On the relationship between sequential and parallel compositions in process algebras

Ivana Černá
cerna@fi.muni.cz

Mojmír Křetínský
mojmir@fi.muni.cz

Antonín Kučera
tony@fi.muni.cz

Faculty of Informatics, Masaryk University
Botanická 68a, 60200 Brno
Czech Republic

Abstract

We compare the classes of behaviours (transition systems) which can be generated by normed BPA and normed BPP processes. We exactly classify the intersection of these two classes - i.e. the class of transition systems which can be equivalently (up to bisimilarity) described by the syntax of normed BPA and normed BPP processes.

Next we show that it is decidable whether for a given normed BPA (resp. BPP) process Δ there is some (unspecified) normed BPP (resp. BPA) process Δ' such that Δ is bisimilar to Δ'. Moreover, if the answer is positive then our algorithm also constructs the process Δ'.

As an immediate (but very important) consequence we also obtain the decidability of bisimilarity in the union of normed BPA and normed BPP processes.

1 Introduction

We consider the relationship between classes of transition systems, which are generated by normed BPA and normed BPP processes. BPA processes can be seen as simple sequential programs (they are equipped with a binary sequential operator). This class of processes has been intensively studied by many researchers. Baeten, Bergstra and Klop proved in [?] that bisimilarity is decidable for normed BPA processes. Much simpler proofs of this were later given by Caucal ([?] and Groote ([?]). In [?] Hüttel and Stirling used a tableau decision method and gave also sound and complete equational theory. This result was later extended to the whole class of BPA processes by Christensen, Hüttel and Stirling ([?]).

If we replace the binary sequential operator with the parallel operator, we obtain BPP processes. They can be thus seen as simple parallel programs. Christensen, Hirsfeld and Moller proved in [?] that bisimilarity is decidable for BPP processes.

An interesting problem is, what is the exact relationship between BPA and BPP processes, i.e. what is the relationship between sequencing and parallelism. It is well-known that there are BPA (resp. BPP) processes for which there is no bisimilar BPP (resp. BPA) process. But there are also behaviours (transition systems), which can be equivalently (up to bisimilarity) described by both BPA and BPP syntax. A natural question is, whether it is possible to classify all such processes (up to bisimilarity). A related problem is whether there is an algorithm, which for a given BPA (resp. BPP) process Δ constructs a bisimilar BPP (resp. BPA) process Δ' if such a Δ' exists, and answers NO otherwise.

In this paper we answer (positively) both these questions for normed subclasses of BPA and BPP processes. As an important consequence we also obtain the decidability of bisimulation equivalence in the union of normed BPA and normed BPP processes.

In many constructions of our paper we use the fact that regularity is decidable for normed BPA and normed BPP processes (a process is regular if it is bisimilar to a process with finitely many states). Regularity of BPA processes was examined for the first time by Mauw and Mulder in [?], but their notion of regularity is different from the usual one. Kučera showed in [?] that the result of Mauw and Mulder can be used to decide regularity of normed BPA processes and that regularity of normed BPP processes is also decidable.

2 Basic definitions, preliminary knowledge

2.1 BPA and BPP processes

Let $Act = \{a, b, c, \ldots\}$ be a countably infinite set of *atomic actions*. Let $Var = \{X, Y, Z, \ldots\}$ be a countably infinite set of *variables*, such that $Var \cap Act = \emptyset$. The classes of recursive BPA and BPP expressions are defined by the following abstract syntax equations:

$$\begin{array}{lcl} E_{BPA} & ::= & a \mid X \mid E_{BPA}.E_{BPA} \mid E_{BPA} + E_{BPA} \\ E_{BPP} & ::= & a \mid X \mid aE_{BPP} \mid E_{BPP} \| E_{BPP} \mid E_{BPP} + E_{BPP} \end{array}$$

Here a ranges over Act and X ranges over Var. The symbol Act^* denotes the set of all finite strings over Act.

As usual, we restrict our attention to guarded expressions. A BPA or BPP expression E is *guarded* if every variable occurence in E is within the scope of an atomic action.

A *guarded BPA (resp. BPP) process* is defined by a finite family Δ of recursive process equations

$$\Delta = \{X_i \stackrel{def}{=} E_i \quad | \quad 1 \leq i \leq n\}$$

where X_i are distinct elements of Var and E_i are guarded BPA (resp. BPP) expressions, containing variables from $\{X_1, \ldots, X_n\}$. The set of variables which appear in Δ is denoted by $Var(\Delta)$.

The variable X_1 plays a special role (X_1 is sometimes called *the leading variable* - it is a root of a labelled transition system, defined by the process Δ and following rules:

$$\frac{}{a \stackrel{a}{\rightarrow} \epsilon} \qquad \frac{E \stackrel{a}{\rightarrow} E'}{E.F \stackrel{a}{\rightarrow} E'.F} \qquad \frac{E \stackrel{a}{\rightarrow} E'}{E + F \stackrel{a}{\rightarrow} E'} \qquad \frac{F \stackrel{a}{\rightarrow} F'}{E + F \stackrel{a}{\rightarrow} F'}$$

$$\frac{E \stackrel{a}{\rightarrow} E'}{E\|F \stackrel{a}{\rightarrow} E'\|F} \qquad \frac{F \stackrel{a}{\rightarrow} F'}{E\|F \stackrel{a}{\rightarrow} E\|F'} \qquad \frac{E \stackrel{a}{\rightarrow} E'}{X \stackrel{a}{\rightarrow} E'} \; (X \stackrel{def}{=} E \in \Delta)$$

The symbol ϵ denotes the empty expression with usual conventions: $\epsilon \| E = E$, $E \| \epsilon = E$ and $\epsilon.E = E$. Nodes of the transition system generated by Δ are BPA (resp. BPP) expressions, which are often called *states of* Δ, or just "states" when Δ is understood from the context. We also define the relation $\stackrel{w}{\rightarrow}$, where $w \in Act^*$, as the reflexive and transitive closure of $\stackrel{a}{\rightarrow}$. Given two states E, F, we say that F is *reachable from* E, if $E \stackrel{w}{\rightarrow} F$ for some $w \in Act^*$. States of Δ which are reachable from X_1 are said to be *reachable*.

Remark 1 *Processes are often identified with their leading variables. Furthermore, if we assume a fixed process Δ, we can view any process expression E (not necessarily guarded) whose variables are defined in Δ as a process too; we simply add a new equation $X \stackrel{def}{=} E'$ to Δ, where X is a variable from Var such that $X \notin Var(\Delta)$ and E' is a process expression which is obtained from E by substituting each variable in E with the right side of its corresponding defining equation in Δ (E' must be guarded now). Moreover, X becomes a new leading variable. All notions originally defined for processes can be used for process expressions in this sense too.*

2.1.1 Bisimulation

The equivalence between process expressions (states) we are interested in here is *bisimilarity* [?], defined as follows:

Definition 1 *A binary relation R over process expressions is a* bisimulation *if whenever $[E, F] \in R$ then for each $a \in Act$*

- *if $E \xrightarrow{a} E'$, then $F \xrightarrow{a} F'$ for some F' such that $[E', F'] \in R$*
- *if $F \xrightarrow{a} F'$, then $E \xrightarrow{a} E'$ for some E' such that $[E', F'] \in R$*

Processes Δ and Δ' are bisimilar*, written $\Delta \sim \Delta'$, if their leading variables are related by some bisimulation.*

2.1.2 Normed processes

An important subclass of BPA (resp. BPP) processes can be obtained by an extra restriction of *normedness*. A variable $X \in Var(\Delta)$ is *normed* if there is $w \in Act^*$ such that $X \xrightarrow{w} \epsilon$. In that case we define the *norm* of X, written $|X|$, to be the length of the shortest such w. Thus $|X| = \min\{Length(w) \mid X \xrightarrow{w} \epsilon\}$. A process Δ is *normed* if all variables of $Var(\Delta)$ are normed. The norm of Δ is then defined to be the norm of X_1.

Remark 2 *As normed processes are intensively studied in this paper, we emphasize some properties of a norm:*

- *A norm of a normed process is easy to compute:*
  $|a| = 1$, $|E + F| = \min\{|E|, |F|\}$, $|E.F| = |E| + |F|$, $|E \| F| = |E| + |F|$, *and if* $X_i \stackrel{def}{=} E_i$ *and* $|E_i| = n$, *then* $|X_i| = n$.
- *Bisimilar processes must have the same norm.*
- *Each transition $E \xrightarrow{a} E'$ where $a \in Act$ can decrease the norm of E at most by one. Moreover, if $|E| = n$ then for each k, $0 \le k < n$ there exists $w \in Act^*$, $Length(w) = k$ such that $E \xrightarrow{w} E'$ where $|E| = n - k$.*

2.1.3 Greibach normal form

Any BPA (resp. BPP) process Δ can be effectively presented in so-called 3-Greibach normal form (see [?] and [?]). Before the definition we need to introduce the set $Var(\Delta)^*$ of all finite sequences of variables from $Var(\Delta)$, and the set $Var(\Delta)^{\otimes}$ of all finite multisets over $Var(\Delta)$. Each multiset of $Var(\Delta)^{\otimes}$ denotes a BPP expression by combining its elements in parallel using the '$\|$' operator.

Definition 2 *A BPA (resp. BPP) process Δ is said to be in* Greibach normal form (GNF) *if all its equations are of the form*

$$X \stackrel{def}{=} \sum_{j=1}^{n} a_j \alpha_j$$

where $n \in N$, $a_j \in Act$ and $\alpha_j \in Var(\Delta)^*$ (resp. $\alpha_j \in Var(\Delta)^{\otimes}$). If $Length(\alpha_j) \le 2$ (resp. $card(\alpha_j) \le 2$) for each j, $1 \le j \le n$, then Δ is said to be in 3-GNF.

$>$From now on we assume that all BPA (resp. BPP) processes we are working with are presented in GNF. This justifies also the assumption that all reachable states of a BPA process Δ are elements of $Var(\Delta)^*$ and all reachable states of a BPP process Δ' are elements of $Var(\Delta')^{\otimes}$.

Remark 3 *If Δ is a normed BPA (resp. BPP) process in GNF and α is a reachable state of Δ such that $|\alpha| = 1$, then α is composed of a single variable from $Var(\Delta)$. With the help of Remark* **??** *we can conclude that for each state β of Δ there are $v \in Act^*$, $Length(v) = |\beta| - 1$ and $V \in Var(\Delta)$, $|V| = 1$ such that $\beta \stackrel{v}{\to} V$.*

Notation remark 1 *In the rest of this paper we let greek letters $\alpha, \beta, \ldots$ to range over reachable states of a BPA resp. BPP process Δ in GNF. Occasionally we will also use the notation α^i with the following meaning:*

$$\alpha^i = \underbrace{\alpha.\alpha\cdots\alpha}_{i} \quad \text{if } \alpha \text{ is a state of some BPA process in GNF}$$
$$\alpha^i = \underbrace{\alpha\|\alpha\cdots\|\alpha}_{i} \quad \text{if } \alpha \text{ is a state of some BPP process in GNF}$$

2.2 Regular processes

Many proofs in this paper take advantage of the fact that regularity of normed BPA (resp. BPP) processes is decidable. The next definition explains what is meant by the notion of regularity:

Definition 3 *A process Δ is* regular *if there is a process Δ' with finitely many states such that $\Delta \sim \Delta'$.*

It is easy to see that a process is regular iff it can reach only finitely many states up to bisimilarity. In [**?**] it is shown, that regular processes can be represented in the following normal form:

Definition 4 *A regular process Δ is said to be in normal form if all its equations are of the form*

$$X \stackrel{def}{=} \sum_{j=1}^{n} a_j X_j$$

where $n \in N$, $a_j \in Act$ and $X_j \in Var(\Delta)$.

Thus a process Δ is regular iff there is a regular process Δ' in normal form such that $\Delta \sim \Delta'$. Now we present several propositions which concern regularity of normed BPA (resp. BPP) processes. Proofs can be found in [**?**].

Proposition 1 *Let Δ be a normed BPA (resp. BPP) process. It is decidable whether Δ is regular. Moreover, if Δ is regular then a regular process Δ' in normal form such that $\Delta \sim \Delta'$ can be effectively constructed.*

Definition 5 (growing variable) *Let Δ be a normed BPA (resp. BPP) process. A variable $Y \in Var(\Delta)$ is* growing *if $Y \stackrel{w}{\to} Y.\alpha$ (resp. $Y \stackrel{w}{\to} Y\|\alpha$) where $w \in Act^*$ and $\alpha \in Var(\Delta)^*$ such that $Length(\alpha) \geq 1$ (resp. $\alpha \in Var(\Delta)^{\otimes}$ such that $card(\alpha) \geq 1$).*

Proposition 2 *A normed BPA (resp. BPP) process Δ is non-regular iff $Var(\Delta)$ contains a growing variable Y such that there is a reachable state of the form $Y.\alpha$ where $\alpha \in Var(\Delta)^*$ (resp. the state Y is reachable).*

Remark 4 *If Δ is a normed BPA (resp. BPP) process and α is a BPA (resp. BPP) expression whose variables are defined in Δ, then Proposition* **??** *can be applied also to α - each such expression denotes a process in the sense of Remark* **??**. *Namely variables of Δ are BPA (resp. BPP) expressions - hence we can also speak about* regular variables.

The following lemma can be easily proved using Proposition **??** and Remark **??**:

Lemma 1 *Let Δ be a normed BPA (resp. BPP) process which is not regular. Then Δ can reach a state of an arbitrary norm.*

3 The classification of nBPA $\cap$ nBPP

In this section we give an exact classification of normed BPA (resp. BPP) processes which can be equivalently defined using BPP (resp. BPA) syntax.

Definition 6 (the class nBPA $\cap$ nBPP) *Let nBPA (resp. nBPP) denote the class of all normed BPA (resp. BPP) processes. We define the class nBPA $\cap$ nBPP in the following way:*

$$\begin{aligned} nBPA \cap nBPP \;\; = \;\; & \{\Delta \in nBPA \;\mid\; \exists \Delta' \in nBPP \text{ such that } \Delta \sim \Delta'\} \cup \\ & \{\Delta \in nBPP \;\mid\; \exists \Delta' \in nBPA \text{ such that } \Delta \sim \Delta'\} \end{aligned}$$

The class nBPA $\cap$ nBPP can be seen as a "semantic intersection" of nBPA and nBPP. To simplify our analysis, we will often assume that processes we are working with are *reduced*:

Definition 7 (reduced processes) *Let Δ be a normed BPA (resp. BPP) process in GNF. We say that Δ is* reduced *if*

1. *variables of $Var(\Delta)$ are pairwise non-bisimilar*
2. *for each $V \in Var(\Delta)$ there is a reachable state of the form $V.\alpha$ (resp. the state V is reachable).*

As bisimilarity is decidable for normed BPA (resp. BPP) processes (see [?],[?],[?]), the first condition can be assumed w.l.o.g. Variables which do not fulfil the second condition cannot contribute to the behaviour of Δ and they can be effectively recognised (and removed). Hence we can assume (w.l.o.g) that a normed BPA (resp.BPP) process Δ is reduced.

As we shall see, each process of nBPA $\cap$ nBPP can be presented in a special normal form:

Definition 8 (I-normal form) *A normed BPA (resp. BPP) process Δ is said to be in* I-normal form *if all its equations are of the form*

$$X \stackrel{def}{=} \sum_{j=1}^{n} a_j X_j^{k_j}$$

where $n \in N$, $a_j \in Act$, $X_j \in Var(\Delta)$ and $k_j \in N \cup \{0\}$. If aZ^k is a summand in a defining equation of Δ such that $k \geq 2$, then the variable Z is called simple *and its defining equation in Δ is of the form*

$$Z \stackrel{def}{=} \sum_{j=1}^{m} a_j Z^{k_j}$$

where $m \in N$, $a_j \in Act$ and $k_j \in N \cup \{0\}$. Moreover, there is $l \in \{1, \ldots, m\}$ such that $k_l \geq 2$. Finally, we also require that all variables from $Var(\Delta)$ are reachable and pairwise non-bisimilar (i.e. Δ is reduced).

A BPA process in I-normal form has an interesting property - if we modify its defining equations by replacing each occurence of the sequential operator with the parallel operator, we obtain a bisimilar BPP process. An analogous statement holds for BPP processes in I-normal form.

Definition 9 (dual processes) *Let Δ be a BPA (resp. BPP) process in I-normal form. We define the* dual *BPP (resp. BPA) process $\overline{\Delta}$ in the following way:*

- *for each $Y \in Var(\Delta)$ we fix a variable $\overline{Y} \in Var$ such that $\overline{Y} \notin Var(\Delta)$.*

- *defining equations in $\overline{\Delta}$ are obtained from defining equations of Δ by replacing each variable Y by $\overline{Y}$ and each occurence of the sequential (resp. parallel) operator with the parallel (resp. sequential) operator.*

Lemma 2 *Let Δ be a BPP (resp. BPA) process in I-normal form. Then $\Delta \sim \overline{\Delta}$.*

Proof: It is easy to check that the relation

$$R = \{[Y, \overline{Y}] \mid Y \in \mathit{Var}(\Delta)\} \cup \{[Z^i, \overline{Z}^i] \mid Z \in \mathit{Var}(\Delta),\ Z \text{ is simple},\ i \in N \cup \{0\}\}$$

is a bisimulation relating leading variables of $\Delta, \overline{\Delta}$ (see Notation remark **??**). □

An immediate consequence of Lemma **??** is the following proposition:

Proposition 3 *Let Δ be a normed BPA (resp. BPP) process in I-normal form. Then $\Delta \in$ nBPA $\cap$ nBPP.*

Now we prove that each process from nBPA $\cap$ nBPP is bisimilar to a process in I-normal form. Several auxiliary definitions and lemmas are needed:

Definition 10 (Assoc sets) *Let Δ be a normed reduced BPP process in 3-GNF. For each growing variable $Y \in \mathit{Var}(\Delta)$ we define the set $\mathit{Assoc}(Y) \subseteq \mathit{Var}(\Delta)$ in the following way:*

$$\begin{array}{rcl} \mathit{Assoc}(Y) & = & \{P \in \mathit{Var}(\Delta),\ Y \stackrel{u}{\rightarrow} P \text{ for some } u \in \mathit{Act}^*\} \cup \\ & & \{P \in \mathit{Var}(\Delta),\ P \| Y \text{ is a reachable state of } \Delta\} \end{array}$$

A variable $L \in \mathit{Var}(\Delta)$ is lonely *if $L \notin \mathit{Assoc}(Y)$ for any growing variable $Y \in \mathit{Var}(\Delta)$.*

Lemma 3 *Let Δ be a normed reduced BPP process in 3-GNF such that $\Delta \in$ nBPA $\cap$ nBPP. Let $Y \in \mathit{Var}(\Delta)$ be a growing variable. Then there is exactly one variable $Z_Y \in \mathit{Var}(\Delta)$ such that:*

- *Z_Y is non-regular and $|Z_Y| = 1$*
- *if $P \in \mathit{Assoc}(Y)$ then $P \sim Z_Y^{|P|}$ and Z_Y is reachable from P*
- *if $a\alpha$ is a summand in the defining equation for Z_Y in Δ then $\alpha \sim Z_Y^{|\alpha|}$*

Proof: As Y is growing, there are $w \in \mathit{Act}^*$ and $\beta \in \mathit{Var}(\Delta)^{\otimes}$, $\beta \neq \emptyset$ such that $Y \stackrel{w}{\rightarrow} Y \| \beta$. As Δ is normed and in GNF, there are $v \in \mathit{Act}^*$ and $Z_Y \in \mathit{Var}(\Delta)$, $|Z_Y = 1|$ such that $\beta \stackrel{v}{\rightarrow} Z_Y$ (see Remark **??**). Note that Z_Y is reachable from Y. Let $P \in \mathit{Assoc}(Y)$. We prove that $P \sim Z_Y^{|P|}$. First we show that the state $P \| Z_Y^i$ is reachable for each $i \in N$. By the definition of Assoc set there are two possibilities:

1. $Y \stackrel{u}{\rightarrow} P$ for some $u \in \mathit{Act}^*$. As Δ is normed and reduced, the state Y is reachable (see Definition **??**). But then $Y \stackrel{w^i}{\rightarrow} Y \| \beta^i \stackrel{u}{\rightarrow} P \| \beta^i \stackrel{v^i}{\rightarrow} P \| Z_Y^i$
2. $P \| Y$ is a reachable state. As Δ is normed, $Y \stackrel{x}{\rightarrow} \epsilon$ for some $x \in \mathit{Act}^*$. Now $P \| Y \stackrel{w^i}{\rightarrow} P \| Y \| \beta^i \stackrel{x}{\rightarrow} P \| \beta^i \stackrel{v^i}{\rightarrow} P \| Z_Y^i$

As $\Delta \in$ nBPA $\cap$ nBPP, there is a normed BPA process Δ' in GNF such that $\Delta \sim \Delta'$. Let $n = |P|$, $m = \max\{|A|,\ A \in \mathit{Var}(\Delta')\}$. The state $P \| Z_Y^{n.m}$ is a reachable state of Δ and therefore there must be a reachable state $\gamma \in \mathit{Var}(\Delta')^*$ of the process Δ' such that $P \| Z_Y^{n.m} \sim \gamma$. Bisimilar states must have the same norm, hence γ is a sequence of at least $n+1$ variables (see Remark **??**) - $\gamma = A_1.A_2 \ldots A_{n+1}.\delta$ where $\delta \in \mathit{Var}(\Delta')^*$. As $|P| = n$, there is $s \in \mathit{Act}^*$, $\mathit{Length}(s) = n$ such that $P \xrightarrow{s} \epsilon$ - hence $P \| Z_Y^{n.m} \xrightarrow{s} Z_Y^{n.m}$. The state $A_1.A_2 \ldots A_{n+1}.\delta$ must be able to match the norm reducing sequence of actions s. As $\mathit{Length}(s) = n$, only the first n variables of $A_1.A_2 \ldots A_{n+1}.\delta$ can contribute to the sequence s, i.e. $A_1.A_2 \ldots A_{n+1}.\delta \xrightarrow{s} \eta.A_{n+1}.\delta$ where $\eta \in \mathit{Var}(\Delta')^*$. As Δ' is normed, there is $t \in \mathit{Act}^*$, $\mathit{Length}(t) = |\eta|$ such that $\eta.A_{n+1}.\delta \xrightarrow{t} A_{n+1}.\delta$. The state $Z_Y^{n.m}$ can match the norm reducing sequence t in only one way - by removing $\mathit{Length}(t)$ copies of Z_Y (each norm reducing action must be matched by a norm reducing action and $|Z_Y| = 1$):

$$\begin{array}{ccc} P \| Z_Y^{n.m} & \sim & A_1 \ldots A_{n+1}.\delta \\ \big\downarrow{\scriptstyle s} & & \big\downarrow{\scriptstyle s} \\ Z_Y^{n.m} & \sim & \eta.A_{n+1}.\delta \\ \big\downarrow{\scriptstyle t} & & \big\downarrow{\scriptstyle t} \\ Z_Y^{n.m-|\eta|} & \sim & A_{n+1}.\delta \end{array}$$

Now let $k = \mathit{Length}(s) + \mathit{Length}(t)$ (i.e. $k = |A_1 \ldots A_n|$). Clearly $k \leq n.m$. As $|Z_Y| = 1$, there is $p \in \mathit{Act}^*$, $\mathit{Length}(p) = k$ such that $P \| Z_Y^{n.m} \xrightarrow{p} P \| Z_Y^{n.m-k}$. The norm reducing sequence p must be matched by $A_1.A_2 \ldots A_{n+1}.\delta$. As $\mathit{Length}(p) = k = |A_1 \ldots A_n|$, we have $A_1.A_2 \ldots A_{n+1}.\delta \xrightarrow{p} A_{n+1}.\delta$ and $P \| Z_Y^{n.m-k} \sim A_{n+1}.\delta$. By a simple transitivity argument we now obtain $P \| Z_Y^{n.m-k} \sim Z_Y^{n.m-|\eta|}$. From this we can easily conclude that $P \sim Z_Y^{|P|}$ (we can remove all copies of Z_Y from $P \| Z_Y^{n.m-k}$ in $n.m-k$ steps and the state $Z_Y^{n.m-|\eta|}$ must be able to match this sequence of norm reducing actions by removing $n.m-k$ copies of Z_Y - it is easy to see that $n.m - |\eta| - (n.m-k) = |P|$).

As the variable Y is non-regular and $Y \sim Z_Y^{|Y|}$, the variable Z_Y is also non-regular. Moreover, Z_Y is a unique variable with the property $P \sim Z_Y^{|P|}$ for each $P \in \mathit{Assoc}(Y)$. It follows from the fact that Δ is reduced - let us assume that there is another variable $C \in \mathit{Var}(\Delta)$ with this property. Then e.g. $Y \sim C^{|Y|}$ and thus $C^{|Y|} \sim Z_Y^{|Y|}$. From this we can conclude $C \sim Z_Y$ and therefore $C \equiv Z_Y$ because variables of Δ are pairwise non-bisimilar.

A similar argument can be used to prove that Z_Y is reachable from each $P \in \mathit{Assoc}(Y)$. As P is normed, there is a norm reducing sequence $s \in \mathit{Act}^*$ such that $P \xrightarrow{s} P'$ where $P' \in \mathit{Var}(\Delta)$, $|P'| = 1$ (see Remark **??**). As $P \sim Z_Y^{|P|}$ and $Z_Y^{|P|}$ can match the sequence s in only one way (s is norm reducing), we have $Z_Y^{|P|} \xrightarrow{s} Z_Y$ with $P' \sim Z_Y$. This implies $P' \equiv Z_Y$ because Δ is reduced.

It remains to check that if $a\alpha$ is a summand of the defining equation for Z_Y in Δ then $\alpha \sim Z_Y^{|\alpha|}$. But each variable $V \in \alpha$ is reachable from Z_Y and Z_Y is reachable from Y - thus V is reachable from Y and hence $V \in \mathit{Assoc}(Y)$. It means that $V \sim Z_Y^{|V|}$ (for each $V \in \alpha$) and therefore $\alpha \sim Z_Y^{|\alpha|}$. $\square$

Notation remark 2 *In the rest of this paper the notation Z_Y where $Y \in \mathit{Var}(\Delta)$ is a growing variable always denotes the unique variable of Lemma* **??**.

Lemma 4 *Let Δ be a normed reduced BPP process in 3-GNF such that $\Delta \in nBPA \cap nBPP$. Let $A \| B$ be a reachable state of Δ such that $A \in \mathit{Assoc}(Y)$ and $B \in \mathit{Assoc}(Q)$ for some growing variables $Y, Q \in \mathit{Var}(\Delta)$. Then $Z_Y \equiv Z_Q$.*

Proof: As Δ is reduced, it suffices to prove that $Z_Y \sim Z_Q$. As $A \in Assoc(Y)$, the state Z_Y is reachable from A (due to Lemma **??**). Similarly, Z_Q is reachable from B and hence the state $Z_Y \| Z_Q$ is a reachable state of Δ. Now we prove that for each $i \in N$ there is a reachable state $Z_Y \| \alpha_i$ of Δ such that $Z_Y \| Z_Q^i \sim Z_Y \| \alpha_i$. The state Z_Q is non-regular and Δ is normed - hence we can use Lemma **??** and conclude that Z_Q can reach a state α_i such that $|\alpha_i| = i$. Thus the state $Z_Y \| \alpha_i$ is a reachable state of Δ. All variables from α_i belong to $Assoc(Y)$ (because they are reachable from Z_Q and Z_Q is reachable from Q - see Lemma **??**), hence $\alpha_i \sim Z_Q^{|\alpha_i|}$.

Let $m = \max\{|V|,\ V \in Var(\Delta')\}$. As $\Delta \in$ nBPA $\cap$ nBPP, there is a normed BPA process Δ' in GNF such that $\Delta \sim \Delta'$. The state $Z_Y \| \alpha_m$ is a reachable state of Δ and therefore there must be a reachable state γ of Δ' such that $Z_Y \| \alpha_m \sim \gamma$ and hence also $Z_Y \| Z_Q^m \sim \gamma$. Moreover, γ is a sequence of at least two variables.

Now we can use a similar construction as in the proof of Lemma **??** and conclude that $Z_Y \| Z_Q^j \sim Z_Q^{j+1}$ for some $j \in N$. This implies $Z_Y \sim Z_Q$. $\square$

Lemma 5 *Let Δ be a normed reduced BPP process in 3-GNF such that $\Delta \in nBPA \cap nBPP$. Let $L \| A$ be a reachable state of Δ such that L is a lonely variable. Then A is a regular process.*

Proof: Let us assume that A is not regular. Then $A \stackrel{w}{\rightarrow} Y$, where $w \in Act^*$ and $Y \in Var(\Delta)$ is a growing variable (see Proposition **??**). But then $L \| A \stackrel{w}{\rightarrow} L \| Y$, thus $L \in Assoc(Y)$ - we have a contradiction. $\square$

Proposition 4 *Let Δ be a process from $nBPA \cap nBPP$. Then there is a process Δ' in I-normal form such that $\Delta \sim \Delta'$.*

Proof: First we prove that if Δ is a normed BPP process from nBPA $\cap$ nBPP then there is a normed BPP process Δ' in I-normal form such that $\Delta \sim \Delta'$. We can assume (w.l.o.g.) that Δ is reduced and in 3-GNF. The process Δ' can be obtained by the following transformation of defining equations of Δ (which can also add completely new variables and equations to Δ'): if $X \stackrel{def}{=} \sum_{j=1}^{m} a_j \alpha_j$ is a defining equation from Δ then $X \stackrel{def}{=} \sum_{j=1}^{m} T(a_j \alpha_j)$ is added to Δ', where T is defined as follows:

- if $card(\alpha_j) \leq 1$ then $T(a_j \alpha_j) = a_j \alpha_j$
- if $card(\alpha_j) = 2$ (i.e. $\alpha_j = A \| B$) then there are three possibilities:
    1. $A \in Assoc(Y) \wedge B \in Assoc(Q)$ for some growing variables $Y, Q \in Var(\Delta)$. Then $A \sim Z_Y^{|A|}$ and $B \sim Z_Q^{|B|}$ (see Lemma **??**). As $A \| B$ is a reachable state, we can conclude (with the help of Lemma **??**) that $Z_Y \equiv Z_Q$ - hence $A \| B \sim Z_Y^{|A|+|B|}$. Thus $T(a(A \| B)) = a(Z_Y^{|A|+|B|})$.
    2. $A \in Assoc(Y)$ for some growing variable $Y \in Var(\Delta) \wedge B$ is lonely. But then $A \sim Z_Y^{|A|}$ and as Z_Y is not regular, A is not regular as well. As the state $A \| B$ is reachable and B is lonely, it contradicts Lemma **??**. Hence this case is in fact impossible (as well as the case when A is lonely and $B \in Assoc(Q)$).
    3. A and B are lonely. Then A and B are regular (due to Lemma **??**) and therefore the state $A \| B$ is also regular. Each regular process can be represented in normal form (see Definition **??**). Let $\Delta_{A\|B}$ be a regular process in normal form which is bisimilar to $A \| B$. We can assume (w.l.o.g.) that $Var(\Delta_{A\|B}) \cap Var(\Delta') = \emptyset$. We add all equations from $\Delta_{A\|B}$ to Δ' and $T(a(A \| B)) = a.N$ where N is the leading variable of $\Delta_{A\|B}$.

The described transformation preserves bisimilarity because T preserves bisimilarity - hence $\Delta \sim \Delta'$. It remains to check that Δ' is in I-normal form. Clearly each summand of each defining equation from Δ' is of the form which is admitted by I-normal form. If aZ^j is a summand of a defining equation in Δ' such that $j \geq 2$, then $Z \equiv Z_Y$ for some growing variable $Y \in \mathit{Var}(\Delta)$. Let $a\alpha$ be a summand in the original defining equation for Z_Y in Δ. We need to show that each such summand must have been tranformed into $aZ_Y^{|\alpha|}$ by T. But it is obvious as each variable from α belongs to $\mathit{Assoc}(Y)$ (each such variable is reachable from Z_Y and Z_Y is reachable from Y - see Lemma **??**). If α is composed of a single variable V, then $V \equiv Z_Y$ because $V \sim Z_Y$ (due to Lemma **??**) and Δ is reduced. Moreover, at least one summand in the defining equation for Z_Y in Δ' is of the form aZ_Y^l where $l \geq 2$, because Z_Y would be regular otherwise.

To complete the proof we need to show that if Δ is a normed BPA process such that $\Delta \in \text{nBPA} \cap \text{nBPP}$ then there is a normed BPA process Δ' in I-normal form such that $\Delta \sim \Delta'$. As $\Delta \in \text{nBPA} \cap \text{nBPP}$, there is a normed BPP process Δ_P such that $\Delta \sim \Delta_P$. As we just proved, there is a normed BPP process Δ'_P in I-normal form such that $\Delta \sim \Delta_P \sim \Delta'_P$. But for each BPP process in I-normal form it is possible to construct its bisimilar dual BPA process in I-normal form (see Lemma **??**) - hence $\overline{\Delta'_P}$ can serve as Δ'. □

Propositions **??** and **??** give us the first main theorem of this paper:

Theorem 1 (the classification of nBPA $\cap$ nBPP) *The class nBPA $\cap$ nBPP contais exactly normed BPA (resp. BPP) processes in I-normal form (up to bisimilarity).*

4 Deciding whether $\Delta \in$ nBPA $\cap$ nBPP

In this section we prove that it is decidable whether a given normed BPA (resp. BPP) process Δ belongs to nBPA $\cap$ nBPP, i.e. whether there is a normed BPP (resp. BPA) process Δ' such that $\Delta \sim \Delta'$. Moreover, our algorithm is constructive.

Lemma 6 *Let Δ be a normed BPP process in 3-GNF and let Δ' be a normed BPP process in I-normal form such that $\Delta \sim \Delta'$. If $A \| B$ is a reachable state of Δ such that A (resp. B) is non-regular, then there is exactly one simple variable $Z \in \mathit{Var}(\Delta')$ such that $A \sim Z^{|A|}$ (resp. $B \sim Z^{|B|}$).*

Proof: Let us assume that e.g. the variable A is non-regular. Let $n = \max\{|V|,\ V \in \mathit{Var}(\Delta')\}$. As A is non-regular, it can reach a state of an arbitrary norm (see Lemma **??**). Hence $A \stackrel{w}{\rightarrow} \alpha$, where $w \in \mathit{Act}^*$, $\alpha \in \mathit{Var}(\Delta)^{\otimes}$ such that $|\alpha| = n$. The state $\alpha \| B$ is a reachable state of Δ and therefore there must be a reachable state β of Δ' such that $\alpha \| B \sim \beta$. As Δ' is in I-normal form, its reachable states are of the form P^i where $P \in \mathit{Var}(\Delta')$ and $i \in N \cup \{0\}$. Moreover, if $i \geq 2$ then P is a simple variable (see Definition **??**). As $|\alpha \| B| > n$ and bisimilar processes must have the same norm, we can conclude that $\beta \equiv Z^{|\alpha \| B|}$ where $Z \in \mathit{Var}(\Delta)$ is a simple variable. As Δ is normed, $\alpha \| B \stackrel{v}{\rightarrow} B$ where $v \in \mathit{Act}^*$ is a norm reducing sequence of actions. The state $Z^{|\alpha \| B|}$ can match the sequence v only by removing $|\alpha|$ copies of Z - hence $Z^{|\alpha \| B|} \stackrel{v}{\rightarrow} Z^{|B|}$ and $B \sim Z^{|B|}$. The variable Z is clearly unique because Δ' is reduced. □

Definition 11 ($S(\Delta)$ set) *Let Δ be a normed BPP process in GNF. The set $S(\Delta) \subseteq \mathit{Var}(\Delta)$ is composed of all variables V such that $|V| = 1$, V is non-regular and if $a\alpha$ is a summand in the defining equation for V in Δ, then $\alpha \sim V^{|\alpha|}$.*

Lemma 7 *Let Δ be a normed reduced BPP process in 3-GNF such that $\Delta \in nBPA \cap nBPP$. If $a(A\|B)$ is a summand in a defining equation of Δ such that A and B are non-regular, then there is exactly one variable $Z \in S(\Delta)$ such that $A\|B \sim Z^{|A\|B|}$*

Proof: As $\Delta \in$ nBPA $\cap$ nBPP, there is a normed BPP process Δ' in I-normal form such that $\Delta \sim \Delta'$. Let $n = \max\{|V|,\ V \in \mathit{Var}(\Delta')\}$. As $A\|B$ is a reachable state of Δ and variables A, B are non-regular, there are simple variables $Z_1, Z_2 \in \mathit{Var}(\Delta')$ such that $A \sim Z_1^{|A|}$ and $B \sim Z_2^{|B|}$ (this is due to Lemma **??**). As Z_1, Z_2 are simple, the state $Z_1^j\|Z_2^j$ is reachable from $Z_1^{|A|}\|Z_2^{|B|}$ for any $j \in N$ (simple variables are growing and thus non-regular). Now choose $j = n$: as $A\|B \sim Z_1^{|A|}\|Z_2^{|B|}$, the state $A\|B$ can reach a state α which is bisimilar to $Z_1^n\|Z_2^n$. As α is a reachable state of Δ, there must be a state β of Δ' such that $\alpha \sim \beta$. Using the same argument as in the proof of Lemma **??** we can conclude that $\beta \equiv Z_3^{|\alpha|}$, where $Z_3 \in \mathit{Var}(\Delta')$ is a simple variable. Hence $Z_1^n\|Z_2^n \sim Z_3^{|\alpha|}$ and thus $Z_1 \sim Z_3 \sim Z_2$. It implies $Z_1 \equiv Z_3 \equiv Z_2$. To complete the proof, it suffices to realize that each simple variable of Δ' must be bisimilar to some variable of $S(\Delta)$ (because each simple variable of Δ' is a reachable state of Δ' and $\Delta \sim \Delta'$). Moreover, this variable is unique because Δ is reduced. $\square$

Lemma 8 *Let Δ be a normed BPP process in 3-GNF. If $a(A\|B)$ is a summand in a defining equation of Δ such that A is regular and B is non-regular (resp. A is non-regular and B is regular), then $\Delta \notin nBPA \cap nBPP$.*

Proof: Let us assume that $\Delta \in$ nBPA $\cap$ nBPP. Then there is a normed BPP process Δ' in I-normal form such that $\Delta \sim \Delta'$. As $A\|B$ is a reachable state of Δ and B is non-regular, we can use Lemma **??** and conclude that $A \sim Z^{|A|}$ where $Z \in \mathit{Var}(\Delta')$ is a simple variable. As simple variables are growing (see Definition **??** and **??**), they are also non-regular (see Proposition **??**) and hence the state $Z^{|A|}$ is also non-regular. It contradicts the regularity of A. The case when A is non-regular and B is regular is handled in a similar way. $\square$

Proposition 5 *Let Δ be a normed BPP process. It is decidable whether $\Delta \in nBPA \cap nBPP$. Moreover, if $\Delta \in nBPA \cap nBPP$ then a normed BPA process Δ' such that $\Delta \sim \Delta'$ can be effectively constructed.*

Proof: Clearly $\Delta \in$ nBPA $\cap$ nBPP iff there is a normed BPP process Δ_I in I-normal form such that $\Delta \sim \Delta_I$. Now we describe an algorithm $\mathcal{A}$ which inputs Δ and outputs Δ_I if such a Δ_I exists, and answers **NO** otherwise. The algorithm $\mathcal{A}$ first checks whether Δ is regular. If so, it constructs a bisimilar regular process in normal form which can serve as Δ_I (see Definition **??**).

Now assume that Δ is not regular. $\mathcal{A}$ first constructs the set $S(\Delta)$ and then starts to transform Δ into Δ_I by the following tranformation of defining equations of Δ: if $X \stackrel{def}{=} \sum_{j=1}^{m} a_j\alpha_j$ is a defining equation of Δ then $X \stackrel{def}{=} \sum_{j=1}^{m} T(a_j\alpha_j)$ is added to Δ_I, where T is defined as follows:

- if $card(\alpha_j) \leq 1$ then $T(a_j\alpha_j) = a_j\alpha_j$
- if $card(\alpha_j) = 2$ (i.e. $\alpha_j = A\|B$) then there are three possibilities ($\mathcal{A}$ can determine which one actually holds):
    1. A and B are regular (see Remark **??**). Then $A\|B$ is regular and $\mathcal{A}$ constructs a bisimilar regular process $\Delta_{A\|B}$ in normal form such that $\mathit{Var}(\Delta) \cap \mathit{Var}(\Delta_{A\|B}) = \emptyset$. Defining equations of $\Delta_{A\|B}$ are added to Δ_I and $T(a_j\alpha_j) = a_jN$ where N is the leading variable of $\Delta_{A\|B}$.

2. A is regular and B is non-regular (resp. A is non-regular and B is regular). Then $\mathcal{A}$ answers **NO** (see Lemma **??**).
3. A and B are non-regular. Then $\mathcal{A}$ checks whether $A\|B \sim Z^{|A\|B|}$ for some $Z \in S(\Delta)$. If not then $\mathcal{A}$ answers **NO** (this is correct due to Lemma **??**). Otherwise $T(a_j\alpha_j) = a_j Z^{|A\|B|}$.

If the described transformation terminates successfully, then Δ_I is in I-normal form and $\Delta \sim \Delta_I$. If $\mathcal{A}$ answers **NO**, then there is *no* normed BPP process in I-normal form which is bisimilar to Δ. To obtain the BPA process Δ' we simply take $\overline{\Delta_I}$ (see Lemma **??**). $\square$

Proposition 6 *Let Δ be a normed BPA process. It is decidable whether $\Delta \in nBPA \cap nBPP$. Moreover, if $\Delta \in nBPA \cap nBPP$ then a normed BPP process Δ' such that $\Delta \sim \Delta'$ can be effectively constructed.*

Proof: The technique is essentially the same as in the proof of Proposition **??** (it is slightly more complicated). The proof is omitted due to the lack of space. $\square$

As an important consequence of Theorem **??** and Proposition **??** we obtain the following:

Theorem 2 *Bisimilarity is decidable in the union of normed BPA and normed BPP processes.*

Proof: Given two normed BPA (resp. BPP) processes, it is possible to check bisimilarity using algorithms which were published in [?],[?],[?]. If we obtain a normed BPA process Δ_1 and a normed BPP process Δ_2, then we run the algorithm $\mathcal{A}$ from Proposition **??** on Δ_2. If $\mathcal{A}$ answers **NO** then $\Delta_1 \not\sim \Delta_2$ (due to Theorem **??**). Otherwise we check bisimilarity between Δ_1 and Δ_2', where Δ_2' is the output of $\mathcal{A}$. $\square$

5 Conclusions and Future Research

We have examined the class nBPA $\cap$ nBPP of those transition systems which can be generated by both normed BPA and BPP processes, i.e. the class of transition systems which can be equivalently (up to bisimilarity) expressed within the syntax of normed BPA processes and that of normed BPP processes as well. We have shown that this class is formed by exactly those processes for which a bisimilar process in I-normal form exists (here 'I' stands for *I*ntersection).

Also we have shown it is decidable whether for a given normed BPA (resp. BPP) process Δ there exists a normed BPP (resp. BPA) process Δ' such that $\Delta \sim \Delta'$. Moreover, if the answer is positive then the provided algorithm constructs this bisimilar process Δ' (in I-normal form). As an immediate consequence the decidability of bisimulation equivalence in the union of normed BPA and normed BPP processes is obtained.

We hope this work can be considered as one of the steps towards a solution of the open problem whether bisimilarity is decidable for PA processes. Furthermore, we would like to examine deeper the relationship between classes of behaviours which are generated by different types of syntax (e.g. Petri nets and BPA) and provide similar results like in the case of normed BPA and normed BPP processes - i.e. to classify the "semantical intersection" and design algorithms which can test (constructively) the membership to this intersection for both types of syntax. Last but not least corresponding complexity results should be provided.

Our result about classification of nBPA $\cap$ nBPP might be of some interest from the point of view of formal languages/automata theory as well. The I-normal form (for normed BPA processes) can be taken as a special type of CF grammars which generate languages of the form $R.(L_1 \cup \ldots \cup L_n)$, where R is regular and each L_i can be generated by a CF grammar having just one nonterminal and rules of the form $Z \to aZ^k,\ k \geq 0$. Considering language equivalence only, it is obvious that languages of the mentioned type $R.(L_1 \cup \ldots \cup L_n)$ can be recognized by nondeterministic one counter automata. Hence our result on classification of nBPA $\cap$ nBPP can be considered as a refinement of the result achieved in [**?**] on the context-freeness of languages generated by Petri nets, as BPP processes form a proper subclass of Petri nets.

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