Petr Hliněný

Approaching Tree-Width of Graphs from a Matroidal Perspective

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Based on joint work with **Geoff Whittle**

Victoria University of Wellington

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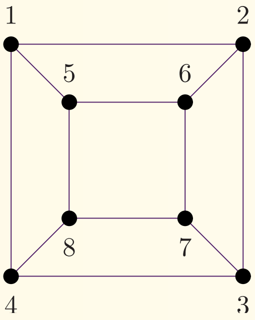
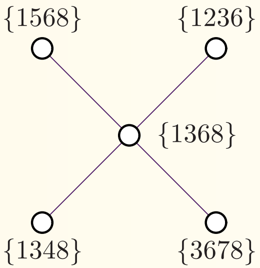
Tree-width $= \mathbf{min}_{\text{decompositions of } \boldsymbol{G}}\ \mathbf{max}\,\{|\boldsymbol{B}| - \mathbf{1} : \boldsymbol{B} \text{ bag in decomp.}\}$





Alternative traditional definition

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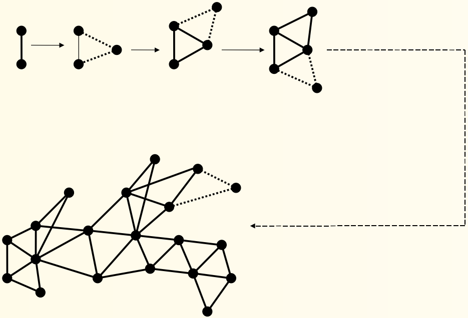
- This can be much easier understood via *k-trees*, see e.g. a 2-tree:



[Beineke & Pippert, 68 – 69], [Rose 74], [Arnborg & Proskurowski, 86].

- A graph G has tree-width $\leq k$ iff G is a partial (subgraph of a) k-tree.

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- Logic side:

  Decidability of *MSO theories* of the graphs of bounded tree-width [Courcelle 88]; a converse by [Seese 91].

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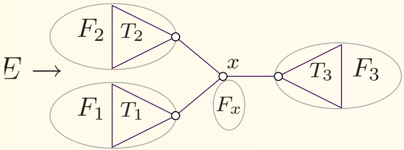

– *Node with of* $x = |V(G)| + (d-1)\cdot c(G) - \sum_{i=1}^{d} c(G - F_i)$,
where F_i are the edges mapped to the subtrees $T - x$,
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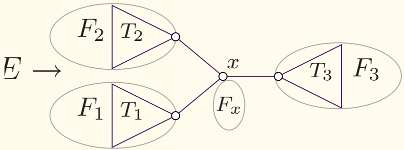

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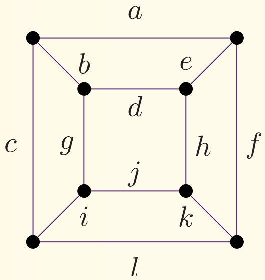
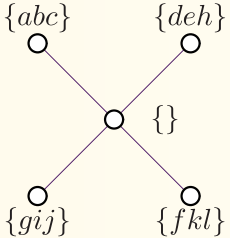
Check the following examples for an illustration...





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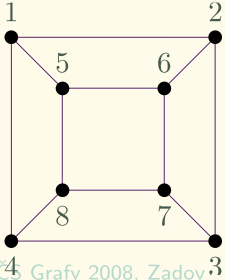

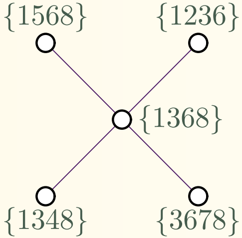

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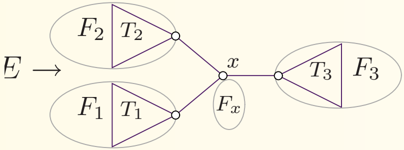

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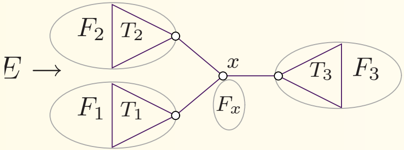

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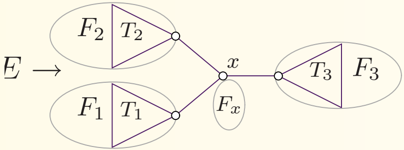

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- BTW, if a matroid M has tree-width k and branch-width b (which readily extends to matroids), then $b-1 \leq k \leq \max(2b-1, 1)$ — that is nice...

Comparing the tree-width parameters

Theorem [PH & Whittle, 03]. Let a graph G has an edge, and M be the cycle matroid of G. Then the tree-width of G equals the tree-width of M.

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  - affine *subspaces* modelling the traditional "bags",
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- An ordinary tree-decomposition can be readily translated into a VF tree-decomposition; just find a bag hosting each edge of G.

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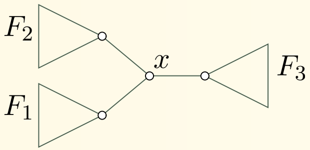
- What happens in the converse direction?
  - Again, any VF tree-decomposition naturally translates into an ordinary decomposition (just apply the interpolation property to the ends of mapped edges).
  - However, the width may increase (dramatically)!

The problem is that edges mapped to a branch in the decomposition may induce a disconnected subgraph, hence further decreasing the node-width in the VF setting...



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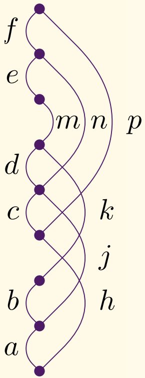
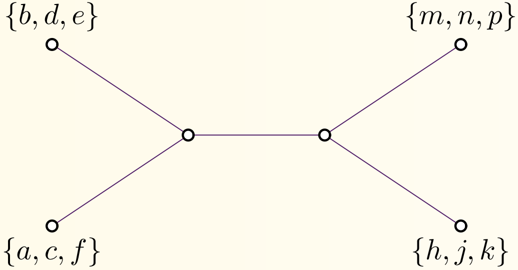
An example of a "disconnected" decomposition





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Easy to check that all six nodes in this VF tree-decomposition have width 4.

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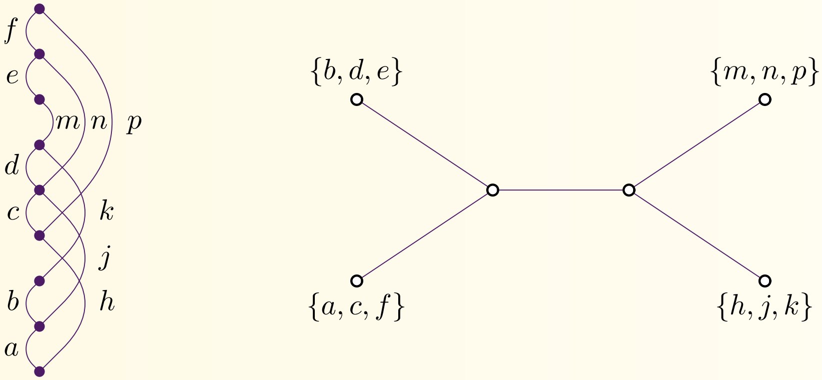

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However, the central two nodes induce bags of size $\mathbf{9}$ in an ordinary tree-decomposition! (tree-width up to 8)

Handling a "disconnected" decomposition

- If we want to get an ordinary tree-decomposition of the same width, we have to alter "disconnected" spots of a VF tree-decomposition. . .
- Actually, the proof complications appear similar to those emerging when proving equality of matroid branch-width to graph branch-width [Hicks & McMurray, 07], [Mazoit & Thomassé].
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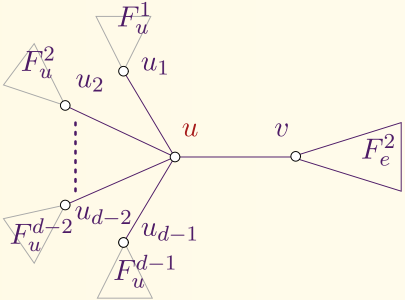
- In response to that, [PH & Whittle, 08] have got an updated, though longer proof.

  We sketch its idea next...

Proof (altering a "disconnected" edge of a VF tree-decomposition T of G).

- We assume an edge $e = uv$ of T such that the G-edges mapped to the u-branch of T form a disconnected subgraph of G, and that the edges mapped to the branches of u-neighbours (not v) stay connected in G.



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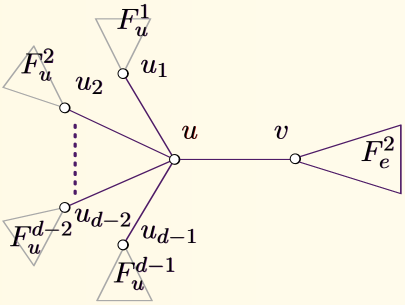

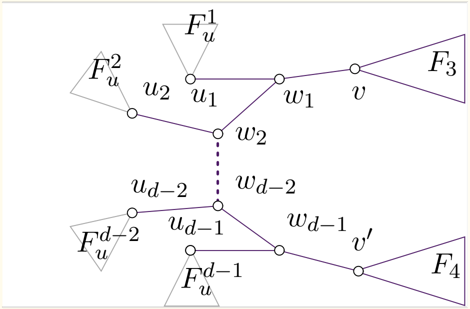

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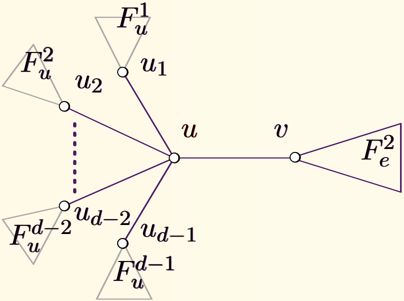

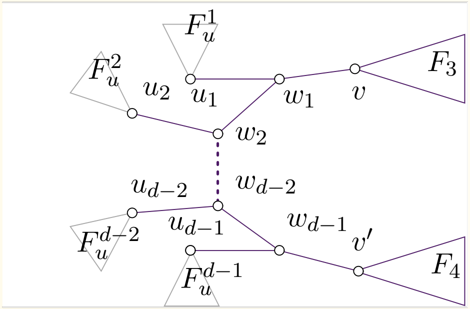

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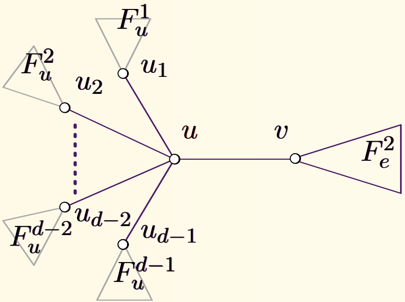

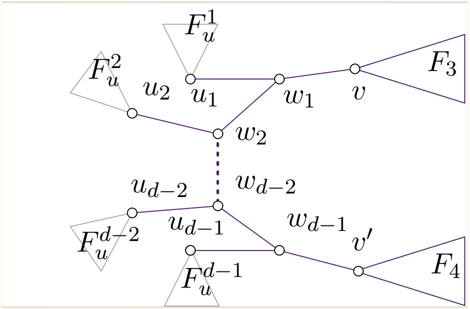

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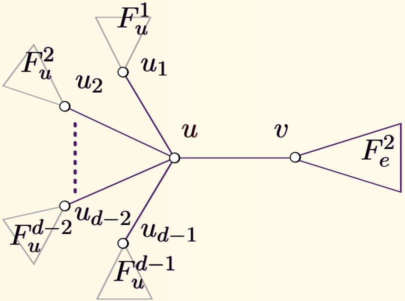

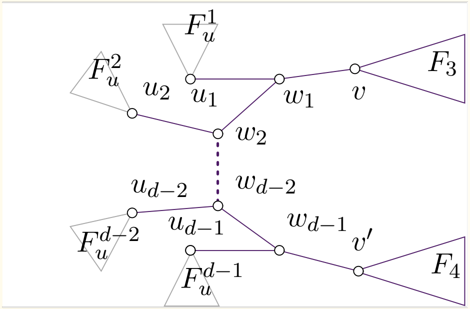

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- After all, there is a "strictly decreasing" sequence of alterations, leading to the connected case in which both tree-width measures are equal.

4 Conclusions

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