On Decidability of MSO Theories of Representable Matroids

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Abstract. We show that, for every finite field $\mathbb{F}$, the class of all $\mathbb{F}$-representable matroids of branch-width at most a constant t has a decidable MSO theory. In the other direction, we prove that every class of $\mathbb{F}$-representable matroids with a decidable MSO theory must have uniformly bounded branch-width.

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1 Introduction

Monadic second order logic, which extends first order logic by allowing quantification over monadic predicates, is famous for their high expressive power in combination with a manageable model theory (see e.g. [12]). For this reason it has found many applications in different areas, as e.g. decidability, model checking, data bases, and computational complexity.

Of special importance in this area are classes of graphs (or other structures) of bounded tree-width, branch-width, or clique-width, since for these classes MSO logic posseses besides the good model theory also very good algorithmic properties. On the structural side, strong interest in tree-width has been motivated by the (now famous) Graph Minor project [19] of Robertson and Seymour which, besides many deep theoretical results, revolutionized the area of algorithm design in computer science. In particular, many problems which are NP-hard for arbitrary structures, could be solved in polynomial and often even linear time if they are restricted to structures of bounded tree-width or bounded clique-width (see e.g. [1], or [8],[7]).

Interestingly, general algorithmic results on efficient computability over structures of bounded tree-width (branch-width, clique-width, etc.) come hand in

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hand with related logic results on decidability of theories. For example, for each $k > 0$, the monadic second order theory of the class of all graphs of tree-width at most k, or of clique-width at most k, respectively, is decidable (see [2, 6], or [22, 23]).

Here we shall concentrate on matroids, as a strong generalization of graphs. Nowadays, one can witness in the matroid community a great effort to extend the above mentioned Robertson-Seymour's theoretical work on graph minors as far as possible to matroids, followed by important new structural results about representable matroids, eg. [10, 11]. Inspired by those advances, we focus on extending the research of related complexity and logic questions from graphs to matroids, building on recent works [13, 14] of the first author.

Since this paper is intended for general computer-science and logic audiences, we provide some basic definitions concerning matroid structure, and decidability and interpretability of theories from mathematical logic, in the next three sections. We bring up the MSO theory of matroids in Section 5, and present some related recent results there; like we show that the MSO theory of the class of all matroids of branch-width at most k is decidable, for every $k > 0$. We present our main result in Section 6, which extends results by the second author from [22]. We prove that, for every finite field $\mathbb{F}$, a class of $\mathbb{F}$-representable matroids with a decidable MSO theory must have uniformly bounded branch-width.

2 Basics of Matroids

We refer to Oxley [16] for matroid terminology. A *matroid* is a pair $M = (E, \mathcal{B})$ where $E = E(M)$ is the ground set of M (elements of M), and $\mathcal{B} \subseteq 2^E$ is a nonempty collection of *bases* of M. Moreover, matroid bases satisfy the "exchange axiom"; if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$, then there is $y \in B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$. We consider only finite matroids. Subsets of bases are called *independent sets*, and the remaining sets are *dependent*. Minimal dependent sets are called *circuits*. All bases have the same cardinality called the *rank* $\mathrm{r}(M)$ of the matroid. The *rank function* $\mathrm{r}_M(X)$ in M is the maximal cardinality of an independent subset of a set $X \subseteq E(M)$.

If G is a (multi)graph, then its *cycle matroid* on the ground set $E(G)$ is denoted by $M(G)$. The independent sets of $M(G)$ are acyclic subsets (forests) in G, and the circuits of $M(G)$ are the cycles in G. Another example of a matroid is a finite set of vectors with usual linear dependency. If $\boldsymbol{A}$ is a matrix, then the matroid formed by the column vectors of $\boldsymbol{A}$ is called the *vector matroid* of $\boldsymbol{A}$, and denoted by $M(\boldsymbol{A})$. The matrix $\boldsymbol{A}$ is a *representation* of a matroid $M \simeq M(\boldsymbol{A})$. We say that the matroid $M(\boldsymbol{A})$ is $\mathbb{F}$-*represented* if $\boldsymbol{A}$ is a matrix over a field $\mathbb{F}$. (Fig. 1.) A *graphic matroid*, i.e. such that it is a cycle matroid of some multigraph, is representable over any field.

The *dual* matroid M^* of M is defined on the same ground set E, and the bases of M^* are the set-complements of the bases of M. A set X is *coindependent* in M if it is independent in M^*. An element e of M is called a *loop* (a *coloop*), if

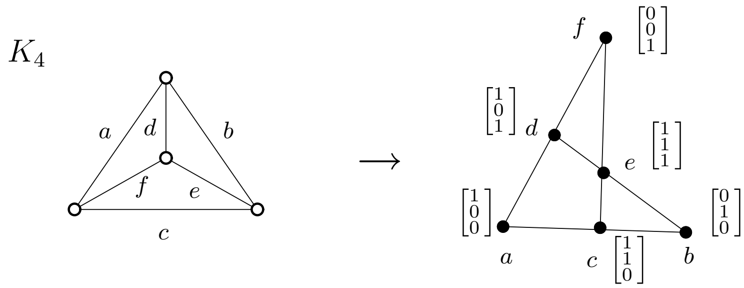

Fig. 1. An example of a vector representation of the cycle matroid $M(K_4)$. The matroid elements are depicted by dots, and their (linear) dependency is shown using lines.

$\{e\}$ is dependent in M (in M^*). The matroid $M \setminus e$ obtained by *deleting* a non-coloop element e is defined as $(E - \{e\}, \mathcal{B}^-)$ where $\mathcal{B}^- = \{B : B \in \mathcal{B},\, e \notin B\}$. The matroid M/e obtained by *contracting* a non-loop element e is defined using duality $M/e = (M^* \setminus e)^*$. (This corresponds to contracting an edge in a graph.) A *minor* of a matroid is obtained by a sequence of deletions and contractions of elements. Since these operations naturally commute, a minor M' of a matroid M can be uniquely expressed as $M' = M \setminus D/C$ where D are the coindependent deleted elements and C are the independent contracted elements. The following claim is folklore in matroid theory:

Lemma 2.1. *Let $N = M \setminus D/C$. Then a set $X \subseteq E(N)$ is dependent in N if and only if there is a dependent set $Y \subseteq E(M)$ in M such that $Y - X \subseteq C$.*

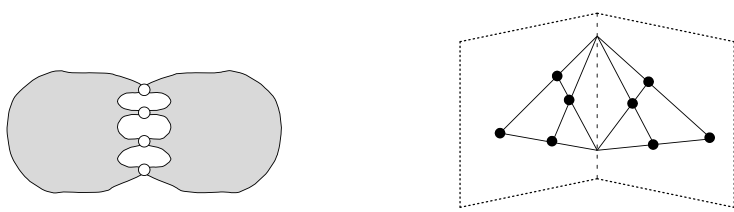

Fig. 2. An illustration to a 4-separation in a graph, and to a 3-separation in a matroid.

Another important concept is matroid connectivity, which is close, but somehow different, to traditional graph connectivity. The *connectivity function* λ_M of a matroid M is defined for all subsets $A \subseteq E$ by

$$\lambda_M(A) = r_M(A) + r_M(E - A) - r(M) + 1 \,.$$

Here $r(M) = r_M(E)$. A subset $A \subseteq E$ is *k-separating* if $\lambda_M(A) \le k$. A partition $(A, E - A)$ is called a *k-separation* if A is k-separating and both $|A|, |E - A| \ge k$. Geometrically, the spans of the two sides of a k-separation intersect in a subspace of rank less than k. See in Fig. 2. In a corresponding graph view, the connectivity function $\lambda_G(F)$ of an edge subset $F \subseteq E(G)$ equals the number of vertices of G incident both with F and with $E(G) - F$. (Then $\lambda_G(F) = \lambda_{M(G)}(F)$ provided both sides of the separation are connected in G.)

3 Tree-Width and Branch-Width

The notion of graph *tree-width* is well known. Let Q_n denote the $n \times n$*-grid graph*, i.e. the graph on $V(Q_n) = \{1, 2, \dots, n\}^2$ and $E(Q_n) = \big\{\{(i,j)(i',j')\} : 1 \le i, j, i', j' \le n,\ \{|i - i'|, |j - j'|\} = \{0, 1\}\big\}$. We say that a class $\mathcal{G}$ of graphs has *bounded tree-width* if there is a constant k such that any graph $G \in \mathcal{G}$ has tree-width at most k. A basic structural result on tree-width is given in [20]:

Theorem 3.1. (Robertson, Seymour) *A graph class $\mathcal{G}$ has bounded tree-width if and only if there exists a constant m such that no graph $G \in \mathcal{G}$ has a minor isomorphic to Q_m.*

The same paper [20] also presents a similar, but less known, parameter called branch-width, and proves that branch-width is within a constant factor of tree-width on graphs.

Assume that λ is a symmetric function on the subsets of a ground set E. (Here $\lambda \equiv \lambda_G$ is the connectivity function of a graph, or $\lambda \equiv \lambda_M$ of a matroid.) A *branch decomposition* of λ is a pair (T, τ) where T is a sub-cubic tree ($\Delta(T) \le 3$), and τ is a bijection of E into the leaves of T. For e being an edge of T, the *width* of e in (T, τ) equals $\lambda(A) = \lambda(E - A)$, where $A \subseteq E$ are the elements mapped by τ to leaves of one of the two connected components of $T - e$. The width of the branch decomposition (T, τ) is maximum of the widths of all edges of T, and *branch-width* of λ is the minimal width over all branch decompositions of λ.

Recall the definitions of graph and matroid connectivity functions from Section 2. Then branch-width of $\lambda \equiv \lambda_G$ is called *branch-width of a graph G*, and that of $\lambda \equiv \lambda_M$ is called *branch-width of a matroid M*. (See examples in Fig. 3.) We remark that it is possible to define matroid tree-width [15] which is within a constant factor of branch-width, but this is not a straightforward extension of traditional graph tree-width. Considering branch-width on matroids, the following recent result [11] is crucial for our paper:

Theorem 3.2. (Geelen, Gerards, Whittle) *For every finite field $\mathbb{F}$; a class $\mathcal{N}$ of $\mathbb{F}$-representable matroids has bounded branch-width if and only if there exists a constant m such that no matroid $N \in \mathcal{N}$ has a minor isomorphic to $M(Q_m)$.*

4 Decidability of Theories

We will use the following notion of a theory. Let $\mathcal{K}$ be a class of structures and let L be a suitable logic for $\mathcal{K}$. A sentence is a set of well-formed L-formulas without free variables. The set of all L-sentences true in $\mathcal{K}$ is denoted as L-theory of $\mathcal{K}$. We use $\mathrm{Th}_L(\mathcal{K})$ as a short notation for this theory. Hence, a theory

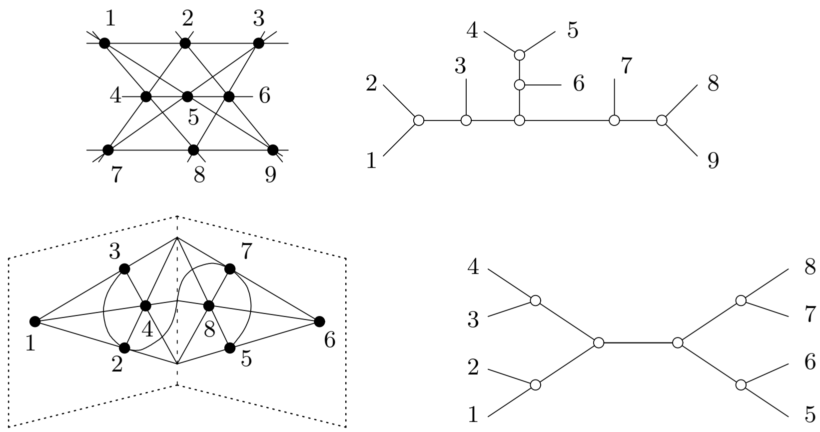

Fig. 3. Two examples of width-3 branch decompositions of the Pappus matroid (top left, in rank 3) and of the binary affine cube (bottom left, in rank 4). The lines in matroid pictures show dependencies among elements.

can be viewed as the set of all properties, expressible in L, which all structures of $\mathcal{K}$ possess. In case that $\mathcal{K} = \{G\}$ we write $\mathrm{Th}_L(G)$ instead of $\mathrm{Th}_L(\mathcal{K})$. Using this definition we obtain $\mathrm{Th}_L(\mathcal{K}) = \bigcap\{\mathrm{Th}_L(G) : G \in \mathcal{K}\}$. We write $\mathrm{Th}(\mathcal{K})$, $\mathrm{Th}_{MSO}(\mathcal{K})$ if L is first order logic, or monadic second order logic (abbreviated as MSO logic), respectively.

For graphs there are actually two variants of MSO logic, commonly denoted by MS_1 and MS_2. In MS_1, set variables only denote sets of vertices. In MS_2, set variables can also denote sets of edges of the considered graph. In other words the difference between both logics is that in MS_1 the domain of the graph consists of the vertices only and the relation is just the usual adjacency between vertices, while in MS_2 the domain is two-sorted and contains vertices as well as edges and the relation is the incidence relation. The expressive power of both logics was studied by Courcelle in [4].

A theory is said to be *decidable* if there is an algorithm deciding, for an arbitrary sentence $\varphi \in L$, whether $\varphi \in \mathrm{Th}_L(\mathcal{K})$ or not, i.e. whether φ is true in all structures of $\mathcal{K}$. Otherwise this theory is said to be *undecidable*. More information concerning the terminology from logic needed in this section can be found in classical textbooks as [9]. A good introduction into the decidability of theories can be found in [18] (see also [12] for monadic theories).

To prove decidability of theories the method of model interpretability, introduced in [17] is often the best tool of choice. To describe the idea of the method assume that two classes of structures $\mathcal{K}$ and $\mathcal{K}'$ are given, and that L and L', respectively, are corresponding languages for the structures of these classes. The basic idea of the interpretability of theory $\mathrm{Th}_L(\mathcal{K})$ into $\mathrm{Th}_{L'}(\mathcal{K}')$ is to transform formulas of L into formulas of L', by translating the nonlogical symbols of L by formulas of L', in such a way that truth is preserved in a certain way. Here we assume that the logics underlying both languages are the same. Otherwise, one has to translate also the logical symbols.

We explain this translation in a simple case of relational structures. First one chooses an L'-formula $\alpha(x)$ intended to define in each L'-structure $G \in \mathcal{K}'$ a set of individuals $G[\alpha] := \{a : a \in dom(G) \text{ and } G \models \alpha(a)\}$, where $dom(G)$ denotes the domain (set of individuals) of G. Then one chooses for each s-ary relational sign R from L an L'-formula $\beta_R(x_1, \ldots, x_s)$, with the intended meaning to define a corresponding relation $G[\beta_R] := \{(a_1, \ldots, a_s) : a_1, \ldots, a_s \in dom(G) \text{ and } G \models \beta_R(a_1, \ldots, a_s)\}$. All these formulas build the formulas of the interpretation $I = \big(\alpha(x), \beta_R(x_1, \ldots, x_s), \ldots\big)$.

With the help of these formulas one can define for each L'-structure G a structure $G^I := \big(G[\alpha], G[\beta_R], \ldots\big)$, which is just the structure defined by the chosen formulas in G. Sometimes G^I is also denoted as $I(G)$ and I is called an (L, L')-interpretation of G^I in G. In case that both L and L' are MSO languages, this interpretation is also denoted as MSO-interpretation. Using these formulas there is also a natural way to translate each L-formula φ into an L'-formula φ^I. This is done by induction on the structure of formulas. The atomic formulas are simply substituted by the corresponding chosen formulas with the corresponding substituted variables. Then one may proceed via induction as follows:

$$(\neg\chi)^I := \neg(\chi^I), \quad (\chi_1 \wedge \chi_2)^I := (\chi_1)^I \wedge (\chi_2)^I,$$

$$\big(\exists x\, \chi(x)\big)^I := \exists x\, \big(\alpha(x) \wedge \chi^I(x)\big),$$

$$(x \in X)^I := x \in X, \quad \big(\exists X\, \chi(X)\big)^I := \exists X\, \chi^I(X).$$

The resulting translation is called an interpretation with respect to L and L'. Its concept could be briefly illustrated with a picture:

$$\begin{array}{ccc} \varphi \in L & \overset{I}{\longrightarrow} & \varphi^I \in L' \\ H \in \mathcal{K} & & G \in \mathcal{K}' \\ & & \\ G^I \simeq H & \overset{I}{\longleftarrow} & G \end{array}$$

For theories, interpretability is now defined as follows. Let $\mathcal{K}$ and $\mathcal{K}'$ be classes of structures and L and L' be corresponding languages. Theory $\mathrm{Th}_L(\mathcal{K})$ is said to be interpretable in $\mathrm{Th}_{L'}(\mathcal{K}')$ if there is an (L, L')-interpretation I translating each L-formula φ into an L'-formula φ^I, and each L'-structure $G \in \mathcal{K}'$ into an L-structure G^I as above, such that the following two conditions are satisfied:

(i) For every structure $H \in \mathcal{K}$, there is a structure $G \in \mathcal{K}'$ such that $G^I \cong H$,
(ii) for every $G \in \mathcal{K}'$, the structure G^I is isomorphic to some structure of $\mathcal{K}$.

It is easy to see that interpretability is transitive. The key result for interpretability of theories is the following theorem [17]:

Theorem 4.1. (Rabin) *Let $\mathcal{K}$ and $\mathcal{K}'$ be classes of structures, and L and L' be suitable languages. If $\mathrm{Th}_L(\mathcal{K})$ is interpretable in $\mathrm{Th}_{L'}(\mathcal{K}')$, then undecidability of $\mathrm{Th}_L(\mathcal{K})$ implies undecidability of $\mathrm{Th}_{L'}(\mathcal{K}')$.*

5 MSO Theory of Matroids

Considering logic point of view, a matroid M on a ground set E is the collection of all subsets 2^E together with a unary predicate *indep* such that $indep(F)$ if and only if $F \subseteq E$ is independent in M. (One may equivalently consider a matroid with a unary predicate for bases or for circuits, see in [13].) We shortly write MS_M to say that the language of *MSO logic is applied to (independence) matroids*. If $\mathcal{N}$ is a class of independence matroids, then the MS_M theory of $\mathcal{N}$ is denoted by $\mathrm{Th}_{MSO}(\mathcal{N})$.

To give readers a better feeling for the expressive power of MS_M on a matroid, we write down a few basic matroid predicates now.

- We write $basis(B) \equiv indep(B) \wedge \forall D\big(B \not\subseteq D \vee B = D \vee \neg\, indep(D)\big)$ to express the fact that a basis is a maximal independent set.
- Similarly, we write $circuit(C) \equiv \neg\, indep(C) \wedge \forall D\big(D \not\subseteq C \vee D = C \vee indep(D)\big)$, saying that C is dependent, but all proper subsets of C are independent.
- A cocircuit is a dual circuit in a matroid (i.e. a bond in a graph). We write $cocircuit(C) \equiv \forall B\big[\, basis(B) \rightarrow \exists x(x \in B \wedge x \in C)\big] \wedge \forall X\big[X \not\subseteq C \vee X = C \vee \exists B\big(\, basis(B) \wedge \forall x(x \notin B \vee x \notin X)\big)\big]$ saying that a cocircuit C intersects every basis, but each proper subset of C is disjoint from some basis.

It is shown that the language of MS_M is at least as powerful as that of MS_2 on graphs. Let $G \uplus H$ denotes the graph obtained from disjoint copies of G and H by adding all edges between them. The following statement is proved in [13]:

Theorem 5.1. (Hliněný) *Let G be a loopless multigraph, and let M be the cycle matroid of $G \uplus K_3$. Then any MSO sentence (in MS_2) about an incidence graph G can be expressed as a sentence about M in MS_M.*

In other words, the MSO theory of (loopless) incidence multigraphs is interpretable in a certain subclass of 3-connected graphic matroids.

The next result we are going to mention speaks about (restricted) recognizability of MS_M-definable matroid properties via tree automata. To formulate this, we have to introduce briefly the concept of *parse trees* for representable matroids of bounded branch-width, which has been first defined in [13]. For a finite field $\mathbb{F}$, an integer $t \geq 1$, and an arbitrary $\mathbb{F}$-represented matroid M of branch-width at most $t + 1$; a t-boundaried parse tree $\bar{T}$ over $\mathbb{F}$ is a rooted ordered binary tree, whose leaves are labeled with elements of M, and the inner nodes are labeled with symbols of a certain finite alphabet (depending on $\mathbb{F}$ and t). Saying roughly, symbols of the alphabet are "small configurations" in the projective geometry over $\mathbb{F}$. The parse tree $\bar{T}$ uniquely determines an $\mathbb{F}$-representation (up to projective transformations) of the matroid $P(\bar{T}) \simeq M$. See [13] for more details and the result:

Theorem 5.2. (Hliněný) *Let $\mathbb{F}$ be a finite field, $t \geq 1$, and let ϕ be a sentence in the language of MS_M. Then there exists a finite tree automaton $\mathcal{A}_t^\phi$ such that the following is true: A t-boundaried parse tree $\bar{T}$ over $\mathbb{F}$ is accepted by $\mathcal{A}_t^\phi$ if and only if $P(\bar{T}) \models \phi$. Moreover, the automaton $\mathcal{A}_t^\phi$ can be constructed (algorithmically) from given $\mathbb{F}$, t, and ϕ.*

Corollary 5.3. *Let $\mathbb{F}$ be a finite field, $t \geq 1$, and let $\mathcal{B}_t$ be the class of all matroids representable over $\mathbb{F}$ of branch-width at most $t+1$. Then the theory* $\mathrm{Th}_{MSO}(\mathcal{B}_t)$ *is decidable.*

Proof. Assume we are given an MS_M-sentence ϕ. We construct the automaton $\mathcal{A}_t^\phi$ from Theorem 5.2. Moreover, there is an (easily constructible [13]) automaton $\mathcal{V}_t$ accepting valid t-boundaried parse trees over $\mathbb{F}$. Then $\mathcal{B}_t \not\models \phi$ if and only if there is a parse tree accepted by $\mathcal{V}_t$, but not accepted by $\mathcal{A}_t^\phi$. We thus, denoting by $-\mathcal{A}_t^\phi$ the complement of $\mathcal{A}_t^\phi$, construct the cartesian product automaton $\mathcal{A} = (-\mathcal{A}_t^\phi) \times \mathcal{V}_t$ accepting the intersection of the languages of $-\mathcal{A}_t^\phi$ and of $\mathcal{V}_t$. Then we check for emptiness of $\mathcal{A}$ using standard tools of automata theory. □

6 Large Grids and Undecidability

We need the following result, which was proved first in a more general form in [21] (see also [22]).

Theorem 6.1. (Seese) *Let $\mathcal{K}$ be a class of adjacency graphs such that for every integer $k > 1$ there is a graph $G \in \mathcal{K}$ such that G has the $k \times k$ grid Q_k as an induced subgraph. Then the MS_1 theory of $\mathcal{K}$ is undecidable.*

Using Theorems 3.1, 6.1 and interpretation, one concludes [22]:

Theorem 6.2. (Seese) *a) If a family $\mathcal{G}$ of planar graphs has a decidable MS_1 theory, then $\mathcal{G}$ has bounded tree-width.*
b) If a graph family $\mathcal{G}$ has a decidable MS_2 theory, then $\mathcal{G}$ has bounded tree-width.

Related results can be found also in [5] and [6]. The troubles, why part (a) of this theorem has to be formulated for planar graphs, lie in the fact that MS_1 logic (unlike MS_2) lacks expressive power to handle minors in arbitrary graphs. However, that is not a problem with our MS_M logic, cf. Theorem 5.1 or [14], and hence we can extend the (now stronger) part (b) to representable matroids as follows:

Theorem 6.3. *Let $\mathbb{F}$ be a finite field, and let $\mathcal{N}$ be a class of matroids that are representable by matrices over $\mathbb{F}$. If the (monadic second-order) MS_M theory* $\mathrm{Th}_{MSO}(\mathcal{N})$ *is decidable, then the class $\mathcal{N}$ has bounded branch-width.*

The key to the proof of this theorem is given in Theorem 3.2, which basically states that the obstructions to small branch-width on matroids are the same as on graphs. Unfortunately, the seemingly straightforward way to prove Theorem 6.3 — via direct interpretation of graphs (Theorem 6.2) in the class of graphic minors of matroids in $\mathcal{N}$, is not so simple due to technical problems with (low) connectivity. That is why we give here a variant of this idea bypassing Theorem 6.2, and using an indirect interpretation of (graph) grids in matroid grid minors.

Remark. A restriction to $\mathbb{F}$-representable matroids in Theorem 6.3 is not really necessary; it comes more from the context of the related matroid structure research. According to [11], it is enough to assume that no member of $\mathcal{N}$ has a $U_{2,m}$- or $U^*_{2,m}$-minor (i.e. an m-point line or an m-point dual line) for some constant m.

We begin the proof of Theorem 6.3 with an interpretation of the MS_M theory of all minors of the class $\mathcal{N}$. To achieve this goal, we use a little technical trick first. Let a *DC-equipped matroid* be a matroid M with two distinguished unary predicates D and C on $E(M)$ (with intended meaning as a pair of sets $D, C \subseteq E(M)$ defining a minor $N = M \setminus D/C$).

Lemma 6.4. *Let $\mathcal{N}$ be a class of matroids, and let $\mathcal{N}_{DC}$ denote the class of all DC-equipped matroids induced by members of $\mathcal{N}$. If $\mathrm{Th}_{MSO}(\mathcal{N})$ is decidable, then so is $\mathrm{Th}_{MSO}(\mathcal{N}_{DC})$.*

Proof. We may equivalently view the distinguished predicates D, C as free set variables in MS_M. Let $\phi(D, C)$ be an MS_M formula, and $N \in \mathcal{N}$. Then, by standard logic arguments, $N_{DC} \models \phi(D, C)$ for all DC-equipped matroids N_{DC} induced by N if and only if $N \models \forall D, C\, \phi(D, C)$. Hence $\mathcal{N}_{DC} \models \phi(D, C)$ if and only if $\mathcal{N} \models \forall D, C\, \phi(D, C)$. Since $\forall D, C\, \phi(D, C)$ is an MSO formula if ϕ is such, the statement follows. □

Lemma 6.5. *Let $\mathcal{N}$ be a class of matroids, and $\mathcal{N}_m$ be the class of all minors of members of $\mathcal{N}$. Then $\mathrm{Th}_{MSO}(\mathcal{N}_m)$ is interpretable in $\mathrm{Th}_{MSO}(\mathcal{N}_{DC})$.*

Proof. We again regard the distinguished predicates D, C of $\mathcal{N}_{DC}$ as free set variables in MS_M. Let us consider a matroid $N_1 \in \mathcal{N}_m$ such that $N_1 = N \setminus D_1/C_1$ for $N \in \mathcal{N}$. We are going to use a "natural" interpretation of N_1 in the DC-equipped matroid N_{DC} which results from N with a particular equipment $D = D_1$, $C = C_1$. (Notice that both theories use the same language of MSO logic, and the individuals of N_1 form a subset of the individuals of N.) Let ψ be an MS_M formula. The translation ψ^I of ψ is obtained inductively:

- For each (bound) element variable x in ψ; it is replaced with

$$\exists x\, \theta(x) \quad \longrightarrow \quad \exists x \left(x \notin C \wedge x \notin D \wedge \theta(x) \right).$$

- For each (bound) set variable X in ψ; it is replaced with

$$\exists X \theta(X) \quad \longrightarrow \quad \exists X \forall z \left((z \notin X \vee z \notin C) \wedge (z \notin X \vee z \notin D) \wedge \theta(X) \right).$$

- Every occurence of the *indep* predicate in ψ is rewritten as (cf. Lemma 2.1)

$$\mathit{indep}^I(X) \equiv \forall Y \left(\mathit{indep}(Y) \vee \exists z (z \in Y \wedge z \notin X \wedge z \notin C) \right).$$

Consider now the structure N^I defined by indep^I in $N_{DC} \in \mathcal{N}_{DC}$. By Lemma 2.1, a set $X \subseteq E(N^I) = E(N_1)$ is independent in N^I if and only if X is independent in N_1, and hence N^I is a matroid isomorphic to $N_1 = N \setminus D/C \in$

$\mathcal{N}_m$. Moreover, it is immediate from the construction of ψ^I that $N_1 \models \psi$ iff $N_{DC} \models \psi^I$. Thus, I is an interpretation of $\mathrm{Th}_{MSO}(\mathcal{N}_m)$ in $\mathrm{Th}_{MSO}(\mathcal{N}_{DC})$. □

Next, we define, for a matroid M, a *4CC-graph of* M as the graph G on the vertex set $E(M)$, and edges of G connecting those pairs of elements $e, f \in E(M)$, such that there are a 4-element circuit C and a 4-element cocircuit C' in M containing both $e, f \in C \cap C'$. (This is *not* the usual way of interpretation in which the ground set of a matroid is formed by graph edges.) The importance of our definition is in that 4CC-graphs "preserve" large grids:

Lemma 6.6. *Let $m \geq 6$ be even, and $M = M(Q_m)$. Denote by G the 4CC-graph of M. Then G has an induced subgraph isomorphic to Q_{m-2}.*

Proof. Recall that circuits in a cycle matroid of a graph correspond to graph cycles, and cocircuits to graph bonds (minimal edge cuts). The only 4-element cycles in a grid clearly are face-cycles in the natural planar drawing of Q_m. The only edge cuts with at most 4 edges in Q_m are formed by sets of edges incident with a single vertex in Q_m, or by edges that are "close to the corners".

Let $E' \subseteq E(Q_m)$ denote the edge set of the subgraph induced on the vertices (i,j) where $1 < i,j < m$. Let G' denotes the corresponding subgraph of G induced on E'. Choose $x \in E'$, and assume up to symmetry $x = \{(i,j),(i',j')\}$ where $i' = i+1$ and $j' = j$. According to the above arguments, the only neighbours of x in G' are in the set

$$E' \cap \big\{ \{(i,j-1),(i,j)\}, \{(i,j),(i,j+1)\}, \{(i',j'-1),(i',j')\}, \{(i',j'),(i',j'+1)\} \big\} .$$

We now define "coordinates" for the elements $x \in E'$ as follows

$$x = \{(i,j),(i',j')\},\ i \leq i', j \leq j' : \qquad k_x = i+j, \qquad \ell_x = i + j' - 2j .$$

As one may easily check from the above description of neighbours, two elements $x, y \in E'$ are adjacent in G' if and only if $\{|k_x - k_y|, |\ell_x - \ell_y|\} = \{0,1\}$. Hence the elements $x \in E'$ such that $\frac{m}{2} + 1 < k_x, \ell_x < \frac{m}{2} + m - 1$ induce in G' a grid isomorphic to Q_{m-2}. □

Now we are to finish a chain of interpretations from Theorem 6.1 to a proof of our Theorem 6.3.

Lemma 6.7. *Let $\mathcal{M}$ be a matroid family, and let $\mathcal{F}_4$ denote the class of all adjacency graphs which are 4CC-graphs of the members of $\mathcal{M}$. Then the MS_1 theory of $\mathcal{F}_4$ is interpretable in the MS_M theory $\mathrm{Th}_{MSO}(\mathcal{M})$.*

Proof. Let us take a graph $G \in \mathcal{F}_4$ which is a 4CC-graph of a matroid $M \in \mathcal{M}$. Now G is regarded as an adjacency graph structure, and so the individuals (the domain) of G are the vertices $V(G)$. These are to be interpreted in the ground set $E(M)$, the domain of M. Let ψ be an MS_1 formula. The translation ψ^I in MS_M of ψ is obtained simply by replacing every occurence of the *adj* predicate in ψ with

$$\mathit{adj}^I(x,y) \equiv \exists C, C'$$
$$\big(|C| = |C'| = 4 \wedge \mathit{circuit}(C) \wedge \mathit{cocircuit}(C) \wedge x,y \in C \wedge x,y \in C'\big)\,,$$

where the matroid MS_M predicates *circuit* and *cocircuit* are defined in Section 5, and $|X| = 4$ has an obvious interpretation in the FO logic.

Consider the structure G^I defined by the predicate adj^I on the domain $E(M)$. It is $G^I \simeq G$ by definition, for all pairs G, M as above. Moreover, adj^I is defined in the MSO logic. Hence we have got an interpretation I of $\mathrm{Th}_{MSO_1}(\mathcal{F}_4)$ in $\mathrm{Th}_{MSO}(\mathcal{M})$. □

Proof of Theorem 6.3. We prove the converse direction of the implication. Assume that a matroid class $\mathcal{N}$ does not have bounded branch-width, and denote by $\mathcal{N}_m$ the class of all matroids which are minors of some member of $\mathcal{N}$. By Theorem 3.2, for every integer $m > 1$, there is a matroid $N \in \mathcal{N}_m$ isomorphic to the cycle matroid of the grid $N \simeq M(Q_m)$. Now denote by $\mathcal{F}_4$ the class of all graphs which are 4CC-graphs of members of $\mathcal{N}_m$. Then, using Lemma 6.6, there exist members of $\mathcal{F}_4$ having induced subgraphs isomorphic to the grid Q_k, for every integer $k > 1$.

Hence the class $\mathcal{K} = \mathcal{F}_4$ satisfies the assumptions of Theorem 6.1, and so the MS_1 theory of $\mathcal{F}_4$ is undecidable. So is the MS_M theory $\mathrm{Th}_{MSO}(\mathcal{N}_m)$ using the interpretation in Lemma 6.7, and Theorem 4.1. We analogously apply the interpretation in Lemma 6.5 to $\mathrm{Th}_{MSO}(\mathcal{N}_m)$, and conclude that also $\mathrm{Th}_{MSO}(\mathcal{N}_{DC})$ is undecidable, where $\mathcal{N}_{DC}$ is the class of all DC-equipped matroids induced by $\mathcal{N}$ as above. Finally, Lemma 6.4 implies that the MS_M theory $\mathrm{Th}_{MSO}(\mathcal{N})$ is undecidable, as needed. □

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