 \overset{a_2}{\to} \cdots \overset{a_k}{\to} A_k.\alpha_k$ such that $k \geq 0$, $A_k \in G(\Delta)$ and $A_k.\alpha_k \not\sim Z^{|A_k.\alpha_k|}$, then $\Delta \notin nBPA \cap nBPP$.

(iii) Let $B \sim Z^{|B|}$ for some $Z \in S(\Delta)$. If for each sequence of transitions $A = A_0 \overset{a_0}{\to} A_1.\alpha_1 \overset{a_1}{\to} A_2.\alpha_2 \overset{a_2}{\to} \cdots \overset{a_k}{\to} A_k.\alpha_k$ such that $A_k \in G(\Delta)$ the state $A_k.\alpha_k$ is bisimilar to $Z^{|A_k.\alpha_k|}$, then the summand $a(A.B)$ can be replaced with aN where $N \notin Var(\Delta)$ and a finite number of new equations in INF can be added to Δ such that the resulting process Δ_2 is bisimilar to Δ.

Our algoritm for nBPA processes differs from the algorithm of Figure 3 in two things—the sets $CL(Y)$ for $Y \in S(\Delta)$ are not computed at all and the last if statement in the loop of step 4 is replaced with the following code:

```
if A, B ∉ R(Δ) then
    if there exist Z ∈ S(Δ) such that B ~ Z^|B|
        then if A can reach a state Q.α where Q ∈ G(Δ) and Q.α ≁ Z^|Q.α|
                 then return NO;
                 else  Construct the process Δ₂ of Lemma 4.13 ;
                       Δ' := Δ₂ ;
        else return NO;
```

The existence of constructive variants of presented algorithms allow us to prove the following theorem:

Theorem 4.14 *Bisimilarity is decidable in the union of nBPA$_\tau$ and nBPP$_\tau$ processes.*

Proof. Given two nBPA$_\tau$ or nBPP$_\tau$ processes, it is possible to check bisimilarity using algorithms which were published in [HJM94a] and [HJM94b]. If we get a nBPP$_\tau$ process Δ_1 and a nBPA$_\tau$ process Δ_2, then we run one of the constructive algorithms presented earlier. We can choose e.g., the first algorithm with Δ_1 on input. If it answers **NO**, then $\Delta_1 \not\sim \Delta_2$. Otherwise we obtain a nBPP$_\tau$ process Δ_1' in INF$_{\text{BPP}}$ which is bisimilar to Δ_1. Now it suffices to check bisimilarity between two nBPA$_\tau$ processes $\overline{\Delta_1'}$ and Δ_2, where $\overline{\Delta_1'}$ is obtained by running the algorithm presented in the proof of Proposition 3.6 with Δ_1' on input. □

Note that the corresponding statement holds for nBPA and nBPP processes by specialization.

5 Related work and future research

The problem whether a given nBPP process belongs to nBPA $\cap$ nBPP has been independently examined by Blanco in [Bla95] where it is shown that given a nBPP process, one can decide whether there is a bisimilar nBPA process. Blanco's approach is based on special properties of BPA transition graphs (see [CM90]). A test whether a given nBPP graph has these properties is given in the work. Consequently, this result does not allow for testing whether a given nBPA process belongs to the intersection. The generalization to nBPA$_\tau$ and nBPP$_\tau$ classes is not considered at all.

Our result on the classification of nBPA $\cap$ nBPP might be of some interest from the point of view of formal languages/automata theory as well. INF (for nBPA processes) can be taken as a special type of CF grammars which generate languages of the form $R.(L_1 \cup \ldots \cup L_n)$, where R is regular and each L_i can be generated by a CF grammar having just one nonterminal and rules of the form $Z \rightarrow aZ^k$, $k \geq 0$. Considering language equivalence only, it is obvious that languages of the mentioned type $R.(L_1 \cup \ldots \cup L_n)$ can be recognized by nondeterministic one-counter automata. Hence our result on the classification of nBPA $\cap$ nBPP can be considered as a refinement of the result achieved in [Sch92] on the context-freeness of languages generated by Petri nets, as BPP processes form a proper subclass of Petri nets.

An obvious question is whether our results can be extended to classes of all (not only normed) BPA and BPP processes. The class BPA $\cap$ BPP contains also processes which cannot be presented in INF. Assume the following BPP process:

$$X \stackrel{def}{=} a(Y \| X)$$
$$Y \stackrel{def}{=} b$$

The process X cannot be presented in INF. But it obviously belongs to BPA $\cap$ BPP; a bisimilar BPA process looks as follows:

$$A \stackrel{def}{=} a(B.A)$$
$$B \stackrel{def}{=} a(B.B) + b$$

Transition systems generated by X and A are isomorphic:

$$\bullet \underset{b}{\overset{a}{\rightleftarrows}} \circ \underset{b}{\overset{a}{\rightleftarrows}} \circ \underset{b}{\overset{a}{\rightleftarrows}} \circ \cdots\cdots$$

This indicates that the problem is actually more complicated. Techniques which were used for normed processes cannot be applied—it seems however, that a deeper study of the structure of BPA and BPP transition graphs could help.

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