Entrance Examination
Mathematics

June 2012
1 Sets, relations, functions

1. Consider a function $F$ of the type $\mathbb{Z} \to \mathbb{Z}$ defined by $F(n) = n^2$. The function $F$

(a) is surjective but not injective
(b) is neither surjective, nor injective ✓
(c) is injective but not surjective
(d) is bijective
(e) is both injective and surjective

2. Let $S = \{a, b\}$, $T = \{b, c\}$ a $U = \{a, c\}$. Which of the following sets is equal to $(R \times S) \cap (S \times U)$ ?

(a) $\{(b, c)\}$ ✓
(b) $\{(b, c), (a, b)\}$
(c) $\{(b, c), (a, b), (a, c)\}$
(d) $\emptyset$
(e) $\{(b, b), (c, c), (a, a)\}$

3. Which of the following sets is equal to $(A \setminus B) \setminus C$ ?

(a) $A \setminus (B \setminus C)$
(b) $A \setminus (B \cap C)$
(c) $A \setminus (B \cup C)$ ✓
(d) $(A \cup B) \setminus C$
(e) $(A \cap B) \setminus C$

(Here $X \setminus Y$ denotes the difference of the sets $X$ and $Y$, i.e. the set of all elements of $X$ that do not belong to $Y$.)

4. The relation $\{(a, a)\}$ on the set $\{a, b, c\}$ is not

(a) symmetric
(b) transitive
(c) antisymmetric
(d) reflexive ✓
(e) non-empty

5. Let $A$ be a set of size $n \in \mathbb{N}$. What is the size of the set $\mathcal{P}(\mathcal{P}(A))$ ?

(Here $\mathcal{P}(A)$ denotes the power set of $A$, i.e. the set of all subsets of $A$.)

(a) $2^{2n}$
(b) $n^2$
(c) $2^{n^2}$
(d) $2^{2n}$ ✓
(e) $2^{2^n} + 2^n$
2 Logic

6. Let us consider the following statement: For every \( x > 0 \) we have that \( P(x) \) holds.
Which of the following statements is obtained by negating the statement above?

(a) For every \( x \leq 0 \) we have that \( P(x) \) holds.
(b) For every \( x \leq 0 \) we have that \( P(x) \) does not hold.
(c) There exists \( x \leq 0 \) such that \( P(x) \) holds.
(d) There exists \( x > 0 \) such that \( P(x) \) does not hold. ✓
(e) There exists \( x > 0 \) such that \( P(x) \) holds.

7. Let us assume that all variables are interpreted as natural numbers (with zero). For which given values of the variable \( x \) is the following formula true?
\[
\exists y \exists z ((x = y + z) \land y \neq 0 \land z \neq 0 \land y \neq z)
\]

(a) 0
(b) 1
(c) 2
(d) 3 ✓
(e) for no value from the set \( \{0, 1, 2, 3\} \)

8. Which of the following formulae is not in the conjunctive normal form (CNF)?

(a) \( A \land B \land C \)
(b) \( A \lor B \lor C \)
(c) \( (A \lor B) \land C \)
(d) \( \neg(A \land B \land C) \) ✓
(e) \( \neg A \lor \neg B \lor \neg C \)

9. Which of the following formulae is a tautology?
(A formula is a tautology if it is true for all valuations of the variables.)

(a) \( A \iff C \)
(b) \( (A \land B) \iff (C \land B) \)
(c) \( (A \land B) \iff (C \lor B) \)
(d) \( (A \land \neg A) \iff (C \land \neg C) \) ✓
(e) \( (A \land \neg A) \iff (C \lor \neg C) \)

10. Let us assume that all variables are interpreted as integers, the symbol \( f \) is interpreted as the function which assigns to each number \( n \) the value \( 2n \), and the symbol \( c \) is interpreted as the constant 99. Which of the following formulae is true under these assumptions?

(a) \( \forall x \ (f(x) = c) \)
(b) \( \forall x \ (x = f(c)) \)
(c) \( \forall x \ (x \neq f(c)) \)
(d) \( \exists x \ (f(x) = c) \)
(e) \( \exists x \ (x = f(c)) \) ✓
3 Mathematical analysis

11. Which of the following functions is odd? (A function \( f \) is odd if for every \( x \in \mathbb{R} \) we have that \( f(-x) = -f(x) \).)
   
   (a) \(|x|\)
   (b) \(\sin(x)\) ✓
   (c) \(x^2\)
   (d) \(e^x\)
   (e) \(1 - x\)

12. Consider the function \( f(x) = e^x \). Which of the following statements is false? (The symbol \( \mathbb{R} \) denotes the set of all real numbers.)
   
   (a) The domain of \( f \) is the set \( \mathbb{R} \).
   (b) \( f'(x) = f(x) \) for every \( x \in \mathbb{R} \)
   (c) The function \( f \) is continuous on \( \mathbb{R} \). ✓
   (d) The image of \( f \) is the set \( \mathbb{R} \).
   (e) \( f(0) = 1 \)

\( f' \) denotes the derivative of function \( f \).

13. Consider the sequence of real numbers \( (a_n)_{n=0}^\infty \) defined as follows:

\[
a_n = \begin{cases} 
\frac{1}{2^n} & \text{if } n \text{ is even} \\
-\frac{1}{2^n} & \text{otherwise}
\end{cases}
\]

What is the value of the limit \( \lim_{n \to \infty} a_n \)?

(a) \(-1\)
(b) \(0\) ✓
(c) \(1\)
(d) \(+\infty\)
(e) The limit does not exist.

14. Which of the following functions is the derivative of \( x \cdot e^{3x} \)?

(a) \(1\)
(b) \(e^{3x}\)
(c) \(1 + 3e^{3x}\)
(d) \(3 \cdot (1 + e^{3x})\)
(e) \(e^{3x} \cdot (1 + 3x)\) ✓

15. Which of the following numbers is equal to \( \int_1^2 3x^2 \, dx \)?

(a) \(1\)
(b) \(3\)
(c) \(7\) ✓
(d) \(9\)
(e) \(21\)
4 Graphs and graph algorithms

16. Consider the following weighted directed graph:

For every pair of vertices \( z, z' \) denote by \( \delta(z, z') \) the length (i.e. the sum of edge weights) of the shortest path from the vertex \( z \) to the vertex \( z' \). Which of the following equalities is true?

(a) \( \delta(s, s) = 1 \)
(b) \( \delta(s, u) = 3 \)
(c) \( \delta(s, v) = 0 \)
(d) \( \delta(s, x) = 4 \)
(e) \( \delta(s, y) = 3 \)

17. The diameter of an undirected graph \( G = (V, E) \) is the number \( \max_{u, v \in V} d(u, v) \), where \( d(u, v) \) denotes the length (i.e. the number of edges) of the shortest path from a vertex \( u \) to a vertex \( v \). What is the diameter of the following graph?

(a) 1
(b) 2
(c) 3
(d) 4
(e) \( \infty \)
18. Consider the following directed graph:

```
  b     a     c
   ↓     |     ↓
   c     d     f
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Which of the following sequences may represent an order in which the breadth-first search algorithm discovers new vertices starting from \( a \)? (We do not assume any implicit ordering of vertices. Thus, the order in which the breadth-first search algorithm discovers new vertices may not be unique.)

(a) \( a, b, d, e, c, f \)
(b) \( a, c, f, b, d, e \)
(c) \( a, b, e, d, c, f \)
(d) \( a, b, d, c, e, f \) ✓
(e) \( a, d, e, f, b, c \)

19. In general, how many edges does a tree with \( n \) vertices have? (In the following, \( \lceil x \rceil \) denotes the largest integer not greater than \( x \).)

(a) \( \lceil \log_2(n) \rceil \)
(b) \( n - 1 \) ✓
(c) \( n \cdot \log_2(n) \)
(d) \( \frac{n(n-1)}{2} \)
(e) \( n^2 \)

20. For which of the following problems no polynomial time algorithm is known?

(a) Given a weighted graph \( G \), find the shortest paths between all pairs of its vertices.
(b) Given a directed graph \( G \) and two of its vertices \( u \) and \( v \), decide whether \( v \) is reachable from \( u \).
(c) Given an undirected graph \( G \), decide whether \( G \) contains a path that visits each vertex exactly once. ✓
(d) Given an undirected graph \( G \), decide whether \( G \) is connected.
(e) Given a weighted undirected graph \( G \), find the minimum spanning tree of \( G \).
5 Linear algebra

21. Which of the following mappings from \( \mathbb{R} \) to \( \mathbb{R} \) is not linear?
(A mapping \( f \) is linear if it satisfies \( f(x + y) = f(x) + f(y) \) and \( f(c \cdot x) = c \cdot f(x) \) for every \( x, y, c \).)

(a) \( f(x) = 0 \)
(b) \( f(x) = 2x \)
(c) \( f(x) = 2x + 3x \)
(d) \( f(x) = 2x - 3x \)
(e) \( f(x) = 2x \cdot 3x \) ✓

22. \[
\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} =
\]

(a) \( \begin{pmatrix} 7 & 10 \end{pmatrix} \)
(b) \( \begin{pmatrix} 1 & 2 \\ 6 & 8 \end{pmatrix} \)
(c) \( \begin{pmatrix} 1 & 4 \\ 3 & 8 \end{pmatrix} \)
(d) \( \begin{pmatrix} 5 \\ 11 \end{pmatrix} \) ✓
(e) \( \begin{pmatrix} 3 \\ 14 \end{pmatrix} \)

23. Consider the following system of linear equations over \( \mathbb{R} \):

\[
\begin{align*}
x + y &= 5 \\
x + 2y &= 6 \\
2x + y &= 8
\end{align*}
\]

Which of the following is true?

(a) The only solution is \( x = 3 \), \( y = 2 \).
(b) The only solution is \( x = 4 \), \( y = 1 \).
(c) There is only one solution (but it is neither \( x = 3 \), \( y = 2 \), nor \( x = 4 \), \( y = 1 \)).
(d) There are multiple solutions.
(e) There is no solution. ✓

24. What is the dimension of the linear span of the set of vectors \{\((1, 1, 0), (0, 0, 1), (1, 1, 1)\)\}?
(The linear span of a set of vectors is the space of all linear combinations of these vectors.)

(a) 0
(b) 1
(c) 2 ✓
(d) 3
(e) \( \infty \)
25. Which of the following matrices determines the linear mapping $A$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ which flips the plane horizontally (as indicated in the figure below)?

\begin{figure}
\centering
\begin{tikzpicture}
\draw[->] (0,0) -- (4,0) node[right] {$x$};
\draw[->] (0,0) -- (0,4) node[above] {$y$};
\draw (0,1) -- (3,1) node[midway,above] {\(A(v)\)};
\fill (0,1) circle (2pt) node[above] {$v$};
\fill (3,1) circle (2pt) node[above] {\(\tilde{v}\)};
\end{tikzpicture}
\end{figure}

(a) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
(b) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\checkmark$
(c) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
(d) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
(e) $\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$

(Every matrix $M$ defines a linear mapping $A$ by $A(\vec{v}) = M\vec{v}$.)