Touching Graphs of Unit Balls*

Petr Hliněný

Dept. of Applied Mathematics, Charles University,
Malostranské nám. 25, 118 00 Praha 1,
hlineny@kam.ms.mff.cuni.cz

Abstract. The touching graph of balls is a graph that admits a representation by non-intersecting balls in the space (of prescribed dimension), so that its edges correspond to touching pairs of balls. By a classical result of Koebe [?], the disc touching graphs are exactly the planar graphs. This paper deals with a recognition of unit-ball touching graphs. The 2–dimensional case was proved to be *NP*-hard by Breu and Kirkpatrick [?]. We show in this paper that also unit-ball touching graphs in dimensions 3 and 4 are *NP*-hard to recognize. By a recent result of Kirkpatrick and Rote, these results may be transferred in ball-touching graphs in one dimension higher.

1 Introduction

The intersection graphs of geometrical objects have been extensively studied for their many applications. Formally the *intersection graph* of a set family $\mathcal{M}$ is defined as a graph G with the vertex set $V(G) = \mathcal{M}$ and the edge set $E(G) = \big\{\{A,B\} \subseteq \mathcal{M} \,|\, A \neq B,\ A \cap B \neq \emptyset\big\}$. Then *geometrical intersection graphs* are those in which the set family $\mathcal{M}$ is determined by some geometrical meaning; in that case we can also think about *touching* (or *contact*) *graphs* if we allow the geometrical objects only to touch each other.

A graph H is called a touching graph of a certain geometrical type, if it is isomorphic to the intersection graph G of some touching set family $\mathcal{M}$ of that type. The *recognition problem* of intersection (spec. touching) graphs is the question whether given graph is isomorphic to an intersection (touching) graph of the specified type.

Probably the first interest in touching graphs is represented by a very nice result of Koebe [?], who proved that touching graphs of arbitrary discs in the plane are exactly the planar graphs (and this result was also rediscovered later). Recently, practical applications led to the introduction of more complex classes of intersection and touching graphs, most of which are *NP*-hard to recognize. Among the touching (or contact) graphs, that have attracted attention recently, we may notice the triangle contact graphs [?] or the contact graphs of straight line segments and of simple curves [?] in the plane.

* The author acknowledges the support of the grant GAUK 193/1996, Czech–US science and technology research grant no. 94051, and the DONET grant.

The interest of our paper is in unit-ball touching graphs in Euclidean space. The d–ball touching graphs are those that admit a representation of vertices by balls in E^d, so that their interiors are disjoint and two vertices are adjacent iff the two corresponding balls touch each other. Specially, if all the balls are of the same diameter, say 1, the *unit-ball touching graphs* are obtained. The touching set of balls is called the (unit) ball-touching representation of the related graph.

If we consider the dimension 1, the touching case is trivial, and the intersection case leads to well known interval graphs [**?**] (or unit-interval graphs). In the dimension 2, the classical result of Koebe is mentioned above. In opposite, the unit-disc touching graphs (and more generally disc touching graphs with bounded ratio of diameters) were considered in [**?**], and turned out to be *NP*-hard to recognize. Let us mention that also intersection graphs of arbitrary discs are *NP*-hard [**?**].

A natural question arises what is the complexity of recognition of ball touching graphs in higher dimensions. It is likely that the unit-ball touching graphs are *NP*-hard in any dimension greater than 1. This paper proves the case of unit balls in dimensions 3 and 4.

The next section also mentions that from this result it follows that recognizing touching graphs of arbitrary balls in dimensions 4 and 5 is *NP*-hard, whereas the case of dimension 3 was already shown to be *NP*-hard by Kirkpatrick.

2 Preliminary results

To get a better feeling about touching graphs of balls or of unit balls in higher dimensions, let us first mention two easy (and probably not original) facts about representability of graphs by balls.

Lemma 2.1 *For every $d \geq 3$, there exists a constant $k(d)$ such that the d–ball touching graphs are $k(d)$–degenerated.*

Proof. Let ϱ be a ball of the smallest radius r in the touching representation of a graph G. Then each ball touching ϱ can be reduced to radius r so that it still touches ϱ at the same point (but possibly not other balls). Since we have a touching representation, the interiors of all such balls around ϱ are pairwise disjoint, and they are, after reduction, all contained in a ball $\bar{\varrho}$ of radius $3r$ concentric with ϱ. Now it suffices to consider the volume of $\bar{\varrho}$, thus $k(d) < 3^d$. □

Realize that the exact value of $k(d)$ is closely related to the so called kissing number problem (how many non-intersecting unit balls can touch one central unit ball) which is not completely solved yet even in dimension 4 (!), so it is probably very difficult to determine.

On the other hand, if the dimension is not fixed, any graph can be represented by unit balls:

Lemma 2.2 *Every graph G on v vertices has a unit-ball touching representation in dimension $v - 1$.*

Proof. Any graph G is an induced subgraph of a d–regular graph G' ($d = \Delta(G)$). Let $\bar{G}$ be obtained from G' by subdividing each edge by a new vertex, and $\bar{M}$ be the adjacency matrix of $\bar{G}$. Then the rows of $\bar{M}$ corresponding to the vertices of G can be taken as centres of balls of the touching representation—two centres x_i, x_j have distance $\sqrt{2d}$ if $\{i,j\} \notin E(G)$, and $\sqrt{2d-2}$ if $\{i,j\} \in E(G)$ (e.g. the "unit distance" equals $\sqrt{2d-2}$). □

Recently, Rote and Kirkpatrick [personal communication] discovered an elegant transformation that allows us to prove the *NP*-hardness of recognizing $(d+1)$-ball touching graphs from the *NP*-hardness of recognizing d-unit-ball touching graphs:

Lemma 2.3 (Kirkpatrick and Rote) *The question whether a given graph G has a unit-ball touching representation in dimension d, can be reduced to the question whether the graph $G \oplus K_2$ has a ball touching representation in dimension $d+1$.*

3 Unit-ball touching graphs in E^3

In this section, a touching graph means a unit-ball touching graph in the 3–dimensional space.

Theorem 1 *The unit-ball touching graphs in 3–dimensional space are NP-hard to recognize.*

Fig. 1. A touching representation of the *FRAME* $_3$ graph

Fig. 2. A detail of a corner of *FRAME* $_3$

The proof of this theorem goes as follows: We form a "firm" global frame (shaped as a large square) that has a unique touching representation, and represent an instance of the $3-SAT$ problem within this frame. These ideas are contained in the following sequence of lemmas.

Let us denote by *FRAME* $_3$ the graph that is shown in Figures **??** and **??**, sufficiently large to represent a given SAT formula (see later). The same picture also shows a touching representation of this graph, consisting of a large square layer of unit balls, with additional balls on the boundary in subsequent layers below and above.

Lemma 3.1 *The graph FRAME* $_3$ *has a unique touching representation in the 3–dimensional space (up to an isometry).*

Fig. 3. How the vertices of F "stick together"

Proof. To prove the lemma, see that the boundary can be decomposed into a sequence of tetra- and octahedrons, each successive pair of them sharing a whole triangle, see Figure **??**. Since an edge in the graph F means a unit distance between the centres of the corresponding balls, and both the regular tetra- and octahedrons are unique, there is only one possible shape of the touching representation of the boundary in E^3. Then the internal balls are arranged in "chains" tightly stretched between the boundary balls, so have unique positions by the triangle inequality. $\square$

Lemma 3.2 *The maximum number of disjoint unit balls that can touch one of the interior balls of* $FRAME_3$ *from one side of the central layer, is equal to* 3.

Proof. Imagine the interior balls of $FRAME_3$ as unit balls with centers in $[x, y, 0]$ where $x, y \in Z$. It is shown how to arrange three disjoint unit balls with a positive third coordinate, all touching the ball B_0 with center $[0, 0, 0]$ and disjoint with other balls of the frame: The centres of these balls are in coordinates $[-\frac{1}{2} + \gamma, \frac{1}{2} - \alpha, z_1]$, $[\frac{1}{2} - \beta, \frac{1}{2} - \gamma, z_2]$, $[\frac{1}{2} - \alpha, -\frac{1}{2} + \gamma, z_3]$, where α, β, γ are very small satisfying $0 < \gamma \lll \beta^2 \lll \alpha^4$ and z_1, z_2, z_3 are determined by unit distance from the origin (close to $\frac{1}{2}\sqrt{2}$).

The proof of the upper bound (i.e. that there cannot be 4 such balls) is too technical and is not presented here. In fact, the upper bound of 3 balls is not critical in the presented *NP* reduction, which may be adapted to any bound greater than 3 (see also the next section). $\square$

For the next classical result see [**?**].

Lemma 3.3 *The* $3 - SAT$ *problem is NP-complete.*

Now we are ready to describe the *NP* reduction to the unit-ball touching problem.

Lemma 3.4 *For every* $3 - SAT$ *formula* φ*, there exists a graph (of polynomial size with respect to* φ*) that has a unit-ball touching representation in* E^3*, if and only if* φ *is satisfiable.*

Proof. Once having built the above firm frame, it is easy to represent a given formula φ within it. Each variable v is replaced by two vertices $v, \neg v$, both adjacent to four chosen neighbouring vertices of the internal layer of $FRAME_3$ (forming an octahedron as on the boundary, see Figure **??**). Each clause c is represented by one chosen vertex of $FRAME_3$. An instance of variable is then connected to its clause by a suitably long path (represented by a chain of successively touching balls). Since we are in 3 dimensions, we do not need to bother with chain crossings. A scheme of this construction is shown in Figure **??**.

Fig. 4. Representing variables and clauses, and their connection

Of course this is not all, we need to distinguish the false and true sides of each clause vertex (the variable vertices are naturally symmetric). This is done globally for all clauses by introducing a special new vertex o adjacent to some four neighbouring vertices of the layer, and connected by sufficiently long paths to all vertices representing clauses, see Figure **??**. This trick also solves the problem of clauses containing only 2 variables, such clauses are connected to o by two paths.

All these paths are then concentrated to the vertex o using a binary tree. The side of a touching representation of $FRAME_3$, which the ball representing o lies in, then becomes the false side for clauses (since every clause has one or two of the 3 touching positions on that side occupied by a path to o).

Fig. 5. The common "obstacle" for clauses on the false side

In general, we first position the variable and clause vertices within the internal layer of $FRAME_3$. Then we determine the length of the paths connecting clauses and variables (or the special vertex o) so that they can be realized without crossing in a possible touching representation. It is clearly enough to use a size of the occupied area and of paths quadratic in the input formula size. Then the whole graph $FRAME_3$ is made so large that the paths cannot go behind it. Of course, the paths also cannot go through the internal layer. Thus they can correctly encode logical values as needed in the reduction.

Suppose now that there is a touching representation of the above constructed graph. Then, by Lemma **??**, each clause vertex must have at least one of the paths connecting it to its variables coming from the true side of the frame layer (i.e. from the side opposite to that containing a ball representing vertex o). The position of the v or $\neg v$ ball representing a variable v on the true or false side then determines the logical value of v, and in that evaluation each clause is true, so φ is satisfied.

Conversely, having a satisfying evaluation for φ, we can construct a touching representation of our graph, using the above-presented ideas in the opposite direction. $\square$

4 Unit-ball touching graphs in E^4

In this section, we briefly show how the ideas of the proof of Theorem **??** can be adapted to 4–dimensional space. However, this is still only a special construction—we cannot proceed into higher dimensions with the same arguments, since the 4–dimensional cube already has a too long diagonal (and we may "insert" another unit ball into it).

Theorem 2 *The unit-ball touching graphs in 4–dimensional space are NP-hard to recognize.*

Sketch of proof. The proof proceeds by the same steps as the previous one, and we only sketch it.

Fig. 6. A scheme of the internal layer of $FRAME_4$

Fig. 7. A touching graph of $FRAME_4$ at the corner

First we construct a (sufficiently large) graph $FRAME_4$, shown in Figures **??**,**??**, with a unique touching representation. This is the most important step of the reduction. The $FRAME_4$ graph is formed from a large "cube" grid (in the sense of edge structure of 3-cube) of vertices. An example of one such cube is shown in Figure **??**, vertices denoted by $c_1, c_2, \ldots, c_8$. Each cube on the boundary is also adjacent to two neighbouring vertices, in the example they are denoted by f_1, f_2. We add more some vertices to the cube faces on the boundary, such as the vertices d_1, d_2, d_3 in the example (but all three of them are used only in corners…).

It is easy to represent this structure using the following coordinates for ball centers: $c_1 = [0,0,0,0]$, $c_2 = [1,0,0,0]$, $c_3 = [1,1,0,0]$, $c_4 = [0,1,0,0]$, $c_5 = [0,0,1,0]$, $c_6 = [1,0,1,0]$, $c_7 = [1,1,1,0]$, $c_8 = [0,1,1,0]$, $f_1 = [\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}]$, $f_2 = [\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}]$, $d_1 = [\frac{1}{2},-\frac{1}{2}\sqrt{2},\frac{1}{2},0]$, $d_2 = [1+\frac{1}{2}\sqrt{2},\frac{1}{2},\frac{1}{2},0]$, $d_3 = [\frac{1}{2},\frac{1}{2},1+\frac{1}{2}\sqrt{2},0]$, and other balls by translating this scheme.

To see that $FRAME_4$ has unique touching representation, realize that all centres of $c_1, \ldots, c_8$ must lie in an intersection of spheres of diameter 2 centred in f_1 and f_2, that is on a $3D$ sphere. With the help of the additional vertices d_1, d_2, d_3, the cube shape of $c_1, \ldots, c_8$ is forced. The same arguments may be applied to other boundary cubes, going from corners. Finally, the internal vertices are then tightly stretched inside the firm border.

The variables and clauses are represented within $FRAME_4$ in a similar way as previously, again connected together by paths. A problem is that we do not know exactly how many balls can touch one clause ball from one side of the frame layer, let this be a constant $t \geq 3$. Then we must connect the special vertex o (see the previous proof) by $t-2$ or $t-1$ paths to each vertex representing a clause.

The rest is very similar and we skip additional technical details here. □

5 Further work

If we want to proceed into higher dimensions, we get more and more difficult views of the situation. It is also related to the sphere-packing and kissing-number problems, that are very difficult in high dimensions and only little is known about them (except very special dimensions 8 and 24). We still do not have any idea how to construct a general reduction for unit-ball touching graphs, working in all dimensions. But since there is no obvious reason why the recognition problem should be easier in higher dimensions, we conjecture:

Conjecture. *The recognition of ball touching graphs and of unit-ball touching graphs is NP-hard in any fixed dimension.*

Another interesting question, which is not discussed in this work at all, is about the minimal size of the touching representation of a ball touching graph (related to the question whether their recognition belongs to *NP*). It seems that not all touching graphs have a representation using only small number of bits for coordinates, since we can easily force strange irrational distances. But we have no results in this area, and the problem may belong to *NP* using quite different certificate.

References

1. H. Breu, D.G. Kirkpatrick, *On the complexity of recognizing intersection and touching graphs of discs*, In: Graph Drawing (F.J. Brandenburg ed.), Proceedings Graph Drawing '95, Passau, September 1995, Lecture Notes in Computer Science 1027, Springer Verlag 1996, 88–98.
2. H. de Fraysseix, P.O. de Mendez, P. Rosenstiehl, *On triangle contact graphs*, Combinatorics, Probability and Computing 3 (1994), 233–246.
3. M.R. Garey, D.S. Johnson, Computers and Intractability, W.H. Freeman and Company, New York 1978.
4. P. Hliněný, *Contact graphs of curves (extended abstract)*, In: Graph Drawing (F.J. Brandenburg ed.), Proceedings Graph Drawing '95, Passau, September 1995; Lecture Notes in Computer Science 1027, Springer Verlag 1996, 312–323.
5. P. Koebe, *Kontaktprobleme der konformen Abbildung,* Berichte über die Verhandlungen der Sächsischen, Akad. d. Wiss., Math.–Physische Klasse 88 (1936), 141–164.
6. J. Kratochvíl, *Intersection graphs of noncrossing arc-connected sets in the plane*, Proceedings Graph Drawing '96, Berkeley, September 1996; Lecture Notes in Computer Science 1190, Springer Verlag, Berlin Heidelberg 1997.
7. C.B. Lekkerkerker, J.C. Boland, *Representation of finite graphs by a set of intervals on the real line*, Fund. Math. 51 (1962), 45–64.

This article was processed using the LaTeX macro package with LLNCS style