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Comparing expressibility of normed BPA and normed BPP processes*

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Summary. We present an exact characterization of those transition systems which can be equivalently (up to bisimilarity) defined by the syntax of normed BPA_τ and normed BPP_τ processes. We give such a characterization for the subclasses of normed BPA and normed BPP processes as well.

Next we demonstrate the decidability of the problem whether for a given normed BPA_τ process Δ there is some unspecified normed BPP_τ process Δ' such that Δ and Δ' are bisimilar. The algorithm is polynomial. Furthermore, we show that if the answer to the previous question is positive, then (an example of) the process Δ' is effectively constructible. Analogous algorithms are provided for normed BPP_τ processes. Simplified versions of the mentioned algorithms which work for normed BPA and normed BPP are given too. As a simple consequence we obtain the decidability of bisimilarity in the union of normed BPA_τ and normed BPP_τ processes.

1 Introduction

The semantics of concurrent processes is often understood in terms of labelled transition systems. The 'sameness' of two processes is then formally defined as an equivalence over the class of transition systems. There are various approaches to this problem, and many 'behavioural' equivalences have been proposed in the literature (see e.g. [17] for an overview). *Bisimulation equivalence* (bisimilarity), due to Park [15] and Milner [14], seems to be of special importance as its accompanying theory has been developed very intensively.

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Bearing in mind bisimulation we study the relationship between the classes of transition systems which are generated by normed BPA$_\tau$ [2] and normed BPP$_\tau$ [6] processes. We also examine such a relationship between their respective proper subclasses formed by normed BPA and normed BPP processes.

BPA processes can be seen as simple sequential programs (they are equipped with a binary sequential operator). This class of processes has been intensively studied by many researchers. Baeten, Bergstra, and Klop proved in [1] that bisimilarity is decidable for normed BPA processes. Within the classical language theory this class corresponds to context-free grammars without irredundant nonterminals and without ϵ-rules. Their proof is based on isolating a complex periodicity in transition graphs of these processes.

Much simpler proofs of this result were later given in [4], [12], and [9], utilizing algebraic properties of this class. Hirshfeld, Jerrum, and Moller demonstrated in [10] that the problem is decidable in polynomial time. The decidability result was later extended to the whole class of BPA processes by Christensen, Hüttel, and Stirling in [8].

If the binary sequential operator is replaced by the parallel one, the class of BPP processes is obtained. Hence, BPP can be seen as a class of simple parallel programs. Christensen, Hirshfeld, and Moller proved in [7] that bisimilarity is decidable for BPP processes. A polynomial decision algorithm for normed BPP processes was presented in [11] by Hirshfeld, Jerrum, and Moller.

If the operator of parallel composition does not specify just merge, but it is enriched to define also an internal synchronous communication between two BPP processes resulting in a special action τ, one obtains the class of BPP$_\tau$ processes [6]. In order to compare this class with its sequential counterpart we employ the class of BPA$_\tau$ processes [2]. Bisimilarity remains decidable in these process classes.

An interesting problem is, what is the exact relationship between BPA$_\tau$ and BPP$_\tau$ processes, i.e. what is the relationship between sequencing and parallelism. We answer this question for normed subclasses of the processes just mentioned. Moreover, we also show how the obtained results can be applied to normed BPA and normed BPP processes (some of these specialized results have been independently achieved by Blanco in [3] – see Sect. 5 for a more detailed discussion).

Our paper is organized as follows. First we introduce basic definitions and recall some known results which are employed in subsequent proofs. An exact characterization of those behaviours (transition systems) which can be equivalently (up to bisimilarity) described by the syntax of normed BPP$_\tau$ and normed BPA$_\tau$ processes is given in Sect. 3. Next we show that if we restrict ourselves to normed BPA and normed BPP processes, a quite simple and (hopefully) nice characterization of those behaviours which are common to these subclasses is obtained. In Sect. 4 we demonstrate decidability of the problem whether for a given normed BPA_τ, BPP_τ, BPA, or BPP process Δ there is some unspecified bisimilar BPP_τ, BPA_τ, BPP, or BPA process Δ', respectively. These algorithms are polynomial. We also show that if the answer to the previous question is positive, then the process Δ' is effectively constructible. Hence, as an important consequence we also obtain decidability of bisimulation equivalence in the union of normed BPA_τ and normed BPP_τ processes.

2 Definitions

2.1 BPA and BPP processes

Let $\Lambda = \{a, b, c, \ldots\}$ be a countably infinite set of *atomic actions* such that for every $a \in \Lambda$ there is its corresponding *dual* action $\overline{a}$ with the convention that $\overline{\overline{a}} = a$. Let $Act = \Lambda \cup \{\tau\}$ where $\tau \notin \Lambda$ is a special (silent) action. Let $Var = \{X, Y, Z, \ldots\}$ be a countably infinite set of *variables* such that $Var \cap Act = \emptyset$. The classes of BPA, BPP, BPA_τ, and BPP_τ expressions are defined by the following abstract syntax equations:

$$
\begin{array}{lcl}
E_{\scriptscriptstyle BPA} & ::= & \epsilon \;\mid\; b \;\mid\; X \;\mid\; bE_{\scriptscriptstyle BPA} \;\mid\; E_{\scriptscriptstyle BPA}.E_{\scriptscriptstyle BPA} \;\mid\; E_{\scriptscriptstyle BPA} + E_{\scriptscriptstyle BPA} \\
E_{\scriptscriptstyle BPP} & ::= & \epsilon \;\mid\; b \;\mid\; X \;\mid\; bE_{\scriptscriptstyle BPP} \;\mid\; E_{\scriptscriptstyle BPP} \| E_{\scriptscriptstyle BPP} \;\mid\; E_{\scriptscriptstyle BPP} + E_{\scriptscriptstyle BPP} \\
E_{\scriptscriptstyle BPA_\tau} & ::= & \epsilon \;\mid\; a \;\mid\; X \;\mid\; aE_{\scriptscriptstyle BPA_\tau} \;\mid\; E_{\scriptscriptstyle BPA_\tau}.E_{\scriptscriptstyle BPA_\tau} \;\mid\; E_{\scriptscriptstyle BPA_\tau} + E_{\scriptscriptstyle BPA_\tau} \\
E_{\scriptscriptstyle BPP_\tau} & ::= & \epsilon \;\mid\; a \;\mid\; X \;\mid\; aE_{\scriptscriptstyle BPP_\tau} \;\mid\; E_{\scriptscriptstyle BPP_\tau} | E_{\scriptscriptstyle BPP_\tau} \;\mid\; E_{\scriptscriptstyle BPP_\tau} + E_{\scriptscriptstyle BPP_\tau}
\end{array}
$$

Here 'b' ranges over Λ, 'a' ranges over Act, and 'X' ranges over Var. Intuitively, 'ϵ' models a successfully terminated process, 'b' is an observable computational step, 'τ' is an internal (not observable) computational step, '.' is sequencing, '$\|$', '$|$' are parallel compositions, and '$+$' is a nondeterministic choice.

In the rest of this paper we do not distinguish between expressions related by *structural congruence* which is the smallest congruence relation over process expressions such that the following laws hold:

- associativity and 'ϵ' as a unit for '.', '$\|$', '$|$', and '$+$'
- commutativity for '$\|$', '$|$', and '$+$'
- $a\epsilon = a$

As usual, we restrict our attention to *guarded* expressions. A process expression E is guarded if every variable occurrence in E is within the scope of an atomic action.

Table 1. SOS rules

$\dfrac{}{aE \xrightarrow{a} E}$ $\dfrac{F \xrightarrow{a} F'}{E+F \xrightarrow{a} F'}$	$\dfrac{E \xrightarrow{a} E'}{E.F \xrightarrow{a} E'.F}$	$\dfrac{E \xrightarrow{a} E'}{E+F \xrightarrow{a} E'}$
$\dfrac{E \xrightarrow{a} E'}{E\Vert F \xrightarrow{a} E'\Vert F}$ $\dfrac{F \xrightarrow{a} F'}{E\vert F \xrightarrow{a} E\vert F'}$	$\dfrac{F \xrightarrow{a} F'}{E\Vert F \xrightarrow{a} E\Vert F'}$	$\dfrac{E \xrightarrow{a} E'}{E\vert F \xrightarrow{a} E'\vert F}$
$\dfrac{E \xrightarrow{b} E' \quad F \xrightarrow{\bar{b}} F'}{E\vert F \xrightarrow{\tau} E'\vert F'} (b \neq \tau)$	$\dfrac{E \xrightarrow{a} E'}{X \xrightarrow{a} E'}\ (X \overset{def}{=} E \in \Delta)$	

A *guarded BPA, BPP, BPA$_\tau$, or BPP$_\tau$ process* is defined by a finite family Δ of recursive process equations

$$\Delta = \{X_i \overset{def}{=} E_i \quad | \quad 1 \leq i \leq n\}$$

where X_i are distinct elements of *Var* and E_i are guarded BPA, BPP, BPA$_\tau$, or BPP$_\tau$ expressions, containing variables of $\{X_1, \ldots, X_n\}$. The set of variables which appear in Δ is denoted by *Var*(Δ).

The variable X_1 plays a special role (X_1 is sometimes called the *leading variable*) – it is a root of a labelled transition system, defined by the process Δ and the rules of Table 1.

Nodes of the transition system generated by Δ are BPA, BPP, BPA$_\tau$, or BPP$_\tau$ expressions, which are often called *states of* Δ, or just 'states' when Δ is understood from the context. We also extend the notation $E \xrightarrow{a} F$ to elements of *Act** in an obvious way (we often write $E \rightarrow^* F$ instead of $E \xrightarrow{w} F$ if $w \in$ *Act** is irrelevant). Given two states E, F, we say that F is *reachable from* E, if $E \rightarrow^* F$. States of Δ which are reachable from X_1 are said to be *reachable*.

Remark 1 Processes are often identified with their leading variables. Furthermore, if we assume a fixed process Δ, we can view any process expression E (not necessarily guarded) whose variables are defined in Δ as a process – if we denote this process by Δ', then the leading equation of Δ' is $X \overset{def}{=} E'$ where $X \notin$ *Var*(Δ) and E' is a process expression obtained from E by substituting each variable in E with the right-hand side of its corresponding defining equation in Δ (E' must be guarded now). Moreover, defining equations of Δ are added to Δ'. All notions originally defined for processes can also be used for process expressions in this sense.

2.1.1 Bisimulation The equivalence between process expressions (states) we are interested in here is *bisimilarity* [15], defined as follows:

Definition 1 *A binary relation R over process expressions is a* bisimulation *if whenever $(E, F) \in R$ then for each $a \in Act$*

- *if $E \stackrel{a}{\to} E'$, then $F \stackrel{a}{\to} F'$ for some F' such that $(E', F') \in R$*
- *if $F \stackrel{a}{\to} F'$, then $E \stackrel{a}{\to} E'$ for some E' such that $(E', F') \in R$*

Processes Δ and Δ' are bisimilar*, written $\Delta \sim \Delta'$, if their leading variables are related by some bisimulation.*

2.1.2 Normed processes Important subclasses of BPA, BPP, BPA$_\tau$, and BPP$_\tau$ processes can be obtained by an extra restriction of *normedness*. A variable $X \in \mathit{Var}(\Delta)$ is *normed* if there is $w \in \mathit{Act}^*$ such that $X \stackrel{w}{\to} \epsilon$. In that case we define the *norm* of X, written $|X|$, to be the length of the shortest such w. In case of BPP$_\tau$ processes we also require that no τ action which appears in w is a result of communication on dual actions in the sense of operational semantics given in Table 1. This is necessary if we want the norm to be additive over the '|' operator (τ may still occur in w, as it can also be used as an action prefix). A process Δ is *normed* if all variables of $\mathit{Var}(\Delta)$ are normed. The norm of Δ is then defined to be the norm of X_1.

Remark 2 As normed processes are intensively studied in this paper, we emphasize some properties of the norm:

- Note the norm of a normed process is easy to compute by the following rules: $|a| = 1$, $|E+F| = \min\{|E|, |F|\}$, $|E.F| = |E|+|F|$, $|E\|F| = |E| + |F|$, $|E|F| = |E| + |F|$, and if $X_i \stackrel{\mathit{def}}{=} E_i$ and $|E_i| = n$, then $|X_i| = n$.
- Bisimilar processes must have the same norm.

In the rest of this paper we denote the normed subclasses of BPA, BPP, BPA$_\tau$, and BPP$_\tau$ processes by nBPA, nBPP, nBPA$_\tau$, and nBPP$_\tau$, respectively.

2.1.3 Greibach normal form Any BPA, BPP, BPA$_\tau$, and BPP$_\tau$ process Δ can be effectively presented in a special normal form which is called 3-Greibach normal form by analogy with CF grammars (see [1] and [6]). Before the definition we need to introduce the set $\mathit{Var}(\Delta)^*$ of all finite sequences of variables from $\mathit{Var}(\Delta)$, and the set $\mathit{Var}(\Delta)^\otimes$ of all finite multisets over $\mathit{Var}(\Delta)$. Each multiset α of $\mathit{Var}(\Delta)^\otimes$ denotes a BPP (or BPP$_\tau$) expression which can be obtained by combining elements of α in parallel using the '$\|$' operator (or the '|' operator).

Definition 2 *A BPA (or BPA$_\tau$) process Δ is said to be in* Greibach normal form (GNF) *if all its equations are of the form*

$$X \stackrel{\mathit{def}}{=} \sum_{j=1}^{n} a_j \alpha_j$$

where $n \in \mathbb{N}$, $a_j \in \Lambda$ *(or* $a_j \in$ *Act), and* $\alpha_j \in$ *Var*$(\Delta)^*$. *We also require that for every* $Y \in$ *Var*(Δ) *there is a reachable state of the form* $Y.\beta$. *If length*$(\alpha_j) \leq 2$ *for each* $j,\ 1 \leq j \leq n$*, then* Δ *is said to be in 3-GNF.*

Definition 3 *A BPP (or BPP*$_\tau$*) process* Δ *is said to be in* Greibach normal form (GNF) *if all its equations are of the form*

$$X \stackrel{\scriptscriptstyle def}{=} \sum_{j=1}^{n} a_j \alpha_j$$

where $n \in \mathbb{N}$, $a_j \in \Lambda$ *(or* $a_j \in$ *Act), and* $\alpha_j \in$ *Var*$(\Delta)^{\otimes}$. *We also require that every* $Y \in$ *Var*(Δ) *appears in some reachable state. If card*$(\alpha_j) \leq 2$ *for each* $j,\ 1 \leq j \leq n$*, then* Δ *is said to be in 3-GNF.*

From now on we assume that all BPA, BPP, BPA$_\tau$, and BPP$_\tau$ processes we work with are presented in GNF. This justifies also the assumption that all reachable states of a BPA or BPA$_\tau$ process Δ are elements of *Var*$(\Delta)^*$, and all reachable states of a BPP or BPP$_\tau$ process Δ' are elements of *Var*$(\Delta')^{\otimes}$.

Remark 3 In the rest of this paper we let Greek letters $\alpha, \beta, \ldots$ range over reachable states of a BPA, BPP, BPA$_\tau$, or BPP$_\tau$ process. Occasionally we also use the notation α^i with the following meaning:

$$\alpha^i = \underbrace{\alpha.\alpha \cdots .\alpha}_{i} \quad \text{if } \alpha \text{ is a state of some BPA or BPA}_\tau \text{ process}$$

$$\alpha^i = \underbrace{\alpha \| \alpha \cdots \| \alpha}_{i} \quad \text{if } \alpha \text{ is a state of some BPP process}$$

$$\alpha^i = \underbrace{\alpha | \alpha \cdots | \alpha}_{i} \quad \text{if } \alpha \text{ is a state of some BPP}_\tau \text{ process}$$

2.2 Regular processes

In this paper some proofs make use of the fact that regularity of nBPA, nBPP, nBPA$_\tau$, and nBPP$_\tau$ processes is decidable in polynomial time. The following definition explains what is meant by the notion of regularity and introduces standard normal form for regular processes.

Definition 4 *A process* Δ *is* regular *if there is a process* Δ' *with finitely many states such that* $\Delta \sim \Delta'$. *A regular process* Δ *is said to be in normal form if all its equations are of the form*

$$X \stackrel{\scriptscriptstyle def}{=} \sum_{j=1}^{n} a_j [X_j]$$

where $n \in \mathbb{N}$, $a_j \in$ *Act, and* $X_j \in$ *Var*(Δ) *(square brackets indicate an optional occurrence).*

It is easy to see that a process is regular iff it can reach only finitely many states up to bisimilarity. In [14] it is shown that regular processes can be represented in the normal form just defined. Thus a process Δ is regular iff there is a regular process Δ' in normal form such that $\Delta \sim \Delta'$. Now we present several propositions which concern regularity of nBPA, nBPP, nBPA$_\tau$, and nBPP$_\tau$ processes. Proofs can be found in [13].

Proposition 1 *Let Δ be a nBPA, nBPP, nBPA$_\tau$, or nBPP$_\tau$ process. The problem whether Δ is regular is decidable in polynomial time. Moreover, if Δ is regular then a regular process Δ' in normal form such that $\Delta \sim \Delta'$ is effectively constructible.*

Definition 5 *Let Δ be a nBPA, nBPP, nBPA$_\tau$, or nBPP$_\tau$ process. A variable $Y \in Var(\Delta)$ is* growing *if $Y \rightarrow^* Y.\alpha$, $Y \rightarrow^* Y\|\alpha$, $Y \rightarrow^* Y.\alpha$, or $Y \rightarrow^* Y|\alpha$, respectively, where α is a nonempty state of Δ.*

Proposition 2 *A nBPA, nBPP, nBPA$_\tau$, or nBPP$_\tau$ process Δ is non-regular iff $Var(\Delta)$ contains a growing variable.*

3 A Characterization of nBPA$_\tau$ $\cap$ nBPP$_\tau$

In this section we give an exact characterization of those normed processes which can be equivalently defined in BPA$_\tau$ and BPP$_\tau$ syntax.

Definition 6 *The* semantical intersection *of nBPA$_\tau$ and nBPP$_\tau$ processes is defined as follows:*

$$\begin{aligned} nBPA_\tau \cap nBPP_\tau = \; & \{\Delta \in nBPA_\tau, \;\mid\; \exists \Delta' \in nBPP_\tau \;\text{ such that } \Delta \sim \Delta'\} \cup \\ & \{\Delta \in nBPP_\tau, \;\mid\; \exists \Delta' \in nBPA_\tau \;\text{ such that } \Delta \sim \Delta'\} \end{aligned}$$

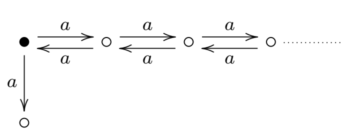
The class nBPA$_\tau$ $\cap$ nBPP$_\tau$ is clearly nonempty as each normed finite-state process belongs to nBPA$_\tau$ $\cap$ nBPP$_\tau$. However, nBPA$_\tau$ $\cap$ nBPP$_\tau$ contains also processes with infinitely many states – consider the following process:

(1) $$X \stackrel{def}{=} a(X|X) + a$$

X is a nBPP$_\tau$ process with infinitely many states. If the '|' operator is replaced by the '.' operator, we obtain a bisimilar nBPA$_\tau$ process:

(2) $$\overline{X} \stackrel{def}{=} a(\overline{X}.\overline{X}) + a$$

Clearly $X \sim \overline{X}$ because transition systems generated by those processes are even isomorphic (see the picture below).



Now we slightly modify the process X.

(3) $$X \overset{\text{def}}{=} a(X|X) + a + \overline{a}$$

Although the process (3) does not differ from the process (1) too much, it is not hard to prove that there is *no* nBPA$_\tau$ process bisimilar to (3).

Now we prove that every nBPP$_\tau$ process of nBPA$_\tau \cap$ nBPP$_\tau$ can be represented in a special normal form, denoted INF$_{\text{BPP}}$ (**I**ntersection **N**ormal **F**orm for nBPP$_\tau$ processes). In order to define INF$_{\text{BPP}}$, we need to introduce the notion of a *reduced* process:

Definition 7 *Let Δ be a nBPA$_\tau$ or nBPP$_\tau$ process. We say that Δ is* reduced *if all its variables are pairwise non-bisimilar.*

As bisimilarity is decidable for nBPA$_\tau$ and nBPP$_\tau$ processes in polynomial time [10,11], every nBPA$_\tau$ and nBPP$_\tau$ process can be effectively transformed to a bisimilar reduced process in polynomial time.

Definition 8 *Let Δ be a reduced nBPP$_\tau$ process.*

1. *A variable $Z \in Var(\Delta)$ is* simple *if all the summands in the defining equation for Z are of the form aZ^i, where $a \in Act$ and $i \in \mathbb{N}_0$. Moreover, at least one of those summands must be of the form aZ^k, where $a \in Act$ and $k \geq 2$. Finally, the defining equation for Z must not contain two summands of the form $b, \overline{b}$, where $b \in \Lambda$.*
2. *The process Δ is said to be in INF$_{BPP}$ if the following condition holds: whenever $a\alpha$ is a summand in a defining equation of Δ such that length$(\alpha) \geq 2$, then $\alpha = Z^i$ for some simple variable Z and $i \geq 2$.*

Note that if Z is a simple variable, then $|Z| = 1$ because Z could not be normed otherwise.

Example 1 Note the process (1) is INF$_{\text{BPP}}$, while the processes (3) is not. Conditions of INF$_{\text{BPP}}$ are also satisfied by the following process:

$$X \overset{\text{def}}{=} aY + b(Z|Z) + b + \overline{b}$$
$$Y \overset{\text{def}}{=} cY + bX + a(Z|Z|Z)$$
$$Z \overset{\text{def}}{=} a(Z|Z) + \overline{a}(Z|Z|Z) + b + \overline{a}$$

Remark 4 The set of all reachable states of a process Δ in INF$_{\text{BPP}}$ looks as follows:

$$Var(\Delta) \cup \{Z^i \mid Z \in Var(\Delta) \text{ is a simple variable and } i \in \mathbb{N}_0\}$$

Proposition 3 *Each process Δ in INF$_{BPP}$ belongs to nBPA$_\tau \cap$ nBPP$_\tau$.*

Proof. We show that a bisimilar nBPA$_\tau$ process $\overline{\Delta}$ is even effectively constructible. First we need to define the notion of a *closed* simple variable – a simple variable $Z \in \mathit{Var}(\Delta)$ is closed if the following condition holds: if the defining equation for Z contains two summands of the form $bZ^i, \overline{b}Z^j$, then it also contains a summand τZ^{i+j-1} (the case $i = j = 0$ is impossible by Definition 8).

The set $\mathit{Var}(\overline{\Delta})$ looks as follows: for each $V \in \mathit{Var}(\Delta)$ we fix a fresh variable $\overline{V}$. Moreover, for each simple non-closed variable $Z \in \mathit{Var}(\Delta)$ we also fix a fresh variable $\overline{Z}_c$. Now we can start to transform Δ to $\overline{\Delta}$. For each equation $Y \stackrel{\mathit{def}}{=} \sum_{i=1}^n a_i\alpha_i$ of Δ we add the equation $\overline{Y} \stackrel{\mathit{def}}{=} \sum_{i=1}^n \mathcal{T}(a_i\alpha_i)$ to $\overline{\Delta}$, where $\mathcal{T}$ is defined as follows:

1. $\mathcal{T}(a_i) = a_i$
2. $\mathcal{T}(a_iV) = a_i\overline{V}$ for each $V \in \mathit{Var}(\Delta)$.
3. If $\alpha_i = Z^j$ where $j \geq 2$ and $Z \in \mathit{Var}(\Delta)$ is a closed simple variable, then $\mathcal{T}(a_iZ^j) = a_i\overline{Z}^j$.
4. If $\alpha_i = Z^j$ where $j \geq 2$ and $Z \in \mathit{Var}(\Delta)$ is a non-closed simple variable, then $\mathcal{T}(a_iZ^j) = a_i\overline{Z}_c^{\,j-1}.\overline{Z}$.

The defining equation for $\overline{Z}_c$ is constructed using the following rules:

1. If aZ^i is a summand in the defining equation for Z, then $a\overline{Z}_c^{\,i}$ is a summand in the defining equation for $\overline{Z}_c$ in $\overline{\Delta}$.
2. If $bZ^i, \overline{b}Z^j$ are summands in the defining equation for Z, then $\tau\overline{Z}_c^{\,i+j-1}$ is a summand in the defining equation for $\overline{Z}_c$ in $\overline{\Delta}$.

The fact $\Delta \sim \overline{\Delta}$ is easy to check. □

Example 2 If we apply the transformation algorithm to the process of Example 1, we obtain the following bisimilar nBPA$_\tau$ process:

$$\begin{aligned}
\overline{X} &\stackrel{\mathit{def}}{=} a\overline{Y} + b(\overline{Z}_c.\overline{Z}) + b + \overline{b}\\
\overline{Y} &\stackrel{\mathit{def}}{=} c\overline{Y} + b\overline{X} + a(\overline{Z}_c.\overline{Z}_c.\overline{Z})\\
\overline{Z} &\stackrel{\mathit{def}}{=} a(\overline{Z}_c.\overline{Z}) + \overline{a}(\overline{Z}_c.\overline{Z}_c.\overline{Z}) + b + \overline{a}\\
\overline{Z}_c &\stackrel{\mathit{def}}{=} a(\overline{Z}_c.\overline{Z}_c) + \overline{a}(\overline{Z}_c.\overline{Z}_c.\overline{Z}_c) + b + \overline{a} + \tau(\overline{Z}_c.\overline{Z}_c.\overline{Z}_c.\overline{Z}_c) + \tau\overline{Z}_c
\end{aligned}$$

Now we prove that every nBPP$_\tau$ process of nBPA$_\tau \cap$ nBPP$_\tau$ is bisimilar to a process in INF$_{\mathrm{BPP}}$. Several auxiliary definitions and lemmas are needed:

Definition 9 *Let Δ be a nBPP$_\tau$ process. For each growing variable $Y \in \mathit{Var}(\Delta)$ we define the set $\mathit{Assoc}(Y) \subseteq \mathit{Var}(\Delta)$ in the following way:*

$$\begin{aligned}
\mathit{Assoc}(Y) = \{&P \in \mathit{Var}(\Delta),\ Y \to^* P\} \cup \\
\{&P \in \mathit{Var}(\Delta),\ P|Y \textit{ is a reachable state of } \Delta\}
\end{aligned}$$

A variable $L \in \mathit{Var}(\Delta)$ is lonely *if $L \notin \mathit{Assoc}(Y)$ for any growing variable $Y \in \mathit{Var}(\Delta)$.*

Lemma 1 *Let Δ be a reduced nBPP$_\tau$ process which belongs to nBPA$_\tau$ $\cap$ nBPP$_\tau$. Let $Y \in Var(\Delta)$ be a growing variable. Then there is exactly one variable $Z_Y \in Var(\Delta)$ such that the following conditions hold:*

- *Z_Y is non-regular and $|Z_Y| = 1$.*
- *If $P \in Assoc(Y)$, then Z_Y is reachable from P and $P \sim Z_Y^{|P|}$.*
- *If $a\alpha$ is a summand in the defining equation for Z_Y in Δ, then $\alpha \sim Z_Y^{|\alpha|}$.*

Proof. As Y is growing, $Y \rightarrow^* Y|\beta$ where $\beta \in Var(\Delta)^{\otimes}$, $\beta \neq \emptyset$. As Δ is normed and in GNF, there is $Z_Y \in Var(\Delta)$, $|Z_Y| = 1$, such that $\beta \rightarrow^* Z_Y$. Hence $Y \rightarrow^* Y|\beta^i \rightarrow^* Y|Z_Y^i$ for every $i \in \mathbb{N}$ (note that Z_Y is reachable from Y). From this and the definition of *Assoc* set we can easily conclude that if $P \in Assoc(Y)$ then the state $P|Z_Y^i$ is reachable for every $i \in \mathbb{N}$.

As $\Delta \in$ nBPA$_\tau \cap$ nBPP$_\tau$, there is a nBPA$_\tau$ process Δ' in GNF such that $\Delta \sim \Delta'$. Let $n = |P|$, $m = \max\{|A|,\ A \in Var(\Delta')\}$. The state $P|Z_Y^{n.m}$ is a reachable state of Δ and therefore there is $\gamma \in Var(\Delta')^*$ such that $P|Z_Y^{n.m} \sim \gamma$. Bisimilar states must have the same norm, hence γ is a sequence of at least $n+1$ variables – $\gamma = A_1.A_2 \ldots A_{n+1}.\delta$ where $\delta \in Var(\Delta')^*$. As $|P| = n$, $P \xrightarrow{s} \epsilon$ for some $s \in Act^*$ with $length(s) = n$, hence $P|Z_Y^{n.m} \xrightarrow{s} Z_Y^{n.m}$. The state $A_1.A_2 \ldots A_{n+1}.\delta$ must be able to match the norm reducing sequence of actions s. As $length(s) = n$, at most the first n variables of $A_1.A_2 \ldots A_{n+1}.\delta$ can contribute to the sequence s, i.e. $A_1.A_2 \ldots A_{n+1}.\delta \xrightarrow{s} \eta.A_{n+1}.\delta$ where $\eta \in Var(\Delta')^*$. As Δ' is normed, $\eta.A_{n+1}.\delta \xrightarrow{t} A_{n+1}.\delta$ for some $t \in Act^*$ with $length(t) = |\eta|$. The state $Z_Y^{n.m}$ can match the sequence t only by removing $length(t)$ copies of Z_Y.

$$\begin{array}{ccc} P|Z_Y^{n.m} & \sim & A_1 \ldots A_{n+1}.\delta \\ \big\downarrow s & & \big\downarrow s \\ Z_Y^{n.m} & \sim & \eta.A_{n+1}.\delta \\ \big\downarrow t & & \big\downarrow t \\ Z_Y^{n.m-|\eta|} & \sim & A_{n+1}.\delta \end{array}$$

Now let $k = length(s) + length(t)$ (i.e. $k = |A_1 \ldots A_n|$). Clearly $k \leq n.m$ and as $|Z_Y| = 1$, $P|Z_Y^{n.m} \xrightarrow{p} P|Z_Y^{n.m-k}$ where $length(p) = k$. The state $A_1.A_2 \ldots A_{n+1}.\delta$ can match the sequence p only by $A_1.A_2 \ldots A_{n+1}.\delta \xrightarrow{p} A_{n+1}.\delta$. By transitivity of $\sim$ we now obtain $P|Z_Y^{n.m-k} \sim Z_Y^{n.m-|\eta|}$, hence $P \sim Z_Y^{|P|}$.

As the variable Y is non-regular and $Y \sim Z_Y^{|Y|}$, the variable Z_Y is also non-regular. Moreover, Z_Y is a unique variable with the property $P \sim Z_Y^{|P|}$ for every $P \in Assoc(Y)$, because Δ is reduced.

A similar argument can be used to prove that Z_Y is reachable from each $P \in \mathit{Assoc}(Y)$. As P is normed, $P \to^* P'$ where $|P'| = 1$. As $P \sim Z_Y^{|P|}$, $P' \sim Z_Y$ and hence $P' = Z_Y$.

It remains to check that if $a\alpha$ is a summand of the defining equation for Z_Y, then $\alpha \sim Z_Y^{|\alpha|}$. But each variable $V \in \alpha$ belongs to $\mathit{Assoc}(Y)$ (because $Y \to^* Z_Y \to^* V$) and thus $V \sim Z_Y^{|V|}$. Hence $\alpha \sim Z_Y^{|\alpha|}$. □

Remark 5 The symbol Z_Y always denotes the unique variable of Lemma 1 in the rest of this paper.

Lemma 2 *Let $\Delta \in nBPA_\tau \cap nBPP_\tau$ be a reduced $nBPP_\tau$ process. Let $A|B$ be a reachable state of Δ such that $A \in Assoc(Y)$ and $B \in Assoc(Q)$. Then $Z_Y = Z_Q$.*

Proof. As Δ is reduced, it suffices to prove that $Z_Y \sim Z_Q$. As $A \in \mathit{Assoc}(Y)$, we have $A \to^* Z_Y$ due to Lemma 1. Similarly, $B \to^* Z_Q$ thus $Z_Y|Z_Q$ is a reachable state of Δ. As Z_Q is non-regular, it can reach a state of an arbitrary norm – for every $i \in \mathbb{N}$ there is $\alpha_i \in \mathit{Var}(\Delta)^{\otimes}$ such that $Z_Q \to^* \alpha_i$ and $|\alpha_i| = i$. Clearly $\alpha_i \sim Z_Q^i$ because each variable of α_i belongs to $\mathit{Assoc}(Q)$. Hence $Z_Y|\alpha_i \sim Z_Y|Z_Q^i$.

As $\Delta \in \mathrm{nBPA}_\tau \cap \mathrm{nBPP}_\tau$, there is a bisimilar nBPA_τ process Δ'. Let $m = \max\{|V|,\ V \in \mathit{Var}(\Delta')\}$. $Z_Y|\alpha_m$ is a reachable state of Δ and therefore there is $\gamma \in \mathit{Var}(\Delta')^*$ such that $Z_Y|\alpha_m \sim \gamma$ and hence also $Z_Y|Z_Q^m \sim \gamma$. Moreover, γ is a sequence of at least two variables.

Now we can use a similar construction as in the proof of Lemma 1 and conclude that $Z_Y|Z_Q^j \sim Z_Q^{j+1}$ for some $j \in \mathbb{N}$. This implies $Z_Y \sim Z_Q$. □

Lemma 3 *Let $\Delta \in nBPA_\tau \cap nBPP_\tau$ be a reduced $nBPP_\tau$ process. Let $L|A$ be a reachable state of Δ such that L is a lonely variable. Then A is a regular process (see Remark 1).*

Proof. Let us assume that A is not regular. Then $A \to^* Y$, where $Y \in \mathit{Var}(\Delta)$ is a growing variable (see Proposition 2). But then $L|A \to^* L|Y$, thus $L \in \mathit{Assoc}(Y)$ and we have a contradiction. □

Proposition 4 *Let Δ be a $nBPP_\tau$ process of $nBPA_\tau \cap nBPP_\tau$. Then there is a process Δ' in INF_{BPP} such that $\Delta \sim \Delta'$.*

Proof. We can assume (w.l.o.g.) that Δ is reduced and in 3-GNF. The process Δ' can be obtained by the following transformation of Δ: If $X \stackrel{def}{=} \sum_{j=1}^{m} a_j\alpha_j$ is a defining equation of Δ, then $X \stackrel{def}{=} \sum_{j=1}^{m} \mathcal{T}(a_j\alpha_j)$ is added to Δ', where $\mathcal{T}$ is defined as follows:

- if $\mathit{card}(\alpha_j) \leq 1$ then $\mathcal{T}(a_j\alpha_j) = a_j\alpha_j$
- if $\mathit{card}(\alpha_j) = 2$ (i.e. $\alpha_j = A|B$), then there are three possibilities:

1. $A \in \mathit{Assoc}(Y)$ and $B \in \mathit{Assoc}(Q)$. Then $A \sim Z_Y^{|A|}$ and $B \sim Z_Q^{|B|}$ (see Lemma 1). As $A|B$ is a reachable state, we can conclude (with a help of Lemma 2) that $Z_Y = Z_Q$, hence $A|B \sim Z_Y^{|A|+|B|}$. Thus $\mathcal{T}(a(A|B)) = a(Z_Y^{|A|+|B|})$.
2. $A \in \mathit{Assoc}(Y)$ and B is lonely. Then $A \sim Z_Y^{|A|}$ and as Z_Y is not regular, A is not regular either. As the state $A|B$ is reachable and B is lonely, it contradicts Lemma 3. Hence this case is in fact impossible (as well as the symmetric case when A is lonely and $B \in \mathit{Assoc}(Q)$).
3. A and B are lonely. Then A and B are regular (due to Lemma 3) and therefore the state $A|B$ is also regular. Each regular process can be represented in normal form (see Definition 4). Let $\Delta_{A|B}$ be a regular process in normal form which is bisimilar to $A|B$. We can assume (w.l.o.g.) that $\mathit{Var}(\Delta_{A|B}) \cap \mathit{Var}(\Delta') = \emptyset$. $\mathcal{T}$ adds all equations of $\Delta_{A|B}$ to Δ' and $\mathcal{T}(a(A|B)) = aN$ where N is the leading variable of $\Delta_{A|B}$.

The transformation $\mathcal{T}$ preserves bisimilarity – hence $\Delta \sim \Delta'$. It remains to check that Δ' is indeed in $\mathrm{INF}_{\mathrm{BPP}}$. Clearly each summand of each defining equation in Δ' is of the form which is admitted by $\mathrm{INF}_{\mathrm{BPP}}$. If aZ^j is a summand of a defining equation in Δ' such that $j \geq 2$, then $Z = Z_Y$ for some growing variable $Y \in \mathit{Var}(\Delta)$. Let $a\alpha$ be a summand in the original defining equation for Z_Y in Δ. We need to show that each such summand must have been transformed into $aZ_Y^{|\alpha|}$ by $\mathcal{T}$. But it is obvious as each variable of α belongs to $\mathit{Assoc}(Y)$. If α is composed of a single variable V, then $V = Z_Y$ because $V \sim Z_Y$ (due to Lemma 1) and Δ is reduced. Moreover, at least one summand in the defining equation for Z_Y in Δ' is of the form aZ_Y^l where $l \geq 2$, because Z_Y would be regular otherwise. To complete the proof we need to show that the defining equation for Z_Y in Δ' cannot contain two summands of the form $b, \overline{b}$. Assume the converse. As $\Delta' \in \mathrm{nBPA}_\tau \cap \mathrm{nBPP}_\tau$, there is a nBPA_τ process Δ_2 such that $\Delta' \sim \Delta_2$. As Z_Y^i is a reachable state of Δ' for every $i \in \mathbb{N}_0$ (see Remark 4), there is $\alpha_i \in \mathit{Var}(\Delta_2)^*$ such that $Z_Y^i \sim \alpha_i$ for every i. Moreover, we can assume (w.l.o.g.) that each α_i is of maximal length, i.e. if $\alpha_i \sim \beta$ for some $\beta \in \mathit{Var}(\Delta_2)^*$, then $\mathit{length}(\alpha_i) \geq \mathit{length}(\beta)$. Let k be the minimal number with the property $\mathit{length}(\alpha_k) \geq 2$. Clearly $\mathit{length}(\alpha_k) = 2$, because otherwise we could easily obtain a contradiction with the minimality of k. Hence $\alpha_k = P.Q$ for some $P, Q \in \mathit{Var}(\Delta_2)$. As $Z_Y^k \xrightarrow{b} Z_Y^{k-1}$, we also have $P.Q \xrightarrow{b} \gamma$ for some $\gamma \sim \alpha_{k-1}$. By definitions of α_i and k, γ must be composed of a single variable. The only such state reachable from $P.Q$ in one step is Q, hence $\alpha_{k-1} \sim Q$. As the defining equation for Z_Y contains two summands $b, \overline{b}$, we also have a transition $Z_Y^k \xrightarrow{\tau} Z_Y^{k-2}$. But $P.Q$ cannot reach a state which is bisimilar to α_{k-2} in one step, because α_{k-2} is (again by definitions of α_i

and k) composed of at most one variable which must be different from Q because $\alpha_{k-1} \not\sim \alpha_{k-2}$. Hence $\alpha_k \not\sim Z_Y^k$ and we have a contradiction. $\square$

Propositions 3 and 4 give us the classification of nBPA$_\tau \cap$ nBPP$_\tau$ in terms of nBPP$_\tau$ syntax.

Theorem 1 *The class nBPA$_\tau \cap$ nBPP$_\tau$ contains exactly (up to bisimilarity) nBPP$_\tau$ processes in INF$_{BPP}$.*

The class nBPA$_\tau \cap$ nBPP$_\tau$ can also be characterized using nBPA$_\tau$ syntax. To do this, we introduce a special normal form for nBPA$_\tau$ processes:

Definition 10 *Let Δ be a reduced nBPA$_\tau$ process in GNF.*

1. *Let $X, Y \in Var(\Delta)$ be non-regular variables. We say that Y is a* communication closure *(C-closure) of X if the following conditions hold:*
   - *All summands in the defining equation for X are either of the form a where $a \in Act$, or $a(Y^i.X)$ where $a \in Act$ and $i \in \mathbb{N}_0$. Moreover, at least one summand is of the form $a(Y^k.X)$ where $k \geq 1$.*
   - *All summands in the defining equation for Y are of the form aY^i, where $a \in Act$ and $i \in \mathbb{N}_0$.*
   - *aY^i is a summand in the defining equation for Y iff one of the following conditions holds:*
     (a) $i = 0$ and a is a summand in the defining equation for X.
     (b) $i \geq 1$ and $a(Y^{i-1}.X)$ is a summand in the defining equation for X.
     (c) $a = \tau$ and there are two summands of the form $b\alpha_1, \overline{b}\alpha_2$ in the defining equation for X such that $i = length(\alpha_1) + length(\alpha_2) - 1$ (note that this condition ensures that defining equations for X, Y do not contain two summands of the form $b, \overline{b}$).
2. *The process Δ is said to be in INF$_{BPA}$ if whenever $a\alpha$ is a summand in a defining equation of Δ such that $length(\alpha) \geq 2$, then $\alpha = Y^i.X$ for some $i \in \mathbb{N}$ and $X, Y \in Var(\Delta)$ such that Y is a C-closure of X. Note that X, Y need not be different – variables which are C-closures of themselves may exist.*

Note that if Y is a C-closure of X, then $|Y| = |X| = 1$. Another interesting property of X and Y is presented in the remark below.

Remark 6 It is easy to check that if Y is a C-closure of X, then $Y^i.X \sim \overline{X}^{i+1}$ where $\overline{X}$ is a nBPP$_\tau$ process composed of a single variable whose defining equation is obtained from the defining equation for X by substituting '.' by '|' and replacing each occurrence of X and Y by $\overline{X}$.

Theorem 2 *The class nBPA$_\tau \cap$ nBPP$_\tau$ contains exactly (up to bisimilarity) nBPA$_\tau$ processes in INF$_{BPA}$.*

Proof. Each $nBPA_\tau$ process in INF_{BPA} belongs to $nBPA_\tau \cap nBPP_\tau$, as a bisimilar $nBPP_\tau$ process is easily constructible by an algorithm which is 'inverse' to the algorithm presented in the proof of Proposition 3 (see Remark 6). The fact that for each $nBPA_\tau$ process of $nBPA_\tau \cap nBPP_\tau$ there is a bisimilar $nBPA_\tau$ process in INF_{BPA} follows directly from Proposition 3 and Proposition 4 (note that the algorithm presented in the proof of Proposition 3 returns a $nBPA_\tau$ process which is almost in INF_{BPA}– the only 'problem' is that it can contain different bisimilar variables and hence it is not reduced in general). □

Our results can be applied to nBPA and nBPP processes as well. So far we have investigated the intersection of $nBPA_\tau$ and $nBPP_\tau$. It was desirable to work with this unrestricted syntax, because we could also examine the problem when the 'real' communications of a $nBPP_\tau$ process can be simulated by a sequential $nBPA_\tau$ process. However, the characterization of nBPA ∩ nBPP is much simpler and therefore we present it explicitly.

Definition 11 *Let Δ be a reduced nBPA (or nBPP) process in GNF.*

1. *A variable $Z \in Var(\Delta)$ is* simple *if all summands in the defining equation for Z are of the form aZ^i, where $a \in Act$ and $i \in \mathbb{N}_0$. Moreover, at least one of those summands must be of the form aZ^k where $a \in Act$ and $k \geq 2$.*
2. *The process Δ is said to be in INF if whenever $a\alpha$ is a summand in a defining equation of Δ such that length$(\alpha) \geq 2$ (or card$(\alpha) \geq 2$), then $\alpha = Z^i$ for some simple variable Z and $i \geq 2$.*

Note that nBPA (or nBPP) processes in INF have a nice property – a bisimilar nBPP (or nBPA) process can be obtained just by replacing the '.' operator by the '‖' operator (or by replacing the '‖' operator by the '.' operator).

Theorem 3 *The class nBPA ∩ nBPP contains exactly (up to bisimilarity) nBPA (or nBPP) processes in INF.*

4 Deciding whether $\Delta \in nBPA_\tau \cap nBPP_\tau$

In this section we prove that the problem whether a given $nBPA_\tau$ or $nBPP_\tau$ process Δ belongs to $nBPA_\tau \cap nBPP_\tau$ is decidable in polynomial time. The technique is essentially similar in both cases – we check if each summand of each defining equation of Δ whose form is not admitted by INF_{BPA} (or INF_{BPP}) can be in principal transformed so that requirements of INF_{BPA} (or INF_{BPP}) are satisfied. We also show that if a $nBPA_\tau$ (or $nBPP_\tau$) process Δ belongs to $nBPA_\tau \cap nBPP_\tau$, then a bisimilar process in INF_{BPA} (or INF_{BPP}) is effectively constructible. Simplified versions of the mentioned algorithms which work for nBPA and nBPP processes are presented as well.

Definition 12 *Let Δ be a nBPA$_\tau$ or nBPP$_\tau$ process in GNF.*

- *The set $S(\Delta) \subseteq Var(\Delta)$ is composed of all variables V such that $|V| = 1$, V is non-regular and if $a\alpha$ is a summand in the defining equation for V in Δ, then $\alpha \sim V^{|\alpha|}$.*
- *The set $R(\Delta) \subseteq Var(\Delta)$ contains all regular variables of Δ.*
- *The set $G(\Delta) \subseteq Var(\Delta)$ contains all growing variables of Δ.*

The sets $S(\Delta)$, $R(\Delta)$, and $G(\Delta)$ can be constructed in polynomial time because bisimilarity and regularity are decidable for nBPA$_\tau$ and nBPP$_\tau$ processes in polynomial time (see [10], [11], and Proposition 1).

If Δ is a nBPA$_\tau$ (or nBPP$_\tau$) process of nBPA$_\tau$ $\cap$ nBPP$_\tau$, then there is Δ' in INF$_{\text{BPA}}$ (or INF$_{\text{BPP}}$) such that $\Delta \sim \Delta'$. In case of nBPP$_\tau$ processes the set $S(\Delta)$ contains in fact variables which can be (potentially) bisimilar to simple variables of Δ'. In case of nBPA$_\tau$ processes the set $S(\Delta)$ contains variables which can be bisimilar to C-closures of variables from $Var(\Delta')$.

Correctness of our algorithm which decides the membership to nBPA$_\tau$ $\cap$ nBPP$_\tau$ for nBPP$_\tau$ processes is shown by the following three lemmas.

Lemma 4 *Let Δ be a reduced nBPP$_\tau$ process in 3-GNF and let $a(A|B)$ be a summand in a defining equation of Δ such that A is regular and B is non-regular. Then $\Delta \notin$ nBPA$_\tau \cap$ nBPP$_\tau$.*

Proof. Assume there is a nBPP$_\tau$ process Δ' in INF$_{\text{BPP}}$ such that $\Delta \sim \Delta'$. Let $n = \max\{|Y|,\ Y \in Var(\Delta')\}$. As B is non-regular, it can reach a state of an arbitrary norm – let $B \rightarrow^* \beta$ where $|\beta| > n$. Then $A|\beta$ is a reachable state of Δ and thus $A|\beta \sim \beta'$ for some reachable state β' of Δ'. As $|\,A|\beta\,| > n$, we can conclude that $\beta' = Z^{|\,A|\beta\,|}$ where $Z \in Var(\Delta')$ is a simple variable (see Remark 4). Hence $A \sim Z^{|A|}$ and as each simple variable is growing (see Definition 8), it contradicts regularity of A. □

Lemma 5 *Let Δ be a reduced nBPP$_\tau$ process in 3-GNF which belongs to nBPA$_\tau \cap$ nBPP$_\tau$. Let $a(A|B)$ be a summand in a defining equation of Δ such that A and B are non-regular. Then there is exactly one variable $Z \in S(\Delta)$ such that $A|B \sim Z^{|\,A|B\,|}$.*

Proof. Let Δ' be a nBPP$_\tau$ process in INF$_{\text{BPP}}$ such that $\Delta \sim \Delta'$. Let $n = \max\{|Y|,\ Y \in Var(\Delta')\}$. Using the same argument as in the proof of Lemma 4 we obtain $A \sim P^{|A|}$, $B \sim Q^{|B|}$ where $P, Q \in Var(\Delta')$ are simple variables. We show that $P = Q$. Let $A \rightarrow^* \alpha$ where $|\alpha| > n$. Then clearly $\alpha \sim P^{|\alpha|}$ and as $\alpha|B$ is a reachable state of Δ, $\alpha|B \sim R^{|\alpha|B|}$ where $R \in Var(\Delta')$ is a simple variable. To sum up, we have $\alpha|B \sim P^{|\alpha|}|Q^{|B|} \sim R^{|\,\alpha|B\,|}$. Hence $P \sim R \sim Q$ and thus $P = R = Q$ because Δ' is reduced. As e.g. P is a reachable state of Δ', there is a reachable state γ of Δ such that $P \sim \gamma$. As $|P| = 1$, we can conclude $\gamma = Z$ for some $Z \in Var(\Delta)$ which clearly belongs to $S(\Delta)$. Moreover, Z is unique because Δ is reduced. □

Lemma 6 *Let Δ be a nBPP$_\tau$ process in GNF and let $X \in S(\Delta)$. If the defining equation for X contains two summands of the form $b, \overline{b}$, then $\Delta \notin$ nBPA$_\tau \cap$ nBPP$_\tau$.*

Proof. Assume there is a nBPP$_\tau$ process Δ' in INF$_{\text{BPP}}$ such that $\Delta \sim \Delta'$. Using the same kind of argument as in the proof of Lemma 4 we obtain $X \sim Z$ for some simple variable $Z \in Var(\Delta')$. As the defining equation for X contains two summands of the form $b, \overline{b}$ and $X \sim Z$, the defining equation for Z must contain those summands too – hence Z is not simple and we have a contradiction. □

The (constructive) algorithm which decides the membership to nBPA$_\tau \cap$ nBPP$_\tau$ for nBPP$_\tau$ processes is presented in Fig. 1. Steps which are executed only by the constructive algorithm are placed within framed boxes – if we omit this code, we obtain a non-constructive polynomial algorithm. The abbreviation "NFR(Δ)" stands for the **N**ormal **F**orm of the **R**egular process Δ, which can be effectively constructed (see Proposition 1). We always assume that NFR(Δ) contains fresh variables which are not contained in any other process we are working with. When the command `return` is executed, the algorithm *halts* and returns the value which follows immediately after the keyword `return`.

The constructive algorithm is not polynomial because the construction of NFR is not polynomial – a regular nBPP$_\tau$ process in 3-GNF with n variables can generally reach exponentially many pairwise non-bisimilar states and each of these states requires its own 'fresh' variable.

Our algorithm for nBPP$_\tau$ processes works for pure nBPP processes as well. It suffices to replace the '$|$' operator with the '$\|$' operator in our description. As there are no communications in nBPP, the notion of dual action is no longer sensible – hence the second step of our algorithm can be removed in case of nBPP processes.

Now we provide an analogous algorithm for nBPA$_\tau$ processes. We start with some auxiliary definitions and lemmas.

Definition 13 *Let Δ be a nBPA$_\tau$ process. For each $Y \in S(\Delta)$ we define the set CL(Y), composed of all $X \in Var(\Delta)$ which satisfy the following conditions:*

- *If $a\alpha$ is a summand in the defining equation for X such that length$(\alpha) \geq 1$, then $\alpha \sim Y^{|\alpha|-1}.X$.*
- *The defining equation for Y contains a summand bisimilar to aY^k, $k \in \mathbb{N}_0$, iff one of the following conditions holds:*
  1. *$k = 0$ and the defining equation for X contains a summand 'a'.*
  2. *$k > 0$ and the defining equation for X contains a summand which is bisimilar to $a(Y^{k-1}.X)$.*

Algorithm: A $\boxed{\text{constructive}}$ test of the membership to nBPA$_\tau$ $\cap$ nBPP$_\tau$ for nBPP$_\tau$ processes.

Input: A reduced nBPP$_\tau$ process Δ in 3-GNF.

Output: **YES** $\boxed{\text{and } \Delta' \text{ in INF}_{\text{BPP}} \text{ such that } \Delta \sim \Delta'}$ if $\Delta \in$ nBPA$_\tau$ $\cap$ nBPP$_\tau$,

NO otherwise.

1. Construct the sets $S(\Delta)$, $R(\Delta)$ and $G(\Delta)$.

2. <u>if</u> there is $X \in S(\Delta)$ whose def. equation contains two summands of the form $b, \bar{b}$
<u>then</u>
    <u>return</u> **NO**;
<u>endif</u>

3. <u>if</u> $G(\Delta) = \emptyset$ <u>then</u>
    $\boxed{\Delta' := \text{NFR}(\Delta)}$;
    <u>return</u> **YES** $\boxed{\text{and } \Delta'}$;
<u>endif</u>

4. $\boxed{\Delta' := \Delta}$;

5. <u>for</u> <u>each</u> summand of the form $a(A|B)$ in defining equations of Δ <u>do</u>
    <u>if</u> $A, B \in R(\Delta)$ <u>then</u>
        $\boxed{\text{Construct NFR}(A|B)}$;
        $\boxed{\text{Replace the summand } a(A|B) \text{ with } aN \text{ in } \Delta'\text{, where } N \text{ is the}}$
        $\boxed{\text{leading variable of NFR}(A|B)}$;
        $\boxed{\Delta' := \Delta' \cup \text{NFR}(A|B)}$;
    <u>endif</u>
    <u>if</u> ($A \in R(\Delta)$ and $B \notin R(\Delta)$) or ($A \notin R(\Delta)$ and $B \in R(\Delta)$) <u>then</u>
        <u>return</u> **NO**;
    <u>endif</u>
    <u>if</u> $A, B \notin R(\Delta)$ <u>then</u>
        <u>if</u> there exists $Z \in S(\Delta)$ such that $A|B \sim Z^{|A|B|}$
            <u>then</u> $\boxed{\text{Replace the summand } a(A|B) \text{ with } a(Z^{|A|B|}) \text{ in } \Delta'}$;
            <u>else</u> <u>return</u> **NO**;
        <u>endif</u>
    <u>endif</u>
<u>endfor</u>

6. <u>return</u> **YES** $\boxed{\text{and } \Delta'}$;

Fig. 1. An algorithm which (constructively) decides the membership to nBPA$_\tau$ $\cap$ nBPP$_\tau$ for nBPP$_\tau$ processes

3. *$a = \tau$ and the defining equation for X contains two summands of the form $b\alpha_1$, $\overline{b}\alpha_2$ such that $k = length(\alpha_1) + length(\alpha_2) - 1$.*

It is easy to see that the set $CL(Y)$ can be constructed in polynomial time for every $Y \in S(\Delta)$. The following lemma is due to D. Caucal (see [4]):

Lemma 7 *Let Δ, Δ' be nBPA$_\tau$ processes in GNF and let $\alpha, \beta \in Var(\Delta)^*$, $\alpha', \beta' \in Var(\Delta')^*$ such that $\beta \sim \beta'$ and $\alpha.\beta \sim \alpha'.\beta'$. Then $\alpha \sim \alpha'$.*

Lemma 8 *Let Δ, Δ' be nBPA$_\tau$ processes. Let $A_1, \ldots, A_k \in Var(\Delta)$, $X, Y \in Var(\Delta')$ such that $|X| = |Y| = 1$ and $A_1. \cdots .A_k \sim Y^l.X$ where $l = |A_1. \cdots .A_k| - 1$. Then $A_k \sim Y^{|A_k|-1}.X$ and $A_i \sim Y^{|A_i|}$ for $1 \leq i < k$.*

Proof. Clearly $A_k \sim Y^{|A_k|-1}.X$. Hence $A_1. \cdots .A_{k-1} \sim Y^{|A_1. \cdots .A_{k-1}|}$ (due to Lemma 7). The fact $A_i \sim Y^{|A_i|}$ for $1 \leq i < k$ can be proved by induction on k. If $k = 2$ then $A_1 \sim Y^{|A_1|}$ and our lemma holds. If $k > 2$, then clearly $A_{k-1} \sim Y^{|A_{k-1}|}$ and due to Lemma 7 we have $A_1. \cdots .A_{k-2} \sim Y^{|A_1. \cdots .A_{k-2}|}$. Now we can use the induction hypothesis and conclude that $A_i \sim Y^{|A_i|}$ for $1 \leq i < (k-2)$. □

Lemma 9 *Let Δ be a reduced nBPA$_\tau$ process in 3-GNF which belongs to nBPA$_\tau \cap$ nBPP$_\tau$. Let $Q.\alpha$ be a reachable state of Δ such that $Q \in G(\Delta)$, $\alpha \neq \epsilon$. Then there are unique variables $Y \in S(\Delta)$, $X \in CL(Y)$ such that $Q.\alpha \sim Y^{|Q.\alpha|-1}.X$.*

Proof. As $\Delta \in$ nBPA$_\tau \cap$ nBPP$_\tau$, there is a nBPA$_\tau$ process Δ' in INF$_{\text{BPA}}$ such that $\Delta \sim \Delta'$. Let $n = \max\{|A|,\ A \in Var(\Delta')\}$. As Q is growing, $Q \to^* Q.\gamma$ where $\gamma \neq \epsilon$. Hence the state $Q.\gamma^n.\alpha$ is a reachable state of Δ and therefore there is a reachable state δ of Δ' such that $Q.\gamma^n.\alpha \sim \delta$. As $|Q.\gamma^n.\alpha| > n$, we can conclude $\delta = R^{|Q.\gamma^n.\alpha|-1}.S$, where R is a C-closure of S (see Definition 10). Hence $Q.\gamma^n.\alpha \sim R^{|Q.\gamma^n.\alpha|-1}.S$ and due to Lemma 8 we have $\alpha \sim R^{|\alpha|-1}.S$ and $Q \sim R^{|Q|}$, thus $Q.\alpha \sim R^{|Q.\alpha|-1}.S$. Now it suffices to show that there are $Y \in S(\Delta)$, $X \in CL(Y)$ such that $Y \sim R$ and $X \sim S$. As Δ is normed, $Q \xrightarrow{s} Y$ where $|Y| = 1$ and s is a norm-decreasing sequence of actions. Then $Q.\alpha \xrightarrow{s} Y.\alpha$ and as $Q.\alpha \sim R^{|Q.\alpha|-1}.S$, the state $R^{|Q.\alpha|-1}.S$ must be able to match the sequence s and enter a state bisimilar to $Y.\alpha$. As s is norm-decreasing and $|R| = 1$, the only such state is $R^{|Y.\alpha|-1}.S$. Hence $Y.\alpha \sim R^{|Y.\alpha|-1}.S$ and due to Lemma 8 we have $Y \sim R$. The fact $Y \in S(\Delta)$ follows directly from Definition 10. As S is a reachable state of Δ', there is a variable $X \in S(\Delta)$ such that $X \sim S$. Clearly $X \in CL(Y)$ (see Definition 10). The variables X, Y are unique because Δ is reduced. □

It is worth noting that the variables X, Y of the previous lemma need not be different. To prove the correctness of our algorithm which decides the membership to nBPA$_\tau \cap$ nBPP$_\tau$ for nBPA$_\tau$ processes we need some lemmas about summands.

Lemma 10 *Let Δ be a reduced nBPA$_\tau$ process in 3-GNF and let $a(A.B)$ be a summand in a defining equation of Δ such that A is non-regular and B is regular. Then $\Delta \notin$ nBPA$_\tau \cap$ nBPP$_\tau$.*

Proof. As $a(A.B)$ is a summand in a defining equation of Δ and Δ is normed and in GNF, there is a reachable state of the form $A.B.\beta$. As A is non-regular, $A \to^* Q.\alpha$ where $Q \in G(\Delta)$. Hence $Q.\alpha.B.\beta$ is a reachable state of Δ and due to Lemma 9 we have $Q.\alpha.B.\beta \sim Y^{|Q.\alpha.B.\beta|-1}.X$ for some $Y \in S(\Delta)$, $X \in CL(Y)$. With a help of Lemma 8 we obtain that $B \sim Y^{|B|}$ or $B \sim Y^{|B|-1}.X$ (the latter possibility holds if $\beta = \epsilon$). As X, Y are non-regular, it contradicts regularity of B. $\square$

Lemma 11 *Let Δ be a reduced nBPA$_\tau$ process in 3-GNF. Let $a(A.B)$ be a summand in a defining equation of Δ such that A is regular and B is non-regular. Then the summand $a(A.B)$ can be replaced with aN where $N \notin Var(\Delta)$, and a finite number of new equations satisfying requirements of INF$_{\text{BPA}}$ can be effectively added to Δ such that the resulting process Δ_1 is bisimilar to Δ.*

Proof. As A is regular, the process $\Delta_A := NFR(A)$ such that $Var(\Delta) \cap Var(\Delta_A) = \emptyset$ is effectively constructible. Now we slightly modify defining equations of Δ_A – each summand of the form a where $a \in Act$ is replaced with aB. The resulting system of equations is in INF$_{\text{BPA}}$. If we add the modified system Δ_A to Δ and replace the summand $a(A.B)$ with aN where N is the leading variable of Δ_A, we obtain a process Δ_1 which is clearly bisimilar to Δ. $\square$

Lemma 12 *Let Δ be a reduced nBPA$_\tau$ process in 3-GNF and let $a(A.B)$ be a summand in a defining equation of Δ such that A and B are non-regular. Then*

1. *If $\Delta \in$ nBPA$_\tau \cap$ nBPP$_\tau$, then there are unique variables $Y \in S(\Delta)$, $X \in CL(Y)$ such that $B \sim Y^{|B|-1}.X$*
2. *Let $B \sim Y^{|B|-1}.X$ for some $Y \in S(\Delta)$ and $X \in CL(Y)$. If there is a sequence of transitions $A = A_0 \overset{a_0}{\to} A_1.\alpha_1 \overset{a_1}{\to} A_2.\alpha_2 \overset{a_2}{\to} \cdots \overset{a_k}{\to} A_k.\alpha_k$ such that $k \geq 0$, $A_k \in G(\Delta)$ and $A_k.\alpha_k \not\sim Y^{|A_k.\alpha_k|}$, then $\Delta \notin$ nBPA$_\tau \cap$ nBPP$_\tau$.*
3. *Let $B \sim Y^{|B|-1}.X$ for some $Y \in S(\Delta)$ and $X \in CL(Y)$. If for each sequence of transitions $A = A_0 \overset{a_0}{\to} A_1.\alpha_1 \overset{a_1}{\to} A_2.\alpha_2 \overset{a_2}{\to} \cdots \overset{a_k}{\to} A_k.\alpha_k$ such that $A_k \in G(\Delta)$ the state $A_k.\alpha_k$ is bisimilar to $Y^{|A_k.\alpha_k|}$, then the summand $a(A.B)$ can be replaced with aN where $N \notin Var(\Delta)$ and a finite number of new equations satisfying requirements of INF$_{\text{BPA}}$ can be effectively added to Δ such that the resulting process Δ_2 is bisimilar to Δ.*

Proof. 1. As A is non-regular, $A \to^* Q.\alpha$ where $Q \in G(\Delta)$. The proof can be easily completed with a help of Lemma 8 and Lemma 9.

2. This is a consequence of Lemma 8 and Lemma 9.
3. It suffices to realize that if $A = A_0 \stackrel{a_0}{\to} A_1.\alpha_1 \stackrel{a_1}{\to} A_2.\alpha_2 \stackrel{a_2}{\to} \cdots \stackrel{a_k}{\to} A_k.\alpha_k$ is a sequence of transitions such that $A_0, \ldots, A_{k-1} \notin G(\Delta)$ and $A_k \in G(\Delta)$, then *length*$(A_i.\alpha_i) \leq$ *card*$($*Var*$(\Delta))$ for $0 \leq i \leq k-1$ (here we use the assumption that Δ is in 3-GNF. Naturally, *length*$(A_i.\alpha_i)$ is bounded also in case of general GNF). As there are only finitely many sequences of variables of this bounded length, we can introduce a fresh variable for each of them. To construct the process Δ_2, we use a similar procedure as in the proof of Lemma 11. □

An existence of a sequence $A = A_0 \stackrel{a_0}{\to} A_1.\alpha_1 \stackrel{a_1}{\to} A_2.\alpha_2 \stackrel{a_2}{\to} \cdots \stackrel{a_k}{\to} A_k.\alpha_k$ such that $A_k \in G(\Delta)$ and $A_k.\alpha_k \not\sim Y^{|A_k.\alpha_k|}$ is decidable in polynomial time, as demonstrated by the following lemma:

Lemma 13 *Let Δ be a reduced nBPA$_\tau$ process in 3-GNF. Let $A \in$ Var(Δ) be a non-regular variable and let $Y \in S(\Delta)$. The problem whether A can reach a state of the form $Q.\alpha$ where $Q \in G(\Delta)$ and $Q.\alpha \not\sim Y^{|Q.\alpha|}$ is decidable in polynomial time.*

Proof. We divide the set *Var*(Δ) into two disjoint subsets of *successful* and *unsuccessful* variables. $P \in$ *Var*(Δ) is unsuccessful if one of the following conditions holds:

- P is growing and $P \not\sim Y^{|P|}$.
- The defining equation for P in Δ contains a summand of the form $a(R.S)$ where R is non-regular and $S \not\sim Y^{|S|}$.

A variable is successful if it is not unsuccessful. Furthermore, we define the binary relation '$\Rightarrow$' on *Var*(Δ): $U \Rightarrow V$ iff U is successful and the defining equation for U in Δ contains a summand which is of one of the following forms:

- aV
- $a(V.W)$ where $W \in$ *Var*(Δ)
- $a(W.V)$ where $W \in$ *Var*(Δ) is regular

Let '$\Rightarrow^*$' be the reflexive and transitive closure of '$\Rightarrow$'. It can be easily proved that A can reach a state of the form $Q.\alpha$ where Q is growing and $Q.\alpha \not\sim Y^{|Q.\alpha|}$ iff $A \Rightarrow^* T$ for some unsuccessful variable T. As the relation '$\Rightarrow^*$' can be constructed in polynomial time, the proof is finished. □

An algorithm which decides the membership to nBPA$_\tau$ $\cap$ nBPP$_\tau$ for nBPA$_\tau$ processes is presented in Fig. 2. We use the same notation as in the case of nBPP$_\tau$.

In case of nBPA processes our algorithm must be slightly modified (and simplified). This is a consequence of the fact that a nBPA process Δ belongs to nBPA $\cap$ nBPP iff it can be represented in INF – and INF is a little different

Algorithm: A $\boxed{\text{constructive}}$ test of the membership to nBPA$_\tau$ $\cap$ nBPP$_\tau$ for nBPA$_\tau$ processes.

Input: A reduced nBPA$_\tau$ process Δ in 3-GNF.

Output: **YES** $\boxed{\text{and } \Delta' \text{ in INF}_{\text{BPA}} \text{ such that } \Delta \sim \Delta'}$ if $\Delta \in$ nBPA$_\tau$ $\cap$ nBPP$_\tau$,

NO otherwise.

1. Construct the sets $S(\Delta), R(\Delta), G(\Delta)$ and for each $Y \in S(\Delta)$ construct the set $CL(Y)$.
2. `if` $(G(\Delta) = \emptyset)$ `then`
    - $\boxed{\Delta' := \text{NFR}(\Delta)}$;
    - `return` **YES** $\boxed{\text{and } \Delta'}$;

    `endif`
3. $\boxed{\Delta' := \Delta}$;
4. `for` each summand of the form $a(A.B)$ in defining equations of Δ `do`

    `if` $A, B \in R(\Delta)$ `then`
    - $\boxed{\text{Construct NFR}(A.B)}$;
    - $\boxed{\text{Replace the summand } a(A.B) \text{ with } aN \text{ in } \Delta'\text{, where } N \text{ is the}}$
    - $\boxed{\text{leading variable of NFR}(A.B)}$;
    - $\boxed{\Delta' := \Delta' \cup \text{NFR}(A.B)}$;

    `endif`

    `if` $A \notin R(\Delta)$ and $B \in R(\Delta)$ `then`
    - `return` **NO**;

    `endif`

    `if` $A \in R(\Delta)$ and $B \notin R(\Delta)$ `then`
    - $\boxed{\text{Construct the process } \Delta_1 \text{ of Lemma 11}}$;
    - $\boxed{\Delta' := \Delta_1}$;

    `endif`

    `if` $A, B \notin R(\Delta)$ `then`
    - `if` there exists $Y \in S(\Delta), X \in CL(Y)$ such that $B \sim Y^{|B|-1}.X$
        - `then` `if` A can reach a state $Q.\alpha$ where $Q \in G(\Delta)$ and $Q.\alpha \not\sim Y^{|Q.\alpha|}$
            - `then` `return` **NO**;
            - `else` $\boxed{\text{Construct the process } \Delta_2 \text{ of Lemma 12; } \Delta' := \Delta_2}$;

            `endif`
        - `else` `return` **NO**;

        `endif`

    `endif`

    `endfor`
5. `return` **YES** $\boxed{\text{and } \Delta'}$;

Fig. 2. An algorithm which (constructively) decides the membership to nBPA$_\tau$ $\cap$ nBPP$_\tau$ for nBPA$_\tau$ processes

```
if A, B ∉ R(Δ) then
    if there exists Z ∈ S(Δ) such that B ∼ Z^|B|
        then if A can reach a state Q.α where Q ∈ G(Δ) and Q.α ≁ Z^|Q.α|
                 then return NO;
                 else [Construct the process Δ2 of Lemma 14];
                      [Δ' := Δ2];
             endif
        else return NO;
    endif
endif
```

Fig. 3. The code for nBPA processes

from $\mathrm{INF}_{\mathrm{BPA}}$ (see Definitions 11 and 10). Lemma 10 and Lemma 11 are valid also for nBPA processes. Instead of Lemma 12 we can prove the following (in a similar way):

Lemma 14 *Let Δ be a reduced nBPA process in 3-GNF and let $a(A.B)$ be a summand in a defining equation of Δ such that A and B are non-regular. Then*

1. *If $\Delta \in$ nBPA $\cap$ nBPP then there is a unique variable $Z \in S(\Delta)$ such that $B \sim Z^{|B|}$*
2. *Let $B \sim Z^{|B|}$ for some $Z \in S(\Delta)$. If there is a sequence of transitions $A = A_0 \stackrel{a_0}{\to} A_1.\alpha_1 \stackrel{a_1}{\to} A_2.\alpha_2 \stackrel{a_2}{\to} \cdots \stackrel{a_k}{\to} A_k.\alpha_k$ such that $k \geq 0$, $A_k \in G(\Delta)$ and $A_k.\alpha_k \not\sim Z^{|A_k.\alpha_k|}$, then $\Delta \notin$ nBPA $\cap$ nBPP.*
3. *Let $B \sim Z^{|B|}$ for some $Z \in S(\Delta)$. If for each sequence of transitions $A = A_0 \stackrel{a_0}{\to} A_1.\alpha_1 \stackrel{a_1}{\to} A_2.\alpha_2 \stackrel{a_2}{\to} \cdots \stackrel{a_k}{\to} A_k.\alpha_k$ such that $A_k \in G(\Delta)$ the state $A_k.\alpha_k$ is bisimilar to $Z^{|A_k.\alpha_k|}$, then the summand $a(A.B)$ can be replaced with aN where $N \notin Var(\Delta)$ and a finite number of new equations satisfying requirements of INF can be effectively added to Δ such that the resulting process Δ_2 is bisimilar to Δ.*

Our algorithm for nBPA processes differs from the algorithm of Fig. 2 in two things – the sets $CL(Y)$ for $Y \in S(\Delta)$ are not computed at all and the last `if` statement in the loop of step 4 is replaced with the code of Fig. 3. Now we can easily prove the following theorem:

Theorem 4 *Bisimilarity is decidable in the union of nBPA$_\tau$ and nBPP$_\tau$ processes.*

Proof. Given two nBPA$_\tau$ or nBPP$_\tau$ processes, it is possible to check bisimilarity using algorithms which were published in [10] and [11]. If we get a nBPP$_\tau$ process Δ_1 and a nBPA$_\tau$ process Δ_2, then we run one of the constructive algorithms presented earlier. We can choose e.g. the first algorithm with Δ_1 on input. If it answers **NO**, then $\Delta_1 \not\sim \Delta_2$. Otherwise we obtain a nBPP$_\tau$ process Δ_1' in INF$_{\text{BPP}}$ which is bisimilar to Δ_1. Now it suffices to check bisimilarity between two nBPA$_\tau$ processes $\overline{\Delta_1'}$ and Δ_2, where $\overline{\Delta_1'}$ is obtained by running the algorithm presented in the proof of Proposition 3 with Δ_1' on input. $\square$

Note that the corresponding statement holds for nBPA and nBPP processes by specialization.

5 Related work and future research

The problem whether a given nBPP process belongs to nBPA $\cap$ nBPP has been independently examined by Blanco in [3] where it is shown that given a nBPP process, one can decide whether there is a bisimilar nBPA process. Blanco's approach is based on special properties of BPA transition graphs (see [5]). A test whether a given nBPP graph has these properties is given in the work. Consequently, this result does not allow for testing whether a given *nBPA* process belongs to the intersection. The generalization to nBPA$_\tau$ and nBPP$_\tau$ classes is not considered.

Our result on the classification of nBPA $\cap$ nBPP might be of some interest from the point of view of formal languages/automata theory as well. The normal form INF for nBPA processes can be taken as a special type of CF grammars which generate languages of the form $R.(L_1 \cup \ldots \cup L_n)$, where R is regular and each $L_i (1 \leq i \leq n)$ can be generated by a CF grammar of the form $G_i = (\{Z_i\}, \Sigma, P, Z_i)$ having just one nonterminal and rules of the form $Z_i \rightarrow aZ_i^k,\ k \geq 0, a \in \Sigma$. It is clear the languages of the mentioned type $R.(L_1 \cup \ldots \cup L_n)$ can be recognized by nondeterministic one-counter automata. Hence our result on the classification of nBPA $\cap$ nBPP can be considered as a refinement of the result achieved in [16] on the context-freeness of languages generated by Petri nets, as BPP processes form a proper subclass of Petri nets.

An obvious question is whether our results can be extended to classes of all (not only normed) BPA and BPP processes. The class BPA $\cap$ BPP contains also processes which cannot be presented in INF. As an example we give the following BPP process:

$$X \stackrel{\text{def}}{=} a(Y \| X)$$
$$Y \stackrel{\text{def}}{=} b$$

The process X cannot be presented in INF. However, it obviously belongs to BPA $\cap$ BPP; a bisimilar BPA process looks as follows:

$$A \stackrel{\text{def}}{=} a(B.A)$$
$$B \stackrel{\text{def}}{=} a(B.B) + b$$

Transition systems generated by X and A are even isomorphic:

$$\bullet \underset{b}{\overset{a}{\rightleftarrows}} \circ \underset{b}{\overset{a}{\rightleftarrows}} \circ \underset{b}{\overset{a}{\rightleftarrows}} \circ \cdots\cdots$$

This indicates that the problem is actually more complicated. Techniques which were used for normed processes cannot be applied – it seems however, that a deeper study of the structure of BPA and BPP transition graphs could help.

References

1. Baeten, J.C.M., Bergstra, J.A., Klop, J.W.: Decidability of bisimulation equivalence for processes generating context-free languages. J Assoc Comput Mach 40:653–682 (1993)
2. Bergstra, J.A., Klop, J.W.: Process theory based on bisimulation semantics. In: Advanced Topics in Artificial Intelligence, LNCS Vol 345, pp 50–122, Berlin Heidelberg New York: Springer 1988
3. Blanco, J.: Normed BPP and BPA. In: Proceedings of ACP'94, Workshops in Computing, pp 242–251, Berlin Heidelberg New York: Springer 1995
4. Caucal, D.: Graphes canoniques de graphes algebriques. Rapport de Recherche 872, INRIA, 1988
5. Caucal, D., Monfort, R.: On the transition graphs of automata and grammars. In: Graph-Theoretic Concepts in Computer Science, LNCS Vol 484, pp 311–337, Berlin Heidelberg New York: Springer 1990
6. Christensen, S.: Decidability and Decomposition in Process Algebras. PhD thesis, The University of Edinburgh, 1993
7. Christensen, S., Hirshfeld, Y., Moller, F.: Bisimulation is decidable for all basic parallel processes. In: Proceedings of CONCUR'93, LNCS Vol 715, pp 143–157, Berlin Heidelberg New York: Springer 1993
8. Christensen, S., Hüttel, H., Stirling, C.: Bisimulation equivalence is decidable for all context-free processes. Inf Comput 121:143–148 (1995)
9. Groote, J.F.: A short proof of the decidability of bisimulation for normed BPA processes. Inf Process Lett 42:167–171 (1992)
10. Hirshfeld, Y., Jerrum, M., Moller, F.: A polynomial algorithm for deciding bisimilarity of normed context-free processes. Theor Comput Sci 158:143–159 (1996)
11. Hirshfeld, Y., Jerrum, M., Moller, F.: A polynomial algorithm for deciding bisimulation equivalence of normed basic parallel processes. Math Struct Comput Sci 6:251–259 (1996)
12. Hüttel, H., Stirling, C.: Actions speak louder than words: Proving bisimilarity for context-free processes. J Log Comput 8(4):485–509 (1998)
13. Kučera, A.: Regularity is decidable for normed PA processes in polynomial time. In: Proceedings of FST&TCS'96, LNCS Vol 1180, pp 111–122, Berlin Heidelberg New York: Springer 1996
14. Milner, R.: Communication and Concurrency. Englewood Cliffs NJ: Prentice-Hall 1989
15. Park, D.M.R.: Concurrency and automata on infinite sequences. In: Proceedings 5^{th} GI Conference, LNCS Vol 104, pp 167–183, Berlin Heidelberg New York: Springer 1981
16. Schwer, S.R.: The context-freeness of the languages associated with vector addition systems is decidable. Theor Comput Sci 98(2):199–247 (1992)
17. van Glabbeek, R.J.: The linear time—branching time spectrum. In: Proceedings of CONCUR'90, LNCS Vol 458, pp 278–297, Berlin Heidelberg New York: Springer 1990